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ON THE CONJECTURE OF JESMANOWICZ CONCERNING PYTHAGOREAN TRIPLES

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Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$. Jeśmanowicz conjectured in 1956 that for any given positive integer n the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $x = y = z = 2$. Building on the work of earlier writers for the case when $n = 1$ and $c = b + 1$, we prove the conjecture when $n > 1$, $c = b + 1$ and certain further divisibility conditions are satisfied. This leads to the proof of the full conjecture for the five triples $(a, b, c) = (3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$ and $(11, 60, 61)$.

1. INTRODUCTION

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$, and let n be a positive integer. Clearly, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z, \tag{1}$$

has the solution $x = y = z = 2$. Whether there are other solutions in positive integers when $n = 1$ has been investigated by a number of writers. Sierpiński [6] showed there were no other solutions when $n = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [2] that there were no others when $n = 1$ and $(a, b, c) = (5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$ and $(11, 60, 61)$. He conjectured that the equation (1) has no positive integer solutions for any n other than $x = y = z = 2$.

The general solution of $a^2 + b^2 = c^2$ in relatively prime positive integers is of course well known to be $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$, where $u > v > 0$, $\gcd(u, v) = 1$ and one of u, v is even, the other odd. A number of other special cases of Jeśmanowicz's conjecture have since been settled. Lu [5] proved it when $v = n = 1$. In 1965, Dem′janenko [1] extended earlier results in several papers by proving the conjecture to be true whenever $n = 1$ or 2 and $u = v + 1$. Takakuwa and Asaeda (see [7]) have proved the conjecture in a number of special cases in which, in particular,

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$n = 1$ and $v \equiv 1 \pmod 4$, and Takakuwa [8] has proved it when u is exactly divisible by 2, $v = 3$, 7, 11 or 15, and $n = 1$. More recently, Le has verified the conjecture if $n = 1$ and 2 exactly divides uv and, in [3], c is a prime power, in [4], $v \equiv 3 \pmod 4$ and $u \geqslant 81v$.

A more general conjecture has been considered by Terai (see [9]). He asks whether the equation (1) with $n = 1$ and $a^p + b^q = c^r$, has any positive integer solutions other than $(x, y, z) = (p, q, r)$. In particular, he has considered $(p, q, r) = (2, 2, 3)$ and $(p, q, r) = (2, 2, 5)$.

Some authors and reviewers have stipulated that $n = 1$ in (1), but this is apparently not part of Jeśmanowicz's conjecture. Nor is it a particularly easy case when $n > 1$. In this paper, we shall take $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, where k is a positive integer, and will obtain by completely elementary means certain conditions on n under which the only positive integer solution of the equation (1) is $x = y = z = 2$. This will lead us to prove Jeśmanowicz's conjecture in full for this generalisation of the original five cases settled by Sierpiński and Jeśmanowicz, that is, for $k \in \{1, 2, 3, 4, 5\}$.

For any integer $N > 1$ with prime factorisation $\prod_{i=1}^{t} p_i^{\alpha_i}$, we write $C(N) = \prod_{i=1}^{t} p_i$. All Greek and Roman letters in this paper denote positive integers unless specified otherwise.

The following two theorems will be proved.

THEOREM 1. *Let $a = 2k+1$, $b = 2k(k + 1)$, $c = 2k(k + 1)+1$, for some positive integer k. Suppose that a is a prime power, and that the positive integer n is such that either $C(b) \mid n$ or $C(n) \nmid b$. Then the only solution of the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ is $x = y = z = 2$.*

THEOREM 2. *For each of the Pythagorean triples $(a, b, c) = (3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, $(11, 60, 61)$, and for any positive integer n, the only solution of the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ is $x = y = z = 2$.*

Theorem 2 is the confirmation of Jeśmanowicz's conjecture in the five cases stated, corresponding to $k = 1$, 2, 3, 4, 5, respectively, in Theorem 1. The first case, when $k = 1$, is an immediate corollary of Theorem 1. The remainder of the proof of Theorem 2 uses Theorem 1 and special arguments in each of the cases $k = 2$, 3, 4, 5, with no pattern apparent. It is plausible that similar approaches will be successful for the next permissible cases $k = 6$, 8, 9, 11, ..., but the details have not been carried out.

