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PENTAVALENT SYMMETRIC GRAPHS OF ORDER 30*p*

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Abstract

A complete classification is given of pentavalent symmetric graphs of order $30p$, where $p \geq 5$ is a prime. It is proved that such a graph Γ exists if and only if $p = 13$ and, up to isomorphism, there is only one such graph. Furthermore, Γ is isomorphic to $\mathcal{C}_{390}$, a coset graph of PSL(2, 25) with $\mathsf{Aut}\,\Gamma = \mathrm{PSL}(2, 25)$, and Γ is 2-regular. The classification involves a new 2-regular pentavalent graph construction with square-free order.

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1. Introduction

In this paper, all graphs are assumed to be finite, simple unless stated otherwise, connected and undirected.

Let Γ be a graph. We denote by $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\,\Gamma$ its vertex set, edge set and automorphism group, respectively. An *arc* in Γ is an ordered pair of adjacent vertices. Let $A\Gamma$ denote the arc set of Γ. Let s be a positive integer. An *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that $(v_{i-1}, v_i) \in A\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. Let X be a subgroup of $\mathsf{Aut}\,\Gamma$. The graph Γ is said to be (X, s)-*arc-transitive* or (X, s)-*regular* if X is transitive or regular on the s-arcs of Γ; and Γ is called (X, s)-*transitive* if it is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive. In the case where $X = \mathsf{Aut}\,\Gamma$, an (X, s)-arc-transitive, (X, s)-regular or (X, s)-transitive graph is said to be an *s-arc-transitive*, *s-regular* or *s-transitive* graph. In particular, a 0-arc-transitive graph is called a *vertex transitive* graph, and a 1-arc-transitive graph is called an *arc-transitive* graph or *symmetric* graph.

Characterising symmetric graphs with small valency is a current topic in the literature. Cubic and tetravalent graphs have been studied extensively, and it is natural to consider pentavalent graphs [1, 8–12, 15, 17, 18, 21]. In particular, [7] classified the symmetric graphs of order 30. In this paper, we classify pentavalent symmetric graphs

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TABLE 1. Pentavalent symmetric graphs of order $30p$ ($p \geq 5$).

Γ	p	$\mathsf{Aut}\,\Gamma$	$(\mathsf{Aut}\,\Gamma)_v$	Transitivity	Remark
C_{390}	13	PSL (2, 25)	F_{20}	2-transitive	Lemma 3.4

of order $30p$, with p a prime. Since the cases $p = 2$ and $p = 3$ have been treated in the classifications of arc-transitive pentavalent graphs of order $12p$ and $18p$ in [10], we consider the case where $p \geq 5$. The main result of this paper is the following theorem, which slightly improves the result in [7].

THEOREM 1.1. *Let Γ be a pentavalent symmetric graph of order $30p$, where $p \geq 5$ is a prime. Then $p = 13$ and, up to isomorphism, there exists only one graph Γ with $\Gamma \cong C_{390}$ as in Construction 3.3. Furthermore, $\mathsf{Aut}(\Gamma)$ and $\mathsf{Aut}(\Gamma)_v$ are described in Table 1, where $v \in V\Gamma$.*

2. Preliminary results

In this section, we give some necessary preliminary results.

For a graph Γ and $X \leq \mathsf{Aut}\,\Gamma$, let N be an intransitive normal subgroup of X on the vertices of Γ. Denote by V_N the set of N-orbits in V. The normal quotient graph Γ_N is defined as the graph with vertex set V_N, and two N-orbits $B, C \in V_N$ are adjacent in Γ_N if some vertex of B is adjacent in Γ to some vertex of C. By [19, Theorem 4.1] and [14, Lemma 2.5], we have the following proposition.

PROPOSITION 2.1. *Let Γ be a connected regular graph of prime valency $p > 2$ and let X be a group of automorphisms of Γ which is arc-transitive on Γ. If a normal subgroup N of X has more than two orbits on $V\Gamma$, then Γ_N is a connected X/N-arc transitive graph of valency p and N is the kernel of the action of X on V_N. Furthermore, N is semiregular on $V\Gamma$.*

Denote by F_{20} the Frobenius group of order 20. The next lemma is about the structure of the vertex-stabilisers of pentavalent symmetric graphs. It is due to [8, 21].

