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CREATION OF MASS PROCESSES AND PERTURBATION THEORY

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1. Introduction. Creation of mass processes were treated lately by several authors. The idea was to find some generalized Markov process that will correspond to a semigroup of operators which are not necessarily contraction operators (or equivalently to a quasi transition function which is not sub-markov). It was G. A. Hunt [**6**] who first suggested the idea of Markov processes in which both the starting time and the terminal time are random. Such processes were constructed by Helms [**4**] and treated also by Nagasawa [**12**] and the author [**10**]. Other approaches are possible, see for example Knight [**9** and **7**; **13**]. Roughly speaking these processes can be described as follows. There exists a state – the precreation state – in which a particle remains until some random starting time β at which time it enters the state space E. Its motion in E is then controlled by some Markov transition function. Again at a random time δ the particle is transferred to an annihilation state where it remains thereafter. Such processes are thus applicable to phenomenon which do not necessarily start at a fixed time but rather each particle can start its motion at a random time. It was noted already in [**4**; **10**; **12**] that there is a connection between creation of mass processes and perturbation theory for infinitesimal generators of Markov processes, and thus also to the theory of perturbations of partial differential equations. Thus the method will be applicable to finding probabilistic solutions (or probabilistic interpretation) to a certain class of partial differential equations.

In this paper we extend the above model to one with a countable number of precreation states, such that the starting time and the initial position of a particle in E both depend on the particular precreation state at which the particle starts. This will allow us, for example, to extend the class of differential equations for which we can find probabilistic solutions. There seem to be some similarities between these processes and branching processes and relations may be found. It may also be possible to extend this model to one with a continuum of precreation states. The approach here will be that of Hunt's potential theory [**5**], namely working with resolvent families and finding semigroups and processes corresponding to them.

Section 2 defines the model, gives an existence proof and some regularity properties of it. Section 3 gives its applications to perturbations of Markov processes. Section 4 takes the well-known special case of perturbation by

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$U \times I$ where U is some function on E, and indicates the way to apply the results to the Cauchy problem in partial differential equations.

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2. The model. Consider the following objects:

(a) Let E be a locally compact Hausdorff space having a countable basis for its topology. Let $\mathscr{E}$ be the σ-algebra of its Borel subsets. Let ∇ be a point not in E which is adjoined to E as a point of infinity of the Alexandroff compactification if E is not compact, or as an isolated point if E is compact. Denote by $E_\nabla = E \cup \{\nabla\}$ and let $\mathscr{E}_\nabla$ be the σ-algebra of subsets of E_∇ generated by $\mathscr{E}$. Consider also a submarkov transition probability $P_t(.\,,.)$, $t \geqq 0$, defined on $E \times \mathscr{E}$. Extend $P_t(.\,,.)$ to a Markov transition probability on $E_\nabla \times \mathscr{E}_\nabla$ in the usual wav

$$(P_t(\nabla, \{\nabla\}) = 1, P_t(x, \{\nabla\}) = 1 - P_t(x, E)).$$

Thus ∇ is an annihilation state. Let $\{\Delta_i : i \geqq 1\}$ be a countable collection of distinct points not in E_∇ and enlarge E_∇ by

$$\hat{E} = E_\nabla \cup \left(\bigcup_{i=1}^{\infty} \Delta_i \right),$$

Δ_i adjoined to E_∇ as isolated points. Let $\hat{\mathscr{E}}$ be the σ-algebra on $\hat{E}$ generated by $\mathscr{E}_\nabla$ and the Δ_i, $i \geqq 1$.

(b) Let $(\Omega, \mathscr{F})$ be a measurable space with a distinguished point $\omega_\nabla \in \Omega$. Let $\{\mathscr{F}_t : t > 0\}$ be an increasing family of σ-subalgebras of $\mathscr{F}$ and let $\mathscr{F}_\infty = \sigma(\bigcup_{t \geqq 0} \mathscr{F}_t)$ be the σ-algebra generated by all of $\mathscr{F}_t$, $t \geqq 0$.

(c) For each $t \in [0, \infty]$, let ξ_t be a measurable map from $(\Omega, \mathscr{F}_t)$ to $(\hat{E}, \hat{\mathscr{E}})$ satisfying for each $\omega \in \Omega$:

(C_1) $\xi_t(\omega) = \nabla$ implies $\xi_s(\omega) = \nabla$ for $s \geqq t$;

(C_2) $\xi_t(\omega) = \Delta_i$ for some $i \geqq 1$ implies $\xi_s(\omega) = \Delta_i$ for $s \leqq t$;

(C_3) $\xi_\infty(\omega) = \nabla$;

(C_4) $\xi_0(\omega_\nabla) = \nabla$; and

(C_5) $\xi_{\beta(\omega)}(\omega) \in E_\nabla$, $\xi_{\delta(\omega)}(\omega) = \nabla$ where

$$\beta(\omega) = \inf\{t : \xi_t(\omega) \in E_\nabla\}, \qquad \delta(\omega) = \inf\{t : \xi_t(\omega) = \nabla\}.$$

(d) For each $t \in [0, \infty]$ let $\theta_t : \Omega \to \Omega$ be a map satisfying $\theta_\infty(\omega) = \omega_\nabla$ for all $\omega \in \Omega$ and $\xi_t \circ \theta_s = \xi_{s+t}$ for $s,t \in [0, \infty]$.