Three lemmas will be required.

LEMMA 1. *Let $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, for some positive integer k. The only solution of the Diophantine equation $a^x + b^y = c^z$ is $x = y = z = 2$.*

This is Dem´janenko's result, mentioned above, when $u = v + 1$ and $n = 1$.

Lemma 2. *If $z \geqslant \max\{x, y\}$, then the Diophantine equation $a^x + b^y = c^z$, where a, b and c are any positive integers (not necessarily relatively prime) such that $a^2 + b^2 = c^2$, has no solution other than $x = y = z = 2$.*

Proof: If $z = 1$ then $x = y = 1$ and $(a+b)^2 > a^2 + b^2 = c^2$, so $a + b > c$. Suppose $z \geqslant 2$ and, without loss of generality, that $x \leqslant y$. If $y = 1$, then $a^x + b^y = a + b < a^2 + b^2 = c^2 \leqslant c^z$. If $y \geqslant 2$, then

$$a^x + b^y \leqslant (a^2)^{y/2} + (b^2)^{y/2} \leqslant (a^2 + b^2)^{y/2} = c^y \leqslant c^z,$$

and there is strict inequality unless $x = y = z = 2$. ∎

Lemma 3. *If p is an odd prime and $\gcd(a, b) = 1$, then*

$$\gcd\left(a + b, \frac{a^p + b^p}{a + b}\right) = 1 \textit{ or } p.$$

Proof: Let q be a prime divisor of $a + b$, so that $q \nmid a$ and $b \equiv -a \pmod{q}$. Then

$$\frac{a^p + b^p}{a + b} = a^{p-1} - a^{p-2}b + \cdots + b^{p-1} \equiv pa^{p-1} \pmod{q}.$$

It follows, if q is a divisor of $(a^p + b^p)/(a + b)$, that $q = p$ and that p is an exact divisor of $(a^p + b^p)/(a + b)$. ∎

2. Proof of Theorem 1

By Lemma 1, we may suppose $n > 1$, and by Lemma 2 that $z < \max\{x, y\}$. Of course, $a^2 + b^2 = c^2$. Notice also that $a^2 = b + c$, $c = b + 1$, $b = k(a + 1)$, $c = k(a - 1) + a$, and a, b, c are relatively prime in pairs. We also suppose that equation (1) holds, and will show that this leads to a contradiction. There are two main cases to the proof, depending on whether $\gcd(n, c) = 1$ or $\gcd(n, c) > 1$, and numerous subcases in each case, indexed by a decimal numbering system.

1 Suppose $\gcd(n, c) = 1$. We cannot have $x = y$, since then $z < x$ and, from (1), we may write $n^{x-z}(a^x + b^x) = c^z$. Then $\gcd(n, c) > 1$, a contradiction.

1.1 Suppose $x > y$, so that we may write $n^y(n^{x-y}a^x + b^y) = n^z c^z$. Then clearly $z \geqslant y$, so that also $z < x$.

1.1.1 Suppose $n \nmid b^y$. Since we may write $n^{x-y}a^x + b^y = n^{z-y}c^z$, then we cannot have $z > y$, so $z = y$ in this case, and $n^{x-z}a^x + b^z = c^z$. Modulo a, we have $k^z \equiv (-k)^z$, and $\gcd(a, k) = 1$, so z is even. Write $z = 2z_1$, so that

$$n^{x-z}a^x = c^z - b^z = (c^{z_1} + b^{z_1})(c^{z_1} - b^{z_1}).$$

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The factors on the right cannot both be divisible by a. Since $a^x > a^z = a^{2z_1} = (c+b)^{z_1} \geqslant c^{z_1} + b^{z_1} > c^{z_1} - b^{z_1}$, we have a contradiction.