LEMMA 2.2. *Let Γ be a pentavalent (X, s)-transitive graph for some $X \leq \mathsf{Aut}\,\Gamma$ and $s \geq 1$. Let $v \in V\Gamma$. If X_v is soluble, then $|X_v| \mid 80$ and $s \leq 3$. If X_v is insoluble, then $|X_v| \mid 2^9 \cdot 3^2 \cdot 5$ and $2 \leq s \leq 5$. Furthermore, one of the following holds:*

(1) $s = 1$, $X_v \cong \mathbb{Z}_5$, D_{10} *or* D_{20};
(2) $s = 2$, $X_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 *or* S_5;
(3) $s = 3$, $X_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $(A_4 \times A_5) : \mathbb{Z}_2$ *or* $S_4 \times S_5$;
(4) $s = 4$, $X_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ *or* $\mathrm{A\Gamma L}(2, 4)$;
(5) $s = 5$, $X_v \cong \mathbb{Z}_2^6 : \Gamma\mathrm{L}(2, 4)$.

From [6, pages 12–14], one may obtain the following proposition by checking the nonabelian simple groups with three prime factors.

PROPOSITION 2.3. *Let G be a nonabelian simple group and $|G| = 2^k \cdot 3^l \cdot 5$. Then $G = A_5$, A_6 or $\mathrm{PSU}(4, 2)$.*

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By checking the orders of nonabelian simple groups, see [6, pages 134–136], we have the following proposition.

PROPOSITION 2.4. *Let $p > 5$ be a prime and let G be a $\{2, 3, 5, p\}$-nonabelian simple group such that $|G|$ divides $2^{10} \cdot 3^3 \cdot 5^2 \cdot p$ and $3 \cdot 5^2 \cdot p$ divides $|G|$. Then $G = \mathrm{PSL}(2, 25)$, $\mathrm{PSU}(3, 4)$, J_2 or $\mathrm{PSp}(4, 4)$.*

From [12], we give some information about pentavalent symmetric graphs of order $10p$ or $6p$ in the following lemma. The graph C_n, following the notation in [12], denotes the corresponding graph of order n, which is a coset graph, and $C\mathcal{D}^l_{10p}$ is defined as a Cayley graph of order $10p$.

LEMMA 2.5. *Let Γ be a pentavalent symmetric graph. Let $p > 5$ be a prime. Then one of the following holds.*

(1) *If $|V\Gamma| = 10p$, then either $\Gamma \cong C_{170}$ with $p = 17$ and $\mathsf{Aut}\,\Gamma \cong \mathsf{Aut}(\mathrm{PSp}(4, 4))$ or $\Gamma \cong C\mathcal{D}^l_{10p}$ with $\mathsf{Aut}\,\Gamma \cong \mathrm{D}_{10p} : \mathbb{Z}_5$.*
(2) *If $|V\Gamma| = 6p$, then $\Gamma \cong C_{42}$ and $\mathsf{Aut}\,\Gamma \cong \mathsf{Aut}(\mathrm{PSL}(3, 4))$, $\Gamma \cong C_{66}$ and $\mathsf{Aut}\,\Gamma \cong \mathsf{Aut}(\mathrm{PGL}(2, 11))$ or $\Gamma \cong C_{114}$ and $\mathsf{Aut}\,\Gamma \cong \mathsf{Aut}(\mathrm{PGL}(2, 19))$.*

In the following, we give some information about pentavalent symmetric graphs of order 50. First we need the definition of bi-coset graph. Let G be a finite group. Given two subgroups L, R of G such that $L \cap R$ is core-free in G, define the *bi-coset graph* $\mathsf{Cos}(G, L, R)$ of G with respect to L and R as the graph with vertex set $[G : R] \cup [G : L]$ such that Lx, Ry are adjacent if and only if $yx^{-1} \in RL$. By [5, Lemma 3.7], a bi-coset graph Γ has the following properties.

LEMMA 2.6. *Let $\Gamma = \mathsf{Cos}(G, L, R)$ be a bi-coset graph. Then:*

(1) *Γ is G-edge transitive and G-vertex intransitive;*
(2) *$|\Gamma(v)| = |L : L \cap R|$ and $|\Gamma(w)| = |R : L \cap R|$, where $v \in [G : L]$ and $w \in [G : R]$.*

Conversely, if Γ is a G-edge-transitive but not G-vertex-transitive graph, then Γ is isomorphic to a bi-coset graph $\mathsf{Cos}(G, G_v, G_w)$, where v and w are two adjacent vertices.