(e) For each $x \in E_\nabla$ let $\mathscr{P}_x$ be a σ-finite measure on $(\Omega, \mathscr{F})$.

Finally, let $\Omega_t = \{\omega : \xi_t(\omega) \in E_\nabla\}$, $t \geqq 0$.

Definition 2.1. The collection $(\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ is called a Markov process with a countable number of precreation states, denoted M.P.C.C.,

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with transition function $P_t(.\,,.)$, $t \geqq 0$, if for every $x \in E_\nabla, A \in \mathscr{E}_\nabla$, $s, t \in [0, \infty]$ the following Markov property holds

(2.1) $$\mathscr{P}_x(\xi_{s+t} \in A \mid \mathscr{F}_t \cap \Omega_t) = P_s(\xi_t, A) \text{ a.e. } \mathscr{P}_x \text{ on } \Omega_t.$$

The space $(\hat{E}, \hat{\mathscr{E}})$ is called the state space, the functions θ_t, $t \in [0, \infty]$ are called the shift operators. The random variables β and δ are called the creation time and annihilation time respectively. The points Δ_i, $i \geqq 1$, are called the precreation states.

Suppose we are given a transition function $P_t(.\,,.)$, $t \geqq 0$, on $E_\nabla \times \mathscr{E}_\nabla$ as defined above. Denote by $\mathscr{B}$ the σ-algebra of Borel subsets of $(0, \infty)$. Suppose we are given, further, for each $x \in E_\nabla$ a sequence of measures $\phi_x^i(.\,,.)$ $i \geqq 1$, defined on $\mathscr{B} \times \mathscr{E}_\nabla$ having the following properties:

(2.2) (i) $\phi_x^i((0, t) \times E_\nabla) < \infty$ for all $x \in E_\nabla$, $t > 0$;
(ii) For each $M \in \mathscr{F} \times \mathscr{E}_\nabla$, $\phi_x^i(M)$ is an $\mathscr{E}_\nabla$-measurable function of x.

We will now construct an M.P.C.C. having transition function $P_t(.\,,.)$ for which each of the measures ϕ_x^i, $i \geqq 1$, determines the amount of time a particle starting at the precreation state Δ_i spends at this state and its initial position in E_∇ once it leaves the state Δ_i. This construction will furnish us with an existence proof of M.P.C.C. Some regularity properties will be discussed later. In the construction we use the notion of Markov process with creation and annihilation defined in [**4**]. These processes can be regarded as a special case of M.P.C.C. if we take, for $i > 1$, the starting time of the ith particle to be identically infinite. Thus only the particle of state Δ_1 is relevant to this process.

Construction of the process. Let Ω^0 denote the set of all maps ω from $[0, \infty)$ to E_∇ that

(1) are right continuous on $[0, \infty)$ and have left limits on $(0, \infty)$,

(2) satisfy $\omega(s) = \nabla$ implies $\omega(t) = \nabla$ for $t \geqq s$,

(3) are bounded on compact subintervals of $[0, \infty)$ and satisfy $\omega(\infty) = \nabla$.

Let $\xi_t(\omega) = \omega(t)$ and let $\mathscr{F}^0 = \sigma\{\xi_t : t \geqq 0\}$ and $\mathscr{F}_t^0 = \sigma(\xi_s : s \leqq t)$, $t \geqq 0$. It is well-known [**3**] that the transition function P_t, $t \geqq 0$, on $E_\nabla \times \mathscr{E}_\nabla$ determines probability measures $\mathscr{P}_x^0$, $x \in E_\nabla$, on $\mathscr{F}^0$ for which

$$\mathscr{P}_x^0(\xi_t \in A) = P_t(x, A), A \in \mathscr{E}_\nabla.$$

For each $i \geqq 1$ fixed consider the following objects: Ω^i, the set of all maps ω from $[0, \infty]$ to $E_\nabla U\{\Delta_i\}$ with $\omega(0) = \Delta_i$ and satisfying the following properties:

(A) ω is right continuous on $[0, \infty)$ having left limits on $(0, \infty)$;

(B) $\omega(t) = \nabla$ implies $\omega(s) = \nabla$, $s \geqq t$;

(C) $\omega(t) = \Delta_i$ implies $\omega(s) = \Delta_i$, $s \leqq t$;

(D) For each ω the function $\xi_t(\omega) = \omega(t)$ is bounded on compact subintervals of $[\beta(\omega), \delta(\omega))$ where $\beta(\omega) = \sup\{t : \xi_t(\omega) = \Delta_i\}$; and

(E) $\xi \circ (\omega) = \nabla$ on $[\delta(\omega), \infty)$, $\xi \circ (\omega) = \Delta_i$ on $[0, \beta(\omega))$.
Let $\mathscr{F}^i = \sigma(\xi_t : t \geqq 0)$ on Ω^i, $\mathscr{F}_t^i = \sigma(\xi_0 : s \leqq t)$ on Ω^i. Denote $\Omega^{i,0} = \Omega^i \cup \Omega^0$ and let $\mathscr{F}^{i,0} = \mathscr{F}^0 \cup \mathscr{F}^i$ and $\mathscr{F}_t^{i,0} = \mathscr{F}_t^0 \cup \mathscr{F}_t^i$ on Ω_0^i.