1.1.2 Suppose $n \mid b^y$. Then it is not the case that $C(n) \nmid b$, so necessarily $C(b) \mid n$. We may take $b = \prod_{i=1}^{s} r_i^{\gamma_i}$ (prime factorisation), so $n = \prod_{i=1}^{s} r_i^{\nu_i}$ with $\nu_i \geqslant 1$ for each $i = 1, \dots, s$. By the division algorithm, we write $\gamma_i y = t_i \nu_i + l_i$, say, where $t_i \geqslant 1$ and $0 \leqslant l_i < \nu_i$, for $i = 1, \dots, s$.

1.1.2.1 If $x > y + t_i$ for all $i = 1, \dots, s$, then we may write

$$\prod_{i=1}^{s} r_i^{\nu_i(y+t_i)} \left(\prod_{i=1}^{s} r_i^{\nu_i(x-y-t_i)} \cdot a^x + \prod_{i=1}^{s} r_i^{l_i} \right) = \prod_{i=1}^{s} r_i^{\nu_i z} \cdot c^z. \tag{2}$$

Since we cannot have $r_i \mid c$ for any $i = 1, \dots, s$, because $\gcd(n, c) = 1$, and since $l_i < \nu_i$ for each i, it follows that we must have $z = y + t_1 = \cdots = y + t_s$, so $t_1 = \cdots = t_s = t$, say, and (2) reduces to $\prod_{i=1}^{s} r_i^{\nu_i(x-y-t)} \cdot a^x + \prod_{i=1}^{s} r_i^{l_i} = c^z$. It is then apparent that $l_i = 0$ for $i = 1, \dots, s$, so that

$$\frac{\nu_1}{\gamma_1} = \cdots = \frac{\nu_s}{\gamma_s} = \frac{y}{t} = \frac{y'}{t'}, \tag{3}$$

say, where $\gcd(y', t') = 1$. Also, (2) further reduces to

$$n^{x-z} a^x + 1 = c^z. \tag{4}$$

If z is even, then, writing $z = 2z_1$, we have $n^{x-z} a^x = (c^{z_1} + 1)(c^{z_1} - 1)$. However, a cannot divide both factors on the right, and

$$a^x > a^z = a^{2z_1} = (b+c)^{z_1} > c^{z_1} + 1 > c^{z_1} - 1,$$

so this is impossible.

Suppose now that z is odd. Using (3), we have $n = b^{y'/t'}$ so that, from (4), $b^{y'(x-z)} a^{xt'} = (c^z - 1)^{t'} = \left((b+1)^z - 1\right)^{t'}$. Since b is even, then $b \nmid z$, so $(b+1)^z - 1$ is divisible by b exactly. Hence $y'(x-z) = t'$. Since $\gcd(y', t') = 1$, then $y' = 1$, $x = z + t'$ and, from (3), $yt' = t$. Since $z = y + t = y(1 + t')$, then t' is even and x is odd. Write $x = 2x_1 + 1$. We have $n^{x-z} = n^{t'} = b^{y'} = b$, so that, from (4),

$$c^z - 1 = ba^x = a(c-1)(b+c)^{x_1} = a(c-1)(2c-1)^{x_1}.$$

Modulo c, we have $a(-1)^{x_1} \equiv 1$, from which $c \mid (a+1)$ or $c \mid (a-1)$. But this is impossible, since $c > a + 1$.

1.1.2.2 If $x \leqslant y + t_i$ for at least one $i = 1, \ldots, s$, then we can quickly obtain a contradiction. The approach may be illustrated by taking $x \leqslant y + t_1$ and $x > y + t_i$ for $i = 2, \ldots, s$ (if $s \geqslant 2$). Then, adjusting (2), we may write

$$r_1^{\nu_1 x} \prod_{i=2}^{s} r_i^{\nu_i(y+t_i)} \left(\prod_{i=2}^{s} r_i^{\nu_i(x-y-t_i)} \cdot a^x + r_1^{\nu_1(y+t_1-x)} \prod_{i=1}^{s} r_i^{l_i} \right) = \prod_{i=1}^{s} r_i^{\nu_i z} \cdot c^z.$$

But since $x > z$, this implies that $r_1 \mid \prod_{i=2}^{s} r_i^{\nu_i z} \cdot c^z$, which is the desired contradiction.

