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LOEWY LENGTHS OF CENTERS OF BLOCKS II

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Abstract. Let ZB be the center of a p-block B of a finite group with defect group D. We show that the Loewy length $LL(ZB)$ of ZB is bounded by $|D|/p+p-1$ provided D is not cyclic. If D is nonabelian, we prove the stronger bound $LL(ZB)<\min\{p^{d-1},4p^{d-2}\}$ where $|D|=p^d$. Conversely, we classify the blocks B with $LL(ZB)\geqslant\min\{p^{d-1},4p^{d-2}\}$. This extends some results previously obtained by the present authors. Moreover, we characterize blocks with uniserial center.

§1. Introduction

The aim of this paper is to extend some results on Loewy lengths of centers of blocks obtained in [9, 12]. In the following we will reuse some of the notation introduced in [9]. In particular, B is a block of a finite group G with respect to an algebraically closed field F of characteristic $p>0$. Moreover, let D be a defect group of B. The second author has shown in [12, Corollary 3.3] that the Loewy length of the center of B is bounded by

$$LL(ZB)\leqslant|D|-\frac{|D|}{\exp(D)}+1$$

where $\exp(D)$ is the exponent of D. It was already known to Okuyama [10] that this bound is best possible if D is cyclic. The first and the third author have given in [9, Theorem 1] the optimal bound $LL(ZB)\leqslant LL(FD)$ for blocks with abelian defect groups. Our main result of the present paper establishes the following bound for blocks with nonabelian defect groups:

$$LL(ZB)<\min\{p^{d-1},4p^{d-2}\}$$

where $|D|=p^d$. As a consequence we obtain

$$LL(ZB)\leqslant p^{d-1}+p-1$$

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for all blocks with noncyclic defect groups. It can be seen that this bound is achieved whenever B is nilpotent and $D \cong C_{p^{d-1}} \times C_p$.

In the second part of the paper we show that $LL(ZB)$ depends more on $\exp(D)$ than on $|D|$. We prove for instance that $LL(ZB) \leqslant d^2 \exp(D)$ unless $d = 0$. Finally, we use the opportunity to improve a result of Willems [15] about blocks with uniserial center.

In addition to the notation used in the papers cited above, we introduce the following objects. Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. A p-subgroup $P \leqslant G$ is called a defect group of $K \in \mathrm{Cl}(G)$ if P is a Sylow p-subgroup of $\mathrm{C}_G(x)$ for some $x \in K$. Let $\mathrm{Cl}_P(G)$ be the set of conjugacy classes with defect group P. Let $K^+ := \sum_{x \in K} x \in FG$ and

$$I_P(G) := \langle K^+ : K \in \mathrm{Cl}_P(G) \rangle \subseteq ZFG,$$
$$I_{\leqslant P}(G) := \sum_{Q \leqslant P} I_Q(G) \trianglelefteq ZFG,$$
$$I_{<P}(G) := \sum_{Q < P} I_Q(G) \trianglelefteq ZFG$$

(here $\trianglelefteq$ means that the subsets are ideals).

§2. Results

We begin by restating a lemma of Passman [13, Lemma 2]. For the convenience of the reader we provide a (slightly easier) proof.

Lemma 1. *Let P be a central p-subgroup of G. Then $I_{\leqslant P}(G) \cdot JZFG = I_{\leqslant P}(G) \cdot JFP$.*

Proof. Let K be a conjugacy class of G with defect group P, and let $x \in K$. Then P is the only Sylow p-subgroup of $\mathrm{C}_G(x)$, and the p-factor u of x centralizes x. Thus $u \in P$. Hence u is the p-factor of every element in K, and $K = uK'$ where K' is a p-regular conjugacy class of G with defect group P. This shows that $I := I_{\leqslant P}(G)$ is a free FP-module with the p-regular class sums with defect group P as an FP-basis. The canonical epimorphism $\nu : FG \to F[G/P]$ maps I into $I_1(G/P) \subseteq SF[G/P]$ (recall that $SF[G/P]$ is the socle of $F[G/P]$). Thus $\nu(I \cdot JZFG) \subseteq SF[G/P] \cdot JZF[G/P] = 0$. Hence $I \cdot JZFG \subseteq I \cdot JFP$. The other inclusion is trivial. □

Lemma 2. *Let $P \leqslant G$ be a p-subgroup of order p^n. Then*

(i) $I_{\leqslant P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))} \subseteq I_{<P}(G)$.
(ii) $I_{\leqslant P}(G) \cdot JZFG^{(p^{n+1}-1)/(p-1)} = 0$.

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Proof.