It is known [**4**] that the transition function P_t, $t \geqq 0$, on $E_\nabla \times \mathscr{E}_\nabla$ and the measures ϕ_x^i, $x \in E_\nabla$, on $\mathscr{B} \times \mathscr{E}_\nabla$, determine a σ-finite measure $\mathscr{P}_x^i$ on $(\Omega^{i,0}, \mathscr{F}^{i,0})$ such that $\{\xi_t, \mathscr{F}_t^i : t \geqq 0\}$ is a Markov process with creation and annihilation on the space $(\Omega^{i,0}, \mathscr{F}^{i,0}, \mathscr{P}_x^i)$, having transition function P_t, $t \geqq 0$. $\mathscr{P}_x^i$ satisfies the relation

$$\mathscr{P}_x^i(\xi_t \in A) = P_t(x, A) + \int_0^t \int_\nabla P_{t-s}(y, A)\phi_x^i(ds, dy) \tag{2.3}$$

for $A \in \mathscr{E}_\nabla$. The measure $\mathscr{P}_x^i$ is a projective limit of a projective system of measures $(\mathscr{P}_x^{i,\tau}, \mathscr{E}^\tau_{\nabla,\Delta})$ ($\mathscr{E}^\tau_{\nabla,\Delta}$-product σ-field) but the expressions for $\mathscr{P}_x^{i,\tau}$ are quite complicated. The idea of the construction is to make $\phi_x^i(ds, dy)$ determine the starting time ds and the initial position dy of a particle in E_∇ and make P_t be the transition function of a particle once it entered the state space E_∇.

Notice that the restriction of $\mathscr{P}_x^i$ to $(\Omega^i, \mathscr{F}^i)$ satisfies (2.3) with the first term on the right omitted.

Define for each $x \in E_\nabla$ a measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$ by letting $\Omega = \bigcup_{i=0}^\infty \Omega^i$, $\mathscr{F} = \{A \subset \Omega : A \cap \Omega^i \in \mathscr{F}^i, i \geqq 0\}$ and for each $A \in \mathscr{F}$ we let

$$\mathscr{P}_x(A) = \sum_{i=0}^\infty \mathscr{P}_x^i(A \cap \Omega^i). \tag{2.4}$$

Since the Ω^i, $i \geqq 0$, are disjoint the measure space is well-defined. Let further $\mathscr{F}_t = \{A \subset \Omega : A \cap \Omega^i \in \mathscr{F}_t^i, i \geqq 0\}$ for each $t \geqq 0$. Define a map ξ_t from Ω to $\hat{E}$ by piecing together the maps ξ_t on Ω^i. Clearly ξ_t is measurable from $(\Omega, \mathscr{F}_t)$ to $(\hat{E}, \hat{\mathscr{E}})$. Further for each $\omega \in \Omega$ $\xi_t(\omega) = \omega(t)$. As a function of t, $\omega(t)$ satisfies (A)-(E). Let θ_t be the shift operator on Ω. θ_t satisfies condition (d) of the definition of M.P.C.C.

LEMMA 2.2. *The measure $\mathscr{P}_x$ defined by* (2.4) *is σ-finite. It is finite or infinite according as*

$$\sum_{i=1}^\infty \phi_x^i((0, \infty) \times E_\nabla)$$

is finite or infinite.

Proof. $\mathscr{P}_x$ is clearly σ-finite since by (i) and (2.4) it can be expressed as a countable sum of σ-finite measures $\mathscr{P}_x^i$, $i \geqq 0$, with disjoined supports Ω^i, $i \geqq 0$, respectively.

The first statement will follow if we prove that

$$P_x(\Omega) = 1 + \sum_{i=1}^\infty \phi_x^i((0, \infty) \times E_\nabla).$$

Indeed $\mathscr{P}_x{}^0$ is a probability measure on $(\Omega^0, \mathscr{F}^0)$; that is,

$$\mathscr{P}_x{}^0(\Omega^0) = 1. \tag{2.5}$$

Let us fix $i \geqq 1$; it is enough to show that $\mathscr{P}_x{}^i(\Omega^i) = \phi_x{}^i((0, \infty) \times E_\nabla)$. Indeed define a sequence $\Omega_j{}^i, j \geqq 0$, of subsets of Ω^i by

$$\Omega_0{}^i = \{\omega \in \Omega^i : \omega(t) = \Delta_i, t \geqq 0\} \quad \text{and} \quad \Omega_j{}^i = \{\omega \in \Omega^i : \omega(j) \in E_\nabla\}, j \geqq 1.$$

Clearly $\mathscr{P}_x{}^i(\Omega_0{}^i) \leqq \mathscr{P}_x{}^i(\xi_k = \Delta_i)$ for each $k \geqq 0$. But by (2.3) the right term equals $\phi_x{}^i((k, x) \times E_\nabla)$ and so

$$\mathscr{P}_x{}^i(\Omega_0{}^i) \leqq \lim_{k\to\infty} \phi_x{}^i((k, \infty) \times E_\nabla) = 0. \tag{2.6}$$

As for $\Omega_j{}^i, j \geqq 1$ we have

$$\mathscr{P}_x{}^i(\Omega_j{}^i) = \mathscr{P}_x{}^i(\xi_j \in E_\nabla) = \int_0^j \int_{E_\nabla} \phi_x{}^i(ds, dy) = \phi_x{}^i((0, j) \times E_\nabla). \tag{2.7}$$

Since $\Omega^i = \bigcup_{j=0}^\infty \Omega_j{}^i$ and $\Omega_j{}^i, j \geqq 1$, increases with j, (2.5) and (2.6) imply

$$\mathscr{P}_x{}^i(\Omega^i) = \lim_{j\to\infty} \phi_x{}^i((0, j) \times E_\nabla) = \phi_x{}^i((0, \infty) \times E_\nabla)$$

which proves the lemma.