By [18], we have the following lemma.

LEMMA 2.7. *Let Γ be a connected pentavalent symmetric graph of order 50. Then $\Gamma \cong C_{50}$, where $C_{50} = \mathsf{Cos}(G, L, R)$ and*

$$G = \langle a, b, c \mid a^5 = b^5 = c^5 = [a, c] = [b, c] = 1, [a, b] = c\rangle$$

is an extra-special group of order 5^3, $L = \langle a\rangle$ and $R = \langle b\rangle$.

REMARKS 2.8. By Magma [2], $\mathsf{Aut}\,C_{50} \cong G : (\mathbb{Z}_4^2 : \mathbb{Z}_2)$, which is arc-transitive on C_{50}, but C_{50} is not G-vertex-transitive. Furthermore, $\mathsf{Aut}\,C_{50}$ is soluble.

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Let G be an extension of N by H, that is, $G/N \cong H$. Recall that an extension is called a central extension if N is the centre of G. A group G is said to be perfect if $G = G'$, the commutator subgroup of G. For a given group H, if N is the largest abelian group such that $G := N.H$ is perfect and the extension is a central extension, then N is called the Schur multiplier of H, written $\mathsf{Mult}(H)$. Since $\mathrm{GL}(2, p)$ contains no nonabelian simple groups (see [4, Lemma 2.7], for example), it is easily shown that the extension $G = N.T$, where $N = \mathbb{Z}_p^2$ and T is a nonabelian simple group, is a central extension. By [13], the following lemma is known.

Lemma 2.9. *Assume that $G = N.T$, where N is cyclic or $|N|$ is prime square, and T is a nonabelian simple group. Then $G = N.T$ is a central extension. Furthermore, $G = NG'$ and $G' = M.T$, where $M \le N$ is a subgroup of* $\mathsf{Mult}(T)$.

3. An example of pentavalent symmetric graph of order 30*p*

In the following, we construct a pentavalent symmetric graph of order 390. To do this, we first introduce the definition of coset graph. Let G be a finite group and let H be a core-free subgroup of G. Define the *coset graph* $\mathsf{Cos}(G, H, HgH)$ of G with respect to H as the graph with vertex set $[G : H]$ such that Hx, Hy are adjacent if and only if $yx^{-1} \in HgH$. The following propositions about coset graphs are well known; see [16, 20].

Lemma 3.1. *Using notation as above, let $\mathsf{val}\,\Gamma$ be the valency of Γ. Then the coset graph $\Gamma = \mathsf{Cos}(G, H, HgH)$ is a G-arc transitive graph and*

(1) $\mathsf{val}\,\Gamma = |H : H \cap H^g|$;

(2) *Γ is undirected if and only if there exists a 2-element $g \in G \setminus H$ such that $g^2 \in H$;*

(3) *Γ is connected if and only if $\langle H, g\rangle = G$.*

Conversely, each G-symmetric graph Σ is isomorphic to the coset graph $\mathsf{Cos}(G, G_v, G_v g G_v)$, where $g \in \mathrm{N}_G(G_{vw})$ is a 2-element such that $g^2 \in G_v$, and $v \in V\Sigma$, $w \in \Sigma(v)$.

Remarks 3.2. For every $\alpha \in \mathsf{Aut}(G)$, $\mathsf{Cos}(G, H, HgH) \cong \mathsf{Cos}(G, H^\alpha, H^\alpha g^\alpha H^\alpha)$.

Construction 3.3. *Let $T \le \mathrm{S}_{26}$ such that $T \cong \mathrm{PSL}(2, 25)$. We may choose the following elements in S_{26}:*

$a = (1\ 20\ 21\ 24\ 11)(3\ 19\ 6\ 9\ 10)(4\ 18\ 14\ 25\ 7)(5\ 16\ 12\ 26\ 23)(8\ 15\ 22\ 13\ 17)$,
$b = (1\ 3\ 15\ 23)(4\ 18\ 25\ 14)(5\ 21\ 10\ 17)(6\ 8\ 12\ 20)(9\ 22\ 16\ 11)(13\ 26\ 24\ 19)$,
$\tau = (2\ 7)(3\ 23)(4\ 11)(5\ 12)(6\ 10)(8\ 21)(9\ 14)(13\ 19)(16\ 18)(17\ 20)(22\ 25)(24\ 26)$.