(i) Let $\mathrm{Br}_P : ZFG \to ZF\,\mathrm{C}_G(P)$ be the Brauer homomorphism. Since $\mathrm{Ker}(\mathrm{Br}_P) \cap I_{\leqslant P}(G) = I_{<P}(G)$, we need to show that $\mathrm{Br}_P(I_{\leqslant P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))}) = 0$. By Lemma 1 we have

$$\begin{aligned}&\mathrm{Br}_P(I_{\leqslant P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))})\\ &\quad\subseteq I_{\leqslant \mathrm{Z}(P)}(\mathrm{C}_G(P)) \cdot JZF\,\mathrm{C}_G(P)^{LL(F\,\mathrm{Z}(P))}\\ &\quad= I_{\leqslant \mathrm{Z}(P)}(\mathrm{C}_G(P)) \cdot JF\,\mathrm{Z}(P)^{LL(F\,\mathrm{Z}(P))} = 0.\end{aligned}$$

(ii) We argue by induction on n. The case $n = 1$ follows from $I_1(G) \subseteq SFG$. Now suppose that the claim holds for $n - 1$. Since $LL(F\,\mathrm{Z}(P)) \leqslant |P| = p^n$, (i) implies

$$\begin{aligned}I_{\leqslant P}(G) \cdot JZFG^{(p^{n+1}-1)/(p-1)} &= I_{\leqslant P}(G) \cdot JZFG^{p^n} JZFG^{(p^n-1)/(p-1)}\\ &\subseteq I_{<P}(G) \cdot JZFG^{(p^n-1)/(p-1)}\\ &= \sum_{Q<P} I_{\leqslant Q}(G) \cdot JZFG^{(p^n-1)/(p-1)} = 0. \qquad \square\end{aligned}$$

Recall from [9, Lemma 9] the following group

$$W_{p^d} := \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x, y] = [x, z] = 1, [y, z] = x^{p^{d-3}} \rangle.$$

Note that W_{p^d} is a central product of $C_{p^{d-2}}$ and an extraspecial group of order p^3. Now we prove our main theorem which improves [9, Theorem 12].

THEOREM 3. *Let B be a block of FG with nonabelian defect group D of order p^d. Then (at least) one of the following holds*

(i) $LL(ZB) < 3p^{d-2}$.
(ii) $p \geqslant 5$, $D \cong W_{p^d}$ *and* $LL(ZB) < 4p^{d-2}$.

In any case we have

$$LL(ZB) < \min\{p^{d-1}, 4p^{d-2}\}.$$

Proof. By [9, Proposition 15], we may assume that $p > 2$. Since D is nonabelian, $|D : \mathrm{Z}(D)| \geqslant p^2$ and $LL(F\,\mathrm{Z}(D)) \leqslant p^{d-2}$. Let Q be a maximal subgroup of D. If Q is cyclic, then $D \cong M_{p^n}$ (see [4, Theorem 5.4.4]) and the claim follows from [9, Proposition 10]. Hence, we may assume

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that Q is not cyclic. Then $LL(F\,\mathrm{Z}(Q)) \leqslant p^{d-2} + p - 1$. Now setting $\lambda := (p^{d-1} - 1)/(p - 1)$ it follows from Lemma 2 that

$$\begin{aligned} JZB^{2p^{d-2}+p-1+\lambda} &\subseteq 1_B JZFG^{2p^{d-2}+p-1+\lambda} \subseteq I_{\leqslant D}(G) \cdot JZFG^{2p^{d-2}+p-1+\lambda} \\ &\subseteq I_{<D}(G) \cdot JZFG^{p^{d-2}+p-1+\lambda} \\ &= \sum_{Q<D} I_{\leqslant Q}(G) \cdot JZFG^{p^{d-2}+p-1+\lambda} \\ &\subseteq \sum_{Q<D} I_{<Q}(G) \cdot JZFG^{\lambda} = 0. \end{aligned}$$

Since $2p^{d-2} + p - 1 + \lambda \leqslant 4p^{d-2}$, we are done in case $p \geqslant 5$ and $D \cong W_{p^d}$. If $p = 3$ and $D \cong W_{p^d}$, then the claim follows from [9, Lemma 11]. Now suppose that $D \ncong W_{p^d}$. If $\mathrm{Z}(D)$ is cyclic of order p^{d-2}, then the claim follows from [9, Lemma 9 and Proposition 10]. Hence, suppose that $\mathrm{Z}(D)$ is noncyclic or $|\,\mathrm{Z}(D)| < p^{d-2}$. Then $d \geqslant 4$ and $LL(F\,\mathrm{Z}(D)) \leqslant p^{d-3} + p - 1$. The arguments above give $LL(ZB) \leqslant p^{d-2} + p^{d-3} + 2p - 2 + \lambda$, hence we are done whenever $p > 3$.