The following lemma follows easily by (2.3) and (2.4).

LEMMA 2.3. *The measure $\mathscr{P}_x$ satisfies*

$$\mathscr{P}_x(\xi_t \in A) = P_t(x, A) + \sum_{i=1}^\infty \int_0^t \int_{E_\nabla} P_{t-s}(y, A)\phi_x{}^i(ds, dy) \tag{2.8}$$

for $A \in \mathscr{E}_\nabla$.

Finally we may state the main theorem.

THEOREM 2.4. *Let $P_t(.\,.), t \geqq 0$, be a transition function on $E_\nabla \times \mathscr{E}_\nabla$ and let $\phi_x{}^i, i \geqq 1$, for each $x \in E_\nabla$, be a sequence of measures on $\mathscr{B} \times \mathscr{E}_\nabla$ satisfying conditions* (i) *and* (ii) *of* (2.2). *Then there exists a Markov process with countable precreation states $(\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ having transition function $P_t(.\,,.)$ and satisfying* (2.8). *The measure $\mathscr{P}_x$ is σ-finite. It is finite if $\sum_{i=1}^\infty \phi_x{}^i((0, \infty) \times E_\nabla)$ is finite. The space Ω can be chosen so that the sample paths $\xi \cdot (\omega)$ are right continuous on $[0, \infty)$, having limits from the left on $(0, \infty)$, and are bounded on compact intervals of $[\beta(\omega), \delta(\omega))$.*

Proof. It only remains to prove the Markov property (2.1). It is enough to show that (2.1) is satisfied on each of the subsets $\Omega_t \cap \Omega^{i,0}, i \geqq 1$, of Ω_t. But on $\Omega^{i,0}$, $\mathscr{P}_x = \mathscr{P}_x{}^i$ which indeed satisfies (2.1) since

$$(\Omega^{i,0}, \mathscr{F}^{i,0}, \mathscr{F}_t{}^{i,0}, \xi_t, \theta_t, \mathscr{P}_x{}^i)$$

is a Markov process with creation and annihilation as defined in [**4**].

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THEOREM 2.5. *The process* $(\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ *satisfies the strong Markov property and is quasi left continuous.*

Proof. We only have to notice that $(\Omega^{i,0}, \mathscr{F}^{i,0}, \mathscr{F}_t^{i,0}, \xi_t, \theta_t, \mathscr{P}_x^{i})$ satisfies both properties for each $i \geqq 0$, which was proved in [**4**]. The theorem then follows since it holds for the restrictions of ξ_t and the stopping time τ to each of the sets $\Omega^{i,0}$.

Remark 2.6. It is important to note that contrary to what one might expect it is usually not possible to combine all the precreation states Δ_i, $i \geqq 1$ into one state $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$ and all the measures ϕ_x^{i}, $i \geqq 1$, into one measure $\phi_x = \sum_{i=1}^{\infty} \phi_x^{i}$ in order to reduce the case of M.P.C.C. into the case of Markov process with creation and annihilation. The reason is that the measure $\mathscr{P}_x^*$ constructed from the combined measure ϕ_x and the transition function $P_t(.\,,.)$ is in general not σ-finite, as the measure ϕ_x will not satisfy the condition required on the creation measure needed for the construction in [**4**].

3. Application to perturbation theory of Marbov processes. In [**10**] we considered the problem of perturbation of Markov processes by an operator B satisfying some smallness conditions relative to the semigroup $\{T_t : t \geqq 0\}$ associated with the process. The perturbation resulted in a Markov process with creation and annihilation having the same transition function as the original process. Probabilistically, the process could be regarded as a small perturbation of the given process. Using this we can, for example, find a solution by probabilistic means to partial differential equations one obtains by small perturbations of equations for which we already know a probabilistic solutions. Now using the extended model of creation of mass processes we can extend the class of perturbing operators for the differential equation and thus extend the class of equations which we can solve by probabilistic methods.

Let us describe briefly the definitions and results in [**10**] needed here. Let E, $\mathscr{E}$ be as before. A kernel $K(.\,,.)$ on $E \times \mathscr{E}$ is a mapping of $E \times \mathscr{E}$ into $[0, \infty]$ such that $K(.\,,A)$ is measurable for each $A \in \mathscr{E}$, and $K(x,.)$ is a measure on $(E, \mathscr{E})$ for each $x \in E$. The kernel is submarkov if $K(x, E) \leqq 1$ for each $x \in E$.

Definition 3.1. 1. A family of kernels $\{S_t : t \geqq 0\}$ is called a semigroup if for each $s, t \geqq 0$, $S_{s+t} = S_s S_t$.