1.2 Suppose $x < y$ and write (1) as $n^x(a^x + n^{y-x}b^y) = n^z c^z$. Then clearly $y > z \geqslant x$.

1.2.1 If $n \nmid a^x$, then we cannot have $z > x$, so $z = x$ and we have $n^{y-z}b^y = c^z - a^z$. Consider this equation modulo 4 if $k = 1$, in which case $a = 3$, $b = 4$ and $c = 5$, and modulo $k+1$ if $k > 1$. In both cases, we conclude that z must be even. Write $z = 2z_1$.

If $k = 1$, then $n^{y-z}4^y = 5^z - 3^z = (5^{z_1} + 3^{z_1})(5^{z_1} - 3^{z_1})$. The factors on the right are both even but cannot both be divisible by 4. Hence one of them is divisible by 2^{2y-1}. But

$$2^{2y-1} > 2^{2z-1} = 2^{4z_1-1} \geqslant 2^{3z_1} = (5+3)^{z_1} \geqslant 5^{z_1} + 3^{z_1} > 5^{z_1} - 3^{z_1}.$$

We have a contradiction.

Suppose $k > 1$. We have $n^{y-z}b^y = (c^{z_1} + a^{z_1})(c^{z_1} - a^{z_1})$, and we observe that $b = 2k(k+1)$, $k \mid (c-a) \mid (c^{z_1} - a^{z_1})$ and $\gcd(c^{z_1} + a^{z_1}, c^{z_1} - a^{z_1}) = 2$. If z_1 is even, or if z_1 is odd and k is even (in which case, $a \equiv c \equiv 1 \pmod 4$), then $c^{z_1} + a^{z_1}$ is divisible by 2 but not by 4, so that $2^{y-1}k^y \mid (c^{z_1} - a^{z_1})$. However,

$$2^{y-1}k^y = \frac{(2k)^y}{2} \geqslant \frac{(2k)^{z+1}}{2} = k\left(4k^2\right)^{z_1} > \left(2k^2 + 2k + 1\right)^{z_1} = c^{z_1} > c^{z_1} - a^{z_1},$$

which is also a contradiction. If z_1 and k are both odd, then, since $c \equiv -a \equiv 1 \pmod{(k+1)}$, we have $(k+1) \mid (c^{z_1} + a^{z_1})$ and $4 \nmid (c^{z_1} - a^{z_1})$. Hence $2^{y-1}(k+1)^y \mid (c^{z_1} + a^{z_1})$. But

$$\begin{aligned} 2^{y-1}(k+1)^y &> \frac{1}{2}\left(2(k+1)\right)^z = \frac{1}{2}\left(4k^2 + 8k + 4\right)^{z_1} \\ &\geqslant \left(2k^2 + 4k + 2\right)^{z_1} = (c+a)^{z_1} \geqslant c^{z_1} + a^{z_1}, \end{aligned}$$

our final contradiction in this case.

1.2.2 Suppose $n \mid a^x$. Write $a = p^\alpha$, where p is prime, and $n = p^\nu$. Also, write $\alpha x = \nu t + l$, where $0 \leqslant l < \nu$.

Suppose $y > x+t$, and write (1) as $n^{x+t}(p^l + n^{y-x-t}b^y) = n^z c^z$. From this, it follows that $z = x+t$ and $l = 0$, so that $n^{y-z}b^y = c^z - 1$. If z is odd, then, as in the last paragraph of **1.1.2.1**, $c^z - 1$ is exactly divisible by b. But $y > z$, so $y \geqslant 2$ and $b^2 \mid (c^z - 1)$. Then z must be even. Write $z = 2z_1$. We have $c^{z_1} + 1 \equiv 2 \pmod{b}$, from which $(c^{z_1}+1, b) = 2$. Since $n^{y-z}b^y = (c^{z_1}+1)(c^{z_1}-1)$, we must then have $b^y/2 \mid (c^{z_1} - 1)$. But

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \frac{1}{2}(c-a)^{z_1}(c+a)^{z_1} \geqslant c^{z_1} + a^{z_1} > c^{z_1} - 1.$$

This is a contradiction.