Then $T = \langle a, b, \tau\rangle$. Let $H = \langle a, b\rangle \cong \mathbb{Z}_5 : \mathbb{Z}_4$. Define the coset graph $C_{390} = \mathsf{Cos}(T, H, H\tau H)$.

Lemma 3.4. *The graph C_{390} is pentavalent symmetric of order 390. Moreover, $\mathsf{Aut}\, C_{390} \cong \mathrm{PSL}(2, 25)$, which acts 2-arc-regularly on Γ.*

Conversely, each pentavalent symmetric graph of order 390 admitting $\mathrm{PSL}(2, 25)$ as an arc-transitive automorphism group is isomorphic to C_{390}.

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Proof. By Magma [2], C_{390} is a connected pentavalent symmetric graph of order 390 and $\mathrm{Aut}(C_{390}) \cong \mathrm{PSL}(2, 25)$. Further, the number of 2-arcs of Γ is $390 \cdot 5 \cdot 4 = |\mathrm{PSL}(2, 25)|$, which implies that Γ is 2-arc regular.

Conversely, let $T = \langle a, b, \tau\rangle \cong \mathrm{PSL}(2, 25)$ and let Γ be a pentavalent symmetric graph of order 390 admitting T as an arc-transitive automorphism group. By Lemma 3.1, Γ is a coset graph of T with respect to a subgroup $H \leq T$ of order 20. Moreover, T has two conjugacy classes of subgroups of H with $H \cong \mathbb{Z}_5 : \mathbb{Z}_4$, which are fused in $\mathrm{Aut}\, T = \mathrm{P\Gamma L}(2, 25)$. By Remark 3.2, we may assume $H = \langle a, b\rangle \cong \mathbb{Z}_5 : \mathbb{Z}_4$. Let $P = \langle b\rangle \cong \mathbb{Z}_4$. Then Γ is isomorphic to a graph of $\mathrm{Cos}(T, H, HgH)$ such that g is a 2-element in $T \setminus H$, $g^2 \in H$ and $g \in \mathrm{N}_T(P) \cong \mathrm{D}_{24}$. Moreover, g satisfies $|H : H \cap H^g| = 5$ and $\langle H, g\rangle = T$. By Magma [2], there are eight choices for g and each such g is an involution. Let S be the set of all such involutions. Note that some of the elements in S are conjugate in $\mathrm{N}_{\mathrm{Aut}\, T}(H)$. By Magma [2], we have two choices g which are not conjugate in $\mathrm{N}_{\mathrm{Aut}\, T}(H)$. Furthermore, their representatives are τ and τ', where

$$\tau' = (1\ 14)(2\ 7)(3\ 25)(4\ 23)(5\ 24)(6\ 12)(9\ 11)(10\ 13)(15\ 18)(16\ 22)(17\ 19)(21\ 26).$$

Again by Magma [2], $\mathrm{Cos}(T, H, H\tau H) \cong \mathrm{Cos}(T, H, H\tau' H)$, as required. □

4. The proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The next simple lemma is helpful to our argument.

Lemma 4.1. *Let Γ be an X-arc-transitive pentavalent graph of order $30p$, where p is a prime and $X \leq \mathrm{Aut}\,\Gamma$. Then for each insoluble normal subgroup $N \trianglelefteq X$, the following hold:*

(1) *N has at most two orbits on $V\Gamma$;*
(2) *For each $v \in V\Gamma$, $5 \mid |N_v^{\Gamma(v)}|$.*

Proof. (1) Suppose that N has at least three orbits on $V\Gamma$. Then, by Proposition 2.1, N is semiregular on $V\Gamma$. Hence $|N| \mid 30p$. Since a group of order $30p$ is soluble, it follows that N is soluble, a contradiction.

(2) For each $v \in V\Gamma$, since $N_v \neq 1$ and X is transitive on $V\Gamma$, we have $|N_v^{\Gamma(v)}| \neq 1$. It follows that $5 \mid |N_v^{\Gamma(v)}|$ since $N_v^{\Gamma(v)} \trianglelefteq X_v^{\Gamma(v)}$ and $X_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$, as required. □

Proof of Theorem 1.1. For the remainder of this paper, we let Γ be a symmetric pentavalent graph of order $30p$, where p is a prime. Let $A = \mathrm{Aut}\,\Gamma$. We first consider the case $p = 5$, beginning with the following lemma.