2. A family of kernels $\{V_\lambda : \lambda > \lambda_0\}$ is called a resolvent family if $V_\lambda V_\mu = V_\mu V_\lambda$ and $V_\mu = (\lambda - \mu) V_\lambda V_\mu$ for every $\lambda > \mu > 0$.

If $\{S_t : t \geqq 0\}$ is a measurable semigroup (i.e., $(t, x) \to S_t f(x)$ is measurable $\mathscr{B} \times \mathscr{E}$ for each $f \geqq 0$, $\mathscr{E}$-measurable), then

$$V_\lambda = \int_0^\infty e^{-\lambda t} S_t dt, \lambda > 0,$$

constitutes a resolvent family of kernels.

Similarly to (1) and (2) one can define these notions for families of operators on some Banach space X. Such a family of bounded operators is called strongly continuous if $||S_t x - x||$ approaches 0 as t tends to zero for each $x \in X$, in case of semigroup, or if $||\lambda V_\lambda x - x||$ approaches 0 as λ goes to infinity for each $x \in X$, in case of resolvent family. For the details and exact definitions of the above terms see [**2**; **11**].

Let C_0 be the Banach space of all bounded continuous functions f on E for which $\lim_{x\to\infty} f(x) = 0$. Let $\{V_\lambda : \lambda > 0\}$ be a strongly continuous submarkov resolvent family of operators on C_0 with associated semigroup of operators $\{T_t : t \geqq 0\}$ and a Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P_x)$ having state space E_∇ and transition function $P_t(. , .)$. We perturb this resolvent family by an operator B satisfying the regularity conditions below to get a family of operators

$$H_\lambda = \sum_{k=0}^{\infty} V_\lambda (B V_\lambda)^k$$

defined for $\lambda > \lambda_0$. The problem then is to find a process having $\{H_\lambda : \lambda > \lambda_0\}$ as its resolvent family, and to relate this process to the original one. The results proved in [**10**] were:

Result 1. Let B be a closed operator on C_0 satisfying: (C_1) its domain $\mathscr{D}(B)$ contains the range of V_λ, (C_2) for each $t > 0$ there exists a constant $N_t < \infty$ such that $||T_t Bf|| \leqq N_t ||f||, f \in \mathscr{D}(B)$, and for which

$$\int_0^1 N_s ds$$

exists and is finite. Then there exist constants $\lambda_0 > 0$, $0 < \epsilon < 1$ such that $||V_\lambda Bf|| \leqq (1 - \epsilon)||f||$, $f \in \mathscr{D}(B)$, and $\{H_\lambda : \lambda > \lambda_0\}$ is a resolvent family of operators on C_0. There also exists a strongly continuous semigroup $\{S_t : t \geqq 0\}$ of bounded linear operators on C_0 satisfying

$$H_\lambda = \int_0^\infty e^{-\lambda t} S_t dt, \ \lambda > \lambda_0.$$

S_t, $t \geqq 0$, is given by

$$S_t = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) = T_t, \ S_n(t) = \int_0^t S_{n-1}(s) B T_{t-s} ds$$

the series being absolutely convergent uniformly with respect to t on each finite interval.

Result 2. Under the above conditions $A + B$ is the infinitesimal generator of the semigroup $\{S_t : t > 0\}$, where A is the infinitesimal generator of $\{T_t : t \geqq 0\}$.

Result 3. Under (C_1), (C_2) and if B is nonnegative (or satisfies a wider comparison criterion) there exists a Markov process with creation and anni-

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hilation $(\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, \xi_t, \mathscr{P}_x)$ having the same transition function $P_t(.,.)$ such that for each $f \in C_0(E_\nabla)$, extended by $f(\Delta) = 0$, $\mathscr{E}_x f(\xi_t) = S_t f(x)$.

In other words there exists a family of measures $\{\beta_s^k : k \geqq 1, s > 0\}$ on $\mathscr{B} \times \mathscr{E}_\nabla$ satisfying $\beta_s^k((0, t) \times E_\nabla) < \infty$ for each $t > 0$, so that for each $A \in \mathscr{E}_\nabla$

$$\mathscr{P}_x(\xi_t \in A) = S_t(x, A) = P_t(x, A) + \sum_{k=1}^{\infty} \int_0^t \int_{E_\Delta} P_{t-s}(y, A)\beta_s^k(x, dy)ds.$$

Let us now extend these results. Denote by $\mathscr{B}_0$ the collection of all Borel measurable functions on E_∇.

Definition 3.2. Let B_i, $i \geqq 1$, be a sequence of nonnegative closed operators on C_0 with a common domain $\mathscr{D}$ each satisfying condition (C_2) of Result 1 with some function N_t^i, $t > 0$. Let L be the class of linear maps B on $\mathscr{B}_0$ with domain containing $V_\lambda(\mathscr{B}_0)$ and such that for each $f \in \mathscr{D}(B), f \geqq 0$, $Bf = \sum_{i=1}^{\infty} B_i f$.