If $y \leqslant x+t$, then write (1) as $n^y(n^{x+t-y}p^l + b^y) = n^z c^z$. Since $y > z$, we have $n \mid c^z$, a contradiction.

2 In the second main case, we suppose $\gcd(n,c) > 1$. Write $c = \prod_{i=1}^{t} q_i^{\alpha_i}$ (prime factorisation).

2.1 Suppose first that $C(n) \mid c$, so that we may write $n = \prod_{i=1}^{s} q_i^{\beta_i}$, say, with $s \leqslant t$ and $\beta_i \geqslant 1$ for $i = 1, \ldots, s$.

2.1.1 Suppose $x = y$, so $z < x$. From (1), we have

$$(a^x + b^x)\prod_{i=1}^{s} q_i^{\beta_i x} = \prod_{i=1}^{s} q_i^{\beta_i z} \cdot \prod_{i=1}^{s} q_i^{\alpha_i z} \cdot \prod_{i=s+1}^{t} q_i^{\alpha_i z},$$

so that

$$a^x + b^x = \prod_{i=1}^{s} q_i^{\alpha_i z - \beta_i(x-z)} \cdot \prod_{i=s+1}^{t} q_i^{\alpha_i z}. \tag{5}$$

It is clear from this that $\alpha_i z - \beta_i(x-z) \geqslant 0$ for each $i = 1, \ldots, s$.

We shall show that $\alpha_1 z - \beta_1(x-z) > \alpha_1$. Suppose this is not true. If $t = 1$ then $s = 1$ and $q_1^{\alpha_1 z - \beta_1(x-z)} \leqslant q_1^{\alpha_1} = c < a^x + b^x$, contradicting (5). If $t > 1$ then, since $q_1^{\alpha_1} \leqslant c/q_2 < c - 1 = b$ and $\prod_{i=2}^{t} q_i^{\alpha_i} \leqslant c/q_1 < c - 1 = b$, we have

$$\prod_{i=1}^{s} q_i^{\alpha_i z - \beta_i(x-z)} \cdot \prod_{i=s+1}^{t} q_i^{\alpha_i z} < q_1^{\alpha_1}\prod_{i=2}^{t} q_i^{\alpha_i z} < b^{z+1} \leqslant b^x < a^x + b^x,$$

another contradiction. Thus $\alpha_1 z - \beta_1(x-z) > \alpha_1$, and similarly $\alpha_i z - \beta_i(x-z) > \alpha_i$ for $i = 2, \ldots, s$. It follows from (5) that

$$a^x + b^x \equiv 0 \pmod{c}. \tag{6}$$

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If x is odd, say $x = 2x_1 + 1$, then

$$a^x + b^x = aa^{2x_1} + bb^{2x_1} \equiv a(-1)^{x_1} - 1 \pmod{c}.$$

By (6), then $c \mid (a-1)$ or $c \mid (a+1)$, which is impossible since $c > a+1$.

If x is even, say $x = 2x_1$, then, from (6), $(-1)^{x_1} + 1 \equiv 0 \pmod{c}$, so x_1 is odd. In that case, $a^x + b^x = (a^2)^{x_1} + (b^2)^{x_1}$ is divisible by $a^2 + b^2$, and, since $z < x$ implies $x > 2$, the quotient must exceed 1. Furthermore, by (5), $(a^x + b^x)/(a^2 + b^2)$ is divisible by q_j, say, for some $j = 1, \dots, t$. Since $a^2 \equiv -1 \equiv b^2 \pmod{c}$, we have

$$\frac{a^x + b^x}{a^2 + b^2} = a^{2(x_1-1)} - a^{2(x_1-2)}b^2 + \cdots + b^{2(x_1-1)} \equiv x_1 \equiv 0 \pmod{q_j},$$

that is, $q_j \mid x_1$. Then $a^{2q_j} + b^{2q_j}$ divides $a^{2x_1} + b^{2x_1}$. Furthermore, $(a^{2q_j} + b^{2q_j})/(a^2 + b^2)$ divides $a^{2x_1} + b^{2x_1}$, and, from (5), must be a product of primes in $\{q_1, \dots, q_t\}$. It follows then from Lemma 3 that $\gcd\left(a^2 + b^2, (a^{2q_j} + b^{2q_j})/(a^2 + b^2)\right) = q_j$. However, it is clear that $(a^{2q_j} + b^{2q_j})/(a^2 + b^2) > q_j$, and $\prod_{i=1}^{t} q_i^2 \mid (a^2 + b^2)$, so we have a contradiction.