Lemma 4.2. *There exists no pentavalent symmetric graph of order 150.*

Proof. Let N be a minimal normal subgroup of A. Assume first that N is soluble. Then N is isomorphic to $\mathbb{Z}_r^d$ for some prime r and integer $d \geq 1$. Since N is half transitive on $V\Gamma$ and $|V\Gamma| = 150$, N has at least three orbits on $V\Gamma$. Thus, by Proposition 2.1, N is

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semiregular. It follows that $N \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$ or $\mathbb{Z}_5^2$. If $N \cong \mathbb{Z}_2$, then by Proposition 2.1, Γ_N is a pentavalent symmetric graph of odd order, a contradiction. If $N \cong \mathbb{Z}_5$, then Γ_N is a pentavalent symmetric graph of order $2 \cdot 3 \cdot 5$, but by [12], there exist no graphs of this order.

Assume that $N \cong \mathbb{Z}_5^2$. In this case, $\Gamma_N \cong \mathsf{K}_6$ and $A/N \le \mathrm{S}_6$. By Proposition 2.1, Γ_N is A/N-arc-transitive and so $5 \cdot 6 \mid |A/N|$. By the structure of subgroups of S_6, A/N is isomorphic to A_5, S_5, A_6 or S_6. For the case $A/N \cong \mathrm{A}_5$ or A_6, by Lemma 2.9, $A = N.T$ is a central extension of N by T, where $T = \mathrm{A}_5$ or A_6; furthermore, $A' = T$ since $\mathsf{Mult}(T) = \mathbb{Z}_2$ or $\mathbb{Z}_6$, which is normal in A. If A' has at least three orbits on $V\Gamma$, then A' is semiregular. It follows that $|A'| \mid |V\Gamma| = 150$, which is impossible. Thus A' has at most two orbits on $V\Gamma$, and so $3 \cdot 5^2 \mid |T|$, which is also impossible. For the case $A/N \cong \mathrm{S}_5$ or S_6, A/N has a normal subgroup $M/N \cong \mathrm{A}_5$ or A_6. Similarly, M is a central extension of N by T, where $T = \mathrm{A}_5$ or A_6, and $M' = T$ which is normal in A. By the above discussion, a contradiction occurs.

We next assume that $N \cong \mathbb{Z}_3$, then Γ_N is a pentavalent symmetric graph with order $2 \cdot 5^2$. By Lemma 2.7, Γ_N is isomorphic to C_{50}. Then A is soluble because $A/N \lesssim \mathsf{Aut}\, C_{50}$. Let F be the Fitting subgroup of A, the subgroup generated by all the normal nilpotent subgroups of A. Since A is soluble, we have $F \ne 1$ and $C_A(F) \le F$.

By the above discussion, A has no nontrivial normal 2-subgroups and 5-subgroups, and so $F = O_3(A)$, the maximal normal 3-subgroup of A. By Proposition 2.1, F is semiregular. Then $|F| = 3$ and so F is abelian and $C_A(F) = F$. It follows that $A/F = A/C_A(F) \lesssim \mathsf{Aut}(F) \cong \mathbb{Z}_2$, which is impossible.

We now suppose that A has no soluble minimal normal subgroups. Then $N = T^d$, where T is a nonabelian simple group. By Lemma 2.2, for a vertex $v \in V\Gamma$, we have $|N_v| \mid 2^9 \cdot 3^2 \cdot 5$ and so $|N| = |T|^d$ divides $2^{10} \cdot 3^3 \cdot 5^3$. Then T is a $\{2,3,5\}$-nonabelian simple group. By Proposition 2.3, T is isomorphic to one of the groups A_5, A_6 or $\mathrm{PSU}(4,2)$. Assume that $d \ge 2$. Then the only possible case is $T = \mathrm{A}_5$ and $d = 2$ or 3. We first suppose that $d = 2$. Then N is an insoluble normal subgroup of A, and by Lemma 4.1, N has at most two orbits on $V\Gamma$ and $5 \mid |N_v|$. However, $|N_v| = |N|/150 = 24$ or $|N_v| = |N|/75 = 48$, giving a contradiction. Now suppose that $d = 3$. Then $N = T_1 \times T_2 \times T_3$ with $T_i \cong \mathrm{A}_5$ and $i = 1,2,3$. By Lemma 4.1, N has at most two orbits on $V\Gamma$ and $5 \mid |N_v^{\Gamma(v)}|$. Suppose that N is transitive on $V\Gamma$. Then N is arc-transitive on Γ. By Lemma 4.1, for every i and each $v \in V\Gamma$, $5 \mid |(T_i)_v|$, and so $5^3 \mid |N_v|$, in contradiction to $|N_v| \mid 2^9 \cdot 3^2 \cdot 5$. Now suppose that N has exactly two orbits on $V\Gamma$. Then $|N_v| = |N|/75 = 2880$. By Lemma 2.2, we have $A_v \cong \mathrm{A\Gamma L}(2,4)$ or $\mathbb{Z}_2^6 : \Gamma\mathrm{L}(2,4)$ since $N_v \lhd A_v$. For the former case, $N_v \cong \mathrm{AGL}(2,4)$. For the later case, by Magma [2], $N_v \cong (\mathrm{A}_6 : \mathbb{Z}_4) : \mathbb{Z}_2$. This is impossible since $N \cong \mathrm{A}_5^3$ has no subgroups isomorphic to $\mathrm{AGL}(2,4)$ or $(\mathrm{A}_6 : \mathbb{Z}_4) : \mathbb{Z}_2$. Hence $d = 1$ and $N = T \trianglelefteq A$ is isomorphic to A_5, A_6 or $\mathrm{PSU}(4,2)$. By Lemma 4.1, N has at most two orbits on $V\Gamma$. It follows that $3 \cdot 5^2 \mid |N|$, which is also impossible. □