From the definition we get $BV_\lambda f = \sum_{i=1}^{\infty} B_i V_\lambda f$ for each $B \in L$ $f \in \mathscr{B}_0^+$, $(f \geqq 0, f \in \mathscr{B}_0)$. Thus, since each of the operators $B_i V_\lambda$ can be associated with a kernel on $E_\nabla \times \mathscr{E}_\nabla$, denoted again by $B_i V_\lambda$, the same is true about BV_λ. So we define the perturbed resolvent family of kernels H_λ, $\lambda > 0$, on $E_\nabla \times \mathscr{E}_\nabla$ as before by letting

$$H_\lambda = V_\lambda + V_\lambda B V_\lambda + V_\lambda (BV_\lambda)^2 + \ldots \tag{3.1}$$

LEMMA 3.3. $\{H_\lambda : \lambda > 0\}$ *is a resolvent family of kernels on* $E_\nabla \times \mathscr{E}_\nabla$.

Proof. As a countable sum of kernels, H_λ, $\lambda > 0$, is clearly a kernel. Let us prove the resolvent equation. If we let A_k be an operator on C_0 given by $A_k = \sum_{i=1}^{k} B_i$, then A_k clearly satisfies conditions (C_1), (C_2) of Result 1 and further $BV_\lambda = \lim_{n\to\infty} A_k V_\lambda$ on C_0. Now for each k let $\{H_\lambda^k : \lambda > 0\}$ be the resolvent family of kernels on $E_\nabla \times \mathscr{E}_\nabla$ obtained by perturbation of $\{V_\lambda : \lambda > 0\}$ by the operator A_k. Then $H_\lambda f = \lim_{k\to\infty} H_\lambda^k f$ for each $f \in \mathscr{B}_0^+$. Indeed by definition

$$H_\lambda^k f = \sum_{n=1}^{\infty} \sum_{(i_1 \ldots i_n) \in L_n} V_\lambda B_{i_1} V_\lambda \ldots B_{i_n} V_\lambda f$$

while

$$H_\lambda f = \sum_{n=0}^{\infty} \sum_{(i_1 \ldots i_n) \in L_n} V_\lambda B_{i_1} V_\lambda \ldots B_{i_n} V_\lambda f$$

where $L_n = \{(i_1, \ldots, i_n) : 1 \leqq i_j \leqq k, j = 1, \ldots, n\}$. Further for each $k \geqq 0$ and $f \in \mathscr{B}_0^+$, $H_\lambda^k f = H_\mu^k f + (\mu - \lambda) H_\lambda^k H_\mu^k f$. Letting k go to infinity we get $H_\lambda^k f \to H_\lambda f$, $H_\mu^k f \to H_\mu f$. As for the last term the double sequence $a_{nk} = H_\lambda^k H_\mu^n f(x)$ is increasing both in n for k fixed and in n for k fixed. Thus the double and iterated limits exist and are equal and so

$$\lim_{k\to\infty} \lim_{n\to\infty} H_\lambda^k H_\mu^n f(x) = \lim_{k\to\infty} H_\lambda^k (\lim_n H_\mu^n f)(x) = \lim_{k\to\infty} H_\lambda^k H_\mu f(x) = H_\lambda H_\mu f(x).$$

which proves the resolvent equation for H_λ.

Our second problem, now, is to find a semigroup associated with $\{H_\lambda : \lambda > 0\}$. Let $\{S_t^k : t \geqq 0\}$ be the semigroup of operators C_0 associated with $\{H_\lambda^k : \lambda > 0\}$ given by Result 1. Again each S_t^k, $t \geqq 0$, $k \geqq 1$, being a bounded nonnegative operator on C_0, corresponds to a kernel on $E_\nabla \times \mathscr{E}_\nabla$ denoted again by S_t^k. Further, since A_k increases with k so does $S_t^k f(x)$ for each $f \in \mathscr{B}_0^+$; therefore we can define a kernel S_t by $S_t f = \lim_{k \to \infty} S_t^k f$, $f \in \mathscr{B}_0^+$.

LEMMA 3.4. *For each* $A \in E_\nabla$, $x \in \mathscr{E}_\nabla$

$$H_\lambda(x, A) = \int_0^\infty e^{-\lambda t} S_t(x, A) dt.$$

Proof. Indeed for each k,

$$H_\lambda^k(x, A) = \int_0^\infty e^{-\lambda t} S_t^k(x, A) dt$$

and both the left side and integrand are increasing with k. Thus the limits exist and are equal.

LEMMA 3.5. $\{S_t : t \geqq 0\}$ *is a semigroup of kernels on* $E_\nabla \times \mathscr{E}_\nabla$.

Proof. Again for each k, $S_{s+t}^k(x, A) = S_s^k S_t^k(x, A)$, $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$. Let k go to infinity and use the same argument as in Lemma 3.3.