2.1.2 Now suppose $x > y$. From (1), we may write

$$\prod_{i=1}^{s} q_i^{\beta_i y}(n^{x-y}a^x + b^y) = \prod_{i=1}^{s} q_i^{\beta_i z} \cdot \prod_{i=1}^{t} q_i^{\alpha_i z}.$$

If $z \geqslant y$ then $q_1 \mid b$, contradicting $\gcd(b, c) = 1$, so $z < y$ and we write

$$n^{x-y}a^x + b^y = \prod_{i=1}^{s} q_i^{\alpha_i z - \beta_i(y-z)} \cdot \prod_{i=s+1}^{t} q_i^{\alpha_i z}. \tag{7}$$

Again we have a contradiction if $\alpha_j z - \beta_j(y-z) > 0$ for some $j = 1, \dots, s$, since then $q_j \mid b$, so $\prod_{i=1}^{s} q_i^{\alpha_i z - \beta_i(y-z)} = 1$. It follows that $s < t$ but, since $\prod_{i=s+1}^{t} q_i^{\alpha_i} < c/q_1 < b$, we have

$$\prod_{i=s+1}^{t} q_i^{\alpha_i z} < b^z < b^y < n^{x-y}a^x + b^y,$$

which is then a contradiction of (7).

Similarly, we cannot have $x < y$.

2.2 If $C(n) \nmid c$, then we may write $n = n_1 n_2$, where $n_1 > 1$ and $\gcd(n_1, n_2) = \gcd(n_1, c) = 1$.

2.2.1 If $x = y$ then (1) becomes $n_1^x n_2^x (a^x + b^x) = n_1^z n_2^z c^z$. Since $z < x$, this implies that $n_1 \mid n_2^z c^z$, a contradiction.

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2.2.2 Suppose $x > y$, and write (1) as $n_1^y n_2^y(n^{x-y}a^x + b^y) = n_1^z n_2^z c^z$. If $z \geqslant y$ then $\gcd(b,c) > 1$, since $\gcd(n,c) > 1$, and this is a contradiction. If $z < y$ then $n_1 \mid c^z$, and this is also impossible.

Similarly, we cannot have $x < y$.

This completes the proof of Theorem 1. □

3. Proof of Theorem 2

In the notation of Theorem 1, we must extend that proof for the special cases $k = 2$, 3, 4 and 5, without the restriction that $C(b) \mid n$ or $C(n) \nmid b$. In effect, we need only look to the case **1.1.2** in the proof of Theorem 1, so that we may assume $\gcd(n,c) = 1$, $x > y$, $n \mid b^y$ and $C(b) \nmid n$. The four values of k must be considered in turn. For this purpose, we continue the previous decimal indexing.

3 Take $k = 2$, so $(a,b,c) = (5,12,13)$. The relevant assumptions are: $x > y$, $n \mid 12^y$ and $6 \nmid n$. Then n is either a power of 3 or a power of 2.

3.1 Let $n = 3^r$, and write $y = tr + l$ where $0 \leqslant l < r$. If $x > y + t$, then we may write (1) as $n^{y+t}(n^{x-y-t}5^x + 3^l 4^y) = n^z 13^z$. It follows that $z = y + t$, so that $n^{x-z}5^x + 3^l 4^y = 13^z$, and then that $l = 0$ since $x > z$. Then, modulo 5, $(-1)^y \equiv 3^z$, from which z must be even. Write $z = 2z_1$, so that $n^{x-z}5^x = (13^{z_1} + 2^y)(13^{z_1} - 2^y)$. The factors on the right cannot both be divisible by 5, and, noting that $z = y + t > y$,

$$5^x > 5^z = 25^{z_1} > 13^{z_1} + 4^{z_1} > 13^{z_1} + 2^y > 13^{z_1} - 2^y,$$

so we have a contradiction. If $x \leqslant y + t$, then we may write (1) as

$$n^x\left(5^x + 3^l n^{y+t-x}4^y\right) = n^z 13^z.$$

This is clearly impossible, since $x > z$.