We now consider the case where $p > 5$. First we suppose that A contains a soluble minimal normal subgroup N. We have the following lemma.

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LEMMA 4.3. *If A has a soluble minimal normal subgroup N, then no graphs appear.*

PROOF. By assumption, $N \cong \mathbb{Z}_q^d$ with q a prime and d a positive integer. It is easy to prove that N has at least three orbits on $V\Gamma$. By Proposition 2.1, N is semiregular on $V\Gamma$, and hence $|N| \mid 30p$. Thus $N \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$ or $\mathbb{Z}_p$. Let us consider these one by one.

If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent symmetric graph of odd order, which is impossible.

If $N \cong \mathbb{Z}_p$, then Γ_N is a pentavalent symmetric graph of order $2 \cdot 3 \cdot 5$, which is also impossible by [12].

Now suppose that $N \cong \mathbb{Z}_3$. Then Γ_N is a pentavalent symmetric graph of order $2 \cdot 5 \cdot p$. By Lemma 2.5, we have $\Gamma_N \cong \mathcal{C}_{170}$ or $\mathcal{CD}_{10p}^l$.

Suppose that $\Gamma_N \cong \mathcal{C}_{170}$. Then $A/N \leq \mathsf{Aut}\,\Gamma_N \cong \mathsf{Aut}(\mathrm{PSp}(4,4))$. Since A/N is arc-transitive on Γ_N, we have $5 \cdot 170 \mid |A/N|$. By [3], $A/N \cong \mathrm{PSp}(4,4).O$, where $O \leq \mathbb{Z}_4$. Then A/N contains a normal subgroup M/N isomorphic to $\mathrm{PSp}(4,4)$. By Lemma 2.9, $M = N.T \cong \mathbb{Z}_3.\mathrm{PSp}(4,4)$ is a central extension of N by T, and $M' \cong \mathrm{PSp}(4,4)$ since $\mathsf{Mult}(T) = 1$, which is a normal subgroup of A. By Lemma 4.1, M' has at most two orbits on $V\Gamma$. If M' is transitive, then $M'N/N \cong \mathrm{PSp}(4,4)$ is transitive on $V\Gamma_N$. Let $\delta \in V\Gamma_N$; we have $|(M'N/N)_\delta| = |\mathrm{PSp}(4,4)|/170 = 5760$, which is impossible as $\mathrm{PSp}(4,4)$ has no subgroups of order 5760. Hence, M' has exactly two orbits on $V\Gamma$ and $|M'_v| = 3840$. By Lemma 4.1, $5 \mid |M_v'^{\Gamma(v)}|$ and $M_v'^{\Gamma(v)}$ is primitive on $\Gamma(v)$. Hence M' is edge-transitive on Γ. By Lemma 2.6, $\Gamma \cong \mathsf{Cos}(M', L, R)$, where $L = M'_v$, $R = M'_w$ and v and w are adjacent vertices. The valency of Γ equals $|L : L \cap R|$. But by Magma [2], all possible cases of $|L \cap R|$ are equal to 16, 256, 60 or 64, a contradiction since Γ is pentavalent.