Finally let us construct the process corresponding to the perturbed semigroup $\{S_t : t \geqq 0\}$. For each $k \geqq 1$, the operator A_k satisfies conditions (C_1) and (C_2) and thus there exists a family of kernels $\{\beta_s^{i,k} : s \geqq 0, i \geqq 1\}$ on $E_\nabla \times \mathscr{E}_\nabla$ such that for each $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$

$$S_t^k(x, A) = P_t(x, A) + \sum_{i=1}^\infty \int_0^t \int_{E_\Delta} P_{t-s}(y, A) \beta_s^{i,k}(x, dy) ds. \tag{3.2}$$

By definition $\{\beta_s^{i,k} : k \geqq 1\}$ is an increasing sequence of kernels, hence we can define the kernels $\phi_s^{i,k} = \beta_s^{i,k} - \beta_s^{i,k}$, $k \geqq 1$, $(\beta_s^{i,0} = 0)$. Re-order the family $\{\phi_s^{i,k} : i, k \geqq 1\}$ to get a family $\{\phi_s^i : i \geqq 1\}$ and we have by (3.2).

$$S_t(x, A) = P_t(x, A) + \sum_{i=1}^\infty \int_0^t \int_{E_\Delta} P_{t-s}(y, A) \phi_s^i(x, dy) ds. \tag{3.3}$$

Clearly the sequence of measures $\phi_x^i(ds, dy) = \phi_s^i(x, dy) ds$ on $\mathscr{B} \times \mathscr{E}_\nabla$ satisfy conditions (i) and (ii) of (2.2). Thus P_t, ϕ_x^i $i \geqq 1$, determine an M.P.C.C. with transition function P_t.

THEOREM 3.6. *Let* $\{V_\lambda : \lambda > 0\}$ *be a strongly continuous submarkov resolvent family of operators on* C_0. *Let* $\{P_t : t \geqq 0\}$ *be its associated semigroup of kernels. Let* B *be a linear map on* $\mathscr{B}_0$ *satisfying Definition* 3.1. *Then there exists a Markov process with countable precreation states* $(\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ *with transition function* P_t. *The measure* $\mathscr{P}_x$ *satisfies the equation* (2.8). *Further for each* $f \in \mathscr{B}_0^+$, *if we let* $f(\Delta_i) = 0$, $i \geqq 1$, *then*

$$\mathscr{E}_x f(\xi_t) = S_t f(x) \tag{3.4}$$

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where $\mathscr{E}_x$ denotes expected value relative to $\mathscr{P}_x$. The space Ω can be chosen so that each path of the process has the continuity and boundedness properties as in Theorems 2.4 *and* 2.5.

Proof. To prove (3.4) let us notice that by (2.4),

$$\mathscr{E}_x f(\xi_t) = \sum_{i=0}^{\infty} \mathscr{E}_x{}^i f(\xi_t)$$

where $\mathscr{E}_x{}^i$ denotes expected value relative to the measure $\mathscr{P}_x{}^i$, $i \geqq 0$. Also $E_x{}^0 f(\xi_t) = T_t f(x)$. Using (2.3) a standard argument shows that for each $i \geqq 1$,

$$\mathscr{E}_x{}^i f(\xi_t) = \int_0^t \int_{E_\nabla} T_{t-s} f(y) \phi_s{}^i(x, dy) ds.$$

Adding and using (3.3) yields (3.4).

Remark 3.7. In connection with Theorem 3.6 one may ask whether $S_t f(x) \to f(x)$ as $t \to 0$. Let $f \in C_0$ (otherwise even $T_t f(x) \to f(x)$ may not hold). For $S_t f(x) \to f(x)$ one needs the condition

$$\sum_{i=0}^{\infty} \int_0^t \int_{E_\nabla} T_{t-s} f(y) \phi_s{}^i(x, dy) ds \to 0 \text{ as } t \to 0;$$

i.e. that the creation rate is small for small time interval. It is enough to assume, for example, that the measure $\phi_s(x, .) = \sum \phi_s{}^i(x, .)$ satisfies

$$\int_0^{t_0} \phi_s(x, E_\nabla) ds < \infty$$

for some $t_0 > 0$.

4. Examples. An interesting special case of perturbing operator B is the operator $B = u \times I$. The case $u \in \mathscr{C}_0{}^-$ is classical and can be found in [**3**]. The case $u \in \mathscr{C}_0{}^+$ was considered by Helms in [**4**]. For a generalization of these results we refer the reader to [**12**]. The results of this paper allow us to extend those results.

Let $E = \mathbf{R}^N$, $N \geqq 1$, be the N-dimensional Euclidian space. Let $\{V_\lambda : \lambda > 0\}$ be the resolvent family of Brownian motion on $\mathbf{R}^N$ or of any subprocess of it. Let $B = u \times I$ where $u \in \mathscr{B}_0{}^+$ on $\mathbf{R}^N$ and where $\lim_{x\to\infty} u(x)$ exists, finite or infinite. (One can consider, of course, more general u's like $u = \sin x + 1$ for $N = 1$ etc.) Since we can write $u = \lim u_n$ where $u_n = u \wedge n$ and since the operators A_n on $C_0(\mathbf{R}^N)$ defined by $A_n f = u_n \times f$ are bounded, they satisfy conditions (C_1) and (C_2) of Result 1.

Thus there exists an M.P.C.C. $(\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ resulting from perturbation of $\{V_\lambda : \lambda > 0\}$ by $B = u \times I$ and satisfying the regularity conditions mentioned in Theorem 3.5. Let us look into some more properties of this process. Using results of Nagasawa [**12**], we can obtain an explicit representation of the semigroup $\{S_t : t \geqq 0\}$ and the creation measures $\{\phi_x{}^i : i \geqq 1\}$.