3.2 Now let $n = 2^s$, and write $2y = ts + l$, where $0 \leqslant l < s$. As in **3.1**, we easily show that we cannot have $x \leqslant y + t$, so $x > y + t$ and we may write $n^{y+t}\left(n^{x-y-t}5^x + 2^l 3^y\right) = n^z 13^z$. This implies that $z = y + t$, and then that $l = 0$, so

$$n^{x-z}5^x + 3^y = 13^z. \tag{8}$$

Then $3^y \equiv 3^z \pmod 5$, so y and z are both even or both odd. If $4 \mid n^{x-z}$, then (8), considered modulo 4, shows that y is even. If $n^{x-z} = 2$, then (8), considered modulo 3, shows that x is odd so that $z = x - 1$ is even. Thus we may put $z = 2z_1$ and $y = 2y_1$, and then $n^{x-z}5^x = (13^{z_1} + 3^{y_1})(13^{z_1} - 3^{y_1})$. As in **3.1**, we may show this to be impossible.

4 Now take $k = 3$, so $(a,b,c) = (7,24,25)$. We are assuming that $x > y$, $n \mid 24^y$ and $6 \nmid n$, so that again n is a power of 3 or a power of 2.

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4.1 Suppose $n = 3^r$, and $y = tr + l$ where $0 \leqslant l < r$. As in **3.1**, we see that we must have $x > y+t$, and, as before, that $z = y+t$ and $l = 0$. Then $n^{x-z}7^x + 8^y = 25^z$. Considering this equation modulo 3, this implies that we may write $y = 2y_1$ so that $n^{x-z}7^x = (5^z + 8^{y_1})(5^z - 8^{y_1})$. However, 7 cannot divide both factors on the right and

$$7^x > 7^z = 7^t 49^{y_1} > 5^t(25^{y_1} + 8^{y_1}) \geqslant 5^{y+t} + 8^{y_1} = 5^z + 8^{y_1} > 5^z - 8^{y_1},$$

so we have a contradiction.

4.2 If $n = 2^s$, then, very much as in **3.2**, we again obtain a contradiction.

5 Next, take $k = 4$, so $(a, b, c) = (9, 40, 41)$. We are assuming that $x > y$, $n \mid 40^y$ and $10 \nmid n$, so that n is a power of 5 or a power of 2.

5.1 Suppose $n = 5^r$, and $y = tr + l$ where $0 \leqslant l < r$. Again, we must have $x > y + t$, so that, from (1), $n^{y+t}\big(n^{x-y-t}9^x + 5^l 8^y\big) = n^z 41^z$, and this implies that $z = y+t$, and then that $l = 0$. The equation $n^{x-z}9^x + 8^y = 41^z$, considered modulo 5, shows that y is even, and then, considered modulo 3, that z is even. Write $y = 2y_1$ and $z = 2z_1$, so that we have $n^{x-z}9^x = (41^{z_1} + 8^{y_1})(41^{z_1} - 8^{y_1})$. The factors on the right cannot both be divisible by 3, and $9^x > 9^z = 81^{z_1} > 41^{z_1} + 8^{z_1} > 41^{z_1} + 8^{y_1} > 41^{z_1} - 8^{y_1}$, so we have a contradiction.

5.2 If $n = 2^s$, then, again as in **3.2**, we obtain a contradiction.

6 In our final case, take $k = 5$, so $(a, b, c) = (11, 60, 61)$. We assume that $x > y$, $n \mid 60^y$ and $30 \nmid n$. We need to consider the possibilities $n = 3^{r_1}$, $n = 5^{r_2}$, $n = 2^s$, $n = 3^{r_1}5^{r_2}$, $n = 2^s 3^{r_1}$, $n = 2^s 5^{r_2}$, and the proofs in each of these cases follow lines similar to the above.

This completes the proof of Theorem 2. □

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