If $\Gamma_N \cong \mathcal{CD}_{10p}^l$, then $A/N \leq \mathsf{Aut}\,\Gamma_N \cong \mathrm{D}_{10p} : \mathbb{Z}_5$. Since A/N is arc-transitive on Γ_N, we have $A/N \cong \mathrm{D}_{10p} : \mathbb{Z}_5$, and it follows that $A = N : H \cong \mathbb{Z}_3 : (\mathrm{D}_{10p} : \mathbb{Z}_5)$. Since H has a normal subgroup K which is isomorphic to $\mathbb{Z}_p$ and centralises $N = \mathbb{Z}_3$, we see that K is a normal subgroup of A. This implies that the corresponding normal quotient graph Γ_K is a pentavalent symmetric graph of order 30. However, by [12], there exists no pentavalent symmetric graph of order 30, a contradiction.

Finally, we assume that $N \cong \mathbb{Z}_5$. By Lemma 2.5, Γ_N is isomorphic to $\mathcal{C}_{42}$, $\mathcal{C}_{66}$ or $\mathcal{C}_{114}$. If $\Gamma_N \cong \mathcal{C}_{42}$, then $A/N \leq \mathsf{Aut}\,\Gamma_N \cong \mathsf{Aut}(\mathrm{PSL}(3,4))$ and $p = 7$. Note that A/N acts arc-transitively on Γ_N and so $5 \cdot 42 \mid |A/N|$. By checking the maximal subgroups of $\mathrm{PSL}(3,4)$, we have $A/N \cong \mathrm{PSL}(3,4).O$, where $O \leq \mathrm{D}_{12}$. Then A/N contains a normal subgroup $M/N \cong \mathrm{PSL}(3,4)$. By Atlas [3], $\mathsf{Mult}(\mathrm{PSL}(3,4)) \cong \mathbb{Z}_4^2 \times \mathbb{Z}_3$. Then, by Lemma 2.9, we have that $M = NM' = N \times M' \cong \mathbb{Z}_5 \times \mathrm{PSL}(3,4)$ is a normal subgroup of A. Since $M' \cong \mathrm{PSL}(3,4)$ is a characteristic subgroup of M, we have $M' \trianglelefteq A$. By Lemma 4.1, M' has at most two orbits on $V\Gamma$ and, for every vertex $v \in V\Gamma$, $5 \mid |M'_v|$. However, $|M'_v| = |M'|/210 = 96$ or $|M'_v| = |M'|/105 = 192$, a contradiction.

Now suppose that $\Gamma_N \cong \mathcal{C}_{66}$. Then $A/N \leq \mathsf{Aut}\,\Gamma_N \cong \mathrm{PGL}(2,11)$ and $p = 11$. In this case, $5 \cdot 66 \mid |A/N|$, and by checking the maximal subgroups of $\mathrm{PGL}(2,11)$, we have $A/N \cong \mathrm{PSL}(2,11).O$, where $O \leq \mathbb{Z}_2$. So A/N contains a normal subgroup M/N isomorphic to $\mathrm{PSL}(2,11)$. Then by Lemma 2.9, $M = N \times M' \cong \mathbb{Z}_5 \times \mathrm{PSL}(2,11)$ since

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$\mathsf{Mult}(\mathrm{PSL}(2, 11)) = \mathbb{Z}_2$. Note that $M' \cong \mathrm{PSL}(2, 11)$ is a normal subgroup of A and so, by Lemma 4.1, M' has at most two orbits on $V\Gamma$ and, for every vertex $v \in V\Gamma$, $5 \mid |M'_v|$. But, $|M'_v| = |M'|/330 = 2$ or $|M'_v| = |M'|/165 = 4$, a contradiction.

Finally, suppose that $\Gamma_N \cong C_{114}$. By Lemma 2.5, $A/N \leq \mathsf{Aut}\,\Gamma_N \cong \mathrm{PGL}(2, 19)$ and $p = 19$. Since A/N is arc-transitive on Γ_N, we have $5 \cdot 114 \mid |A/N|$. By checking the maximal subgroups of $\mathrm{PGL}(2, 19)$, we see that A/N contains a normal subgroup $M/N \cong \mathrm{PSL}(2, 19)$. Then by Lemma 2.9, $M = NM' = N \times M' = \mathbb{Z}_5 \times \mathrm{PSL}(2, 19)$ because $\mathsf{Mult}(\mathrm{PSL}(2, 19)) = \mathbb{Z}_2$. Hence $M' = \mathrm{PSL}(2, 19) \trianglelefteq A$. By Lemma 4.1, M' has at most two orbits on $V\Gamma$ and, for every $v \in V\Gamma$, $5 \mid |M'_v|$. This is impossible since $|M'_v| = |M'|/570 = 6$ or $|M'_v| = |M'|/285 = 12$. □

We now turn to the case where A has no soluble minimal normal subgroups. The next lemma completes the proof of Theorem 1.1.