First let $v_n = u_n - u_{n-1}$ ($u_0 = 0$) and let $B_n f = v_n \times f$. Then $B = \sum_{i=1}^{\infty} B_i$ and $B \in L$, i.e. B satisfies Definition 3.2. Since v_n is bounded and continuous, Nagasawa's results clearly apply for each $n \geqq 1$ to give

$$\phi_x^{(n)}(ds, dy) = E_x\left[\exp \int_0^s v_n(X_r)drd\left(\int_0^s v_n(X_r)dr\right); X_s \in dy \cap E\right]$$

and

$$\int_0^t \phi_s^{(n)}(X, E_\nabla)ds = E_x\left[\exp \int_0^t V_n(X_s)ds\right]$$

where E_x denotes expected value relative the Brownian notion measure P_x. Further for $f \in \mathscr{B}_0^+$, $f(\Delta_i) = 0$, $i \geqq 1$

$$S_t^n f(x) = E_x\left[f(X_t) \exp \int_0^t v_n(X_s)ds\right],$$

where X_s is the path function of Brownian motion on $\mathbf{R}^N$. Letting n go to infinity we get the representation

$$\mathscr{E}_x f(\xi_t) = S_t f(x) = E_x\left[f(X_t) \exp \int_0^t u(X_s)ds\right]. \tag{4.1}$$

This is the same representation one gets in the classical case of perturbation by $B = u \times I$, $u \in C_0^+$, $u \leqq 0$, only in the latter case the resulting process is again a Markov process (which is a subprocess of the original one).

Furthermore, using Remark 3.6 we can state a condition that will guarantee the continuity property $\lim_{t \to 0} S_t f(x) = f(x)$ for $f \in C_0$. One such condition is that for some $t_0 > 0$

$$E_x\left[\exp \int_0^{t_0} u(X_s)ds\right] < \infty. \tag{4.2}$$

As we mentioned earlier the results of Section 3 can be applied, using the theory of infinitesimal generators of processes, to finding probabilistic solutions for a certain class of partial differential equations. Consider for example the following Cauchy problem, related to the example above. Given $f \in C_0$, find a solution $w(t, x)$, $t \in [0, T)$, $x \in \mathbf{R}^N$ satisfying

$$\frac{\partial w}{\partial t} = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 w}{\partial X_i^2} + u \times w \tag{4.3}$$

$$w|_{t=0} = f$$

where u is as above, thus not necessarily bounded or continuous on E^N. It follows from the arguments above that a candidate for a solution to (4.3) is $\mathscr{E}_x f(\xi_t)$ where $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ is the M.P.C.C. we get by perturbing the Brownian motion by $B = u \times I$. Let us take for example the unbounded function $u(x) = |x|^\gamma$, $\gamma < 2$. Then it is known [**8**] that the expression on the right of (4.1) is indeed a solution of (4.3). Thus we have found a process

$\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on a σ-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$, $x \in E_\nabla$, having the Markov property (2.1) and for which $\mathscr{E}_x f(\xi_t)$ yields a solution to (4.3). Recall that $E_x f(X_t)$ is the well-known probabilistic solution of 4.3 with $u = 0$ (i.e. of the unperturbed equation). Both processes have the same transition function and the M.P.C.C. can be regarded probabilistically as a perturbation of the second.

If we consider the example $u(x) = |x|^2$ then it is known [**8**] that the expression on the right side of (4.1) yields a solution to (4.3) as long as $t < \pi/\sqrt{2}$. Thus again $w(t, x) = \mathscr{E}_x f(\xi_t)$ is a probabilistic solution of (4.3) for any $T < \pi/\sqrt{2}$. Notice that Remark 2.6 applies to this case and the resulting process cannot be reduced to a process with one precreation state. We note that Nagasawa's results [**12**] can not be applied to this example since they assume (4.1) for each $t_0 > 0$ to guarantee the σ-finiteness of the measure space.

References

1. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory* (Academic Press, New York and London, 1969).

2. N. Dunford and J. Schwartz, *Linear operators. Part I: General theory* (Interscience Publication Inc., New York, 1964).

3. E. B. Dynkin, *Markov processes.* Vol. I (Springer-Verlag, Berlin, 1965).

4. L. L. Helms, *Markov processes with creation of mass.* I, II; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete *7* (1967), 225–234; *15* (1970), 208–218.

5. G. A. Hunt, *Markov processes and potentials.* II, Illinois J. Math. *1* (1957), 316–369.

6. ——— *Markov chains and Martin boundaries*, Illinois J. Math. *4* (1960), 313–340.

7. N. Ikeda, M. Nagasawa, and S. Watanabe, *Branching Markov processes*, Math. Kyoto Univ. *8* (1968), 233–278, 365–410; *9* (1969), 95–160.

8. K. Ito and H. P. Mckean, Jr., *Diffusion processes and their sample path* (Springer-Verlag, Berlin, 1965).

9. F. B. Knight, *A path space for positive semigroups*, Illinois J. Math. *13* (1969), 542–563.

10. T. Leviatan, *Perturbations of Markov processes*, J. Functional Analysis *10* (1972), 309–325.

11. P. A. Meyer, *Probability and potentials* (Blandsdell Pub. 10, Waltham, Mass., 1966).

12. M. Nagasawa, *Markov processes with creation and annihilation*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete *14* (1969), 49–70.

13. M. L. Silverstein, *Markov processes with creation of particles*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete *9* (1968), 235–257.

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