Lemma 4.4. *If A has no soluble minimal normal subgroups, then $\Gamma \cong C_{390}$ as in Construction 3.3, and, up to isomorphism, there exists only this one graph.*

Proof. Let N be a insoluble minimal normal subgroup of A. Then $N = T^d$ with T a nonabelian simple group. By Lemma 4.1, N has at most two orbits on $V\Gamma$. Thus $15p$ divides $|N : N_v|$, and so $p \mid |T|$. Suppose that $d \geq 2$. Then $p^d \mid |N|$. However, by Lemma 2.2, $|A_v| \mid 2^9 \cdot 3^2 \cdot 5$, and so $|N| \mid |A| \mid 2^{10} \cdot 3^3 \cdot 5^2 \cdot p$, a contradiction. Hence $d = 1$ and $N = T \trianglelefteq A$. Let $C := \mathrm{C}_A(T)$. Then $C \lhd A$ and $CT = C \times T$. If $C \neq 1$, then C is insoluble because A has no soluble minimal normal subgroups. By Lemma 4.1, we have $5 \mid |C_v|$. On the other hand, $5 \mid |T_v|$, thus $5^2 \mid |A_v|$, but by Lemma 2.2 this is impossible. Hence $C = 1$ and A is an almost simple group.

Note that T has at most two orbits on $V\Gamma$, hence $|T_v| = |T|/30p$ or $|T_v| = |T|/15p$. Furthermore, $5 \mid |T_v|$. Now $|T| \mid |A| \mid 2^{10} \cdot 3^3 \cdot 5^2 \cdot p$ and $3 \cdot 5^2 \cdot p \mid |T|$. By Proposition 2.4, T is isomorphic to $\mathrm{PSL}(2, 25)$, $\mathrm{PSU}(3, 4)$, J_2 or $\mathrm{PSp}(4, 4)$.

Suppose that $T \cong \mathrm{PSU}(3, 4)$. Then $p = 13$ and $T \leq A \leq \mathsf{Aut}\,T = T.\mathbb{Z}_4$, and so $|A_v|$ divides $|\mathsf{Aut}\,T|/30 \cdot 13 = 640$. However, $|T_v| = 160$ or 320. Since $T_v \leq A_v$, by Lemma 2.2, $3 \mid |A_v|$, a contradiction.

Suppose that $T \cong \mathrm{J}_2$. Then $p = 7$ and $|T_v| = 2880$ or 5760. But by Atlas [3], J_2 has no subgroups of order 2880 or 5760.

Suppose that $T \cong \mathrm{PSp}(4, 4)$. Then $p = 17$ and $|T_v| = 1920$ or 3840. For the former case, T is transitive on $V\Gamma$ and, by Lemma 4.1, $5 \mid |T_v|$. It follows that T is arc-transitive on Γ. On the one hand, by Atlas, the subgroup of T with order 1920 is soluble. On the other hand by Lemma 2.2, we have $|T_v| \mid 80$, a contradiction. For the latter case, by Lemma 2.2, we have $A_v \cong 2^6 : \Gamma\mathrm{L}(2, 4)$, and so $|A| = 30 \cdot 17 \cdot |A_v| = 2^{10} \cdot 3^3 \cdot 5 \cdot 17$, which is impossible since $A \leq \mathsf{Aut}\,T$ and $|\mathsf{Aut}\,T| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$.

Suppose that $T \cong \mathrm{PSL}(2, 25)$. Then $p = 13$. If T has two orbits on $V\Gamma$, then $|T_v| = |T|/15 \cdot 13 = 40$. By Atlas [3], T has no subgroups of order 40. Hence T is transitive on $V\Gamma$. Further Γ is a pentavalent T-arc-transitive graph of order 390. So the graph is $\Gamma = C_{390}$ as in Construction 3.3. By Lemma 3.4, the proof is complete. □

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