In the following we assume that $p = 3$. Here we have $LL(ZB) \leqslant 3^{d-2} + 3^{d-3} + 4 + \frac{1}{2}(3^{d-1} - 1)$ and it suffices to handle the case $d = 4$. By [12, Theorem 3.2], there exists a nontrivial B-subsection (u, b) such that

$$LL(ZB) \leqslant (|\langle u\rangle| - 1)LL(Z\overline{b}) + 1$$

where $\overline{b}$ is the unique block of $F\,\mathrm{C}_G(u)/\langle u\rangle$ dominated by b. We may assume that $\overline{b}$ has defect group $\mathrm{C}_D(u)/\langle u\rangle$ (see [14, Lemma 1.34]). If $u \notin \mathrm{Z}(D)$, we obtain $LL(ZB) < |\mathrm{C}_D(u)| \leqslant 27$ as desired. Hence, let $u \in \mathrm{Z}(D)$. Then $D/\langle u\rangle$ is not cyclic. Moreover, by our assumption on $\mathrm{Z}(D)$, we have $|\langle u\rangle| = 3$. Now it follows from [9, Theorem 1, Proposition 10 and Lemma 11] applied to $\overline{b}$ that

$$LL(ZB) \leqslant 2LL(Z\overline{b}) + 1 \leqslant 23 < 27. \qquad \square$$

We do not expect that the bounds in Theorem 3 are sharp. In fact, we do not know if there are p-blocks B with nonabelian defect groups of order p^d such that $p > 2$ and $LL(ZB) > p^{d-2}$. See also Proposition 7 below.

Corollary 4. *Let B be a block of FG with noncyclic defect group of order p^d. Then*

$$LL(ZB) \leqslant p^{d-1} + p - 1.$$

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Proof. By Theorem 3, we may assume that B has abelian defect group D. Then [9, Theorem 1] implies $LL(ZB) \leqslant LL(FD) \leqslant p^{d-1} + p - 1$. ∎

We are now in a position to generalize [9, Corollary 16].

COROLLARY 5. *Let B be a block of FG with defect group D of order p^d such that $LL(ZB) \geqslant \min\{p^{d-1}, 4p^{d-2}\}$. Then one of the following holds*

(i) *D is cyclic.*
(ii) *$D \cong C_{p^{d-1}} \times C_p$.*
(iii) *$D \cong C_2 \times C_2 \times C_2$ and B is nilpotent.*

Proof. Again by Theorem 3 we may assume that D is abelian. By [9, Corollary 16], we may assume that $p > 2$. Suppose that D is of type $(p^{a_1}, \ldots, p^{a_s})$ such that $s \geqslant 3$. Then

$$\begin{aligned}\min\{p^{d-1}, 4p^{d-2}\} &\leqslant LL(ZB) = p^{a_1} + \cdots + p^{a_s} - s + 1 \\ &\leqslant p^{a_1} + p^{a_2} + p^{a_3 + \cdots + a_s} - 2 \leqslant p^{d-2} + 2(p-1).\end{aligned}$$

This clearly leads to a contradiction. Therefore, $s \leqslant 2$ and the claim follows. ∎

In case (i) of Corollary 5 it is known conversely that $LL(ZB) = (p^d - 1)/l(B) + 1 > p^{d-1}$ (see [7, Corollary 2.8]).

Our next result gives a more precise bound by invoking the exponent of a defect group.

THEOREM 6. *Let B be a block of FG with defect group D of order $p^d > 1$ and exponent p^e. Then*

$$LL(ZB) \leqslant \left(\frac{d}{e} + 1\right)\left(\frac{d}{2} + \frac{1}{p-1}\right)(p^e - 1).$$

In particular, $LL(ZB) \leqslant d^2 p^e$.

Proof. Let $\alpha := \lfloor d/e \rfloor$. Let $P \leqslant D$ be abelian of order p^{ie+j} with $0 \leqslant i \leqslant \alpha$ and $0 \leqslant j < e$. If P has type $(p^{a_1}, \ldots, p^{a_r})$, then $a_i \leqslant e$ for $i = 1, \ldots, r$ and

$$LL(FP) = (p^{a_1} - 1) + \cdots + (p^{a_r} - 1) + 1 \leqslant i(p^e - 1) + p^j.$$

Arguing as in Theorem 3, we obtain

$$LL(ZB) \leqslant \sum_{i=0}^{\alpha} \sum_{j=0}^{e-1} i(p^e - 1) + p^j = e(p^e - 1)\left(\sum_{i=0}^{\alpha} i\right) + (\alpha + 1)\frac{p^e - 1}{p - 1}$$

$$= e(p^e-1)\frac{\alpha(\alpha+1)}{2}+(\alpha+1)\frac{p^e-1}{p-1}$$
$$\leqslant \left(\frac{d}{e}+1\right)\left(\frac{d}{2}+\frac{1}{p-1}\right)(p^e-1).$$

This proves the first claim. For the second claim we note that

$$\left(\frac{d}{e}+1\right)\left(\frac{d}{2}+\frac{1}{p-1}\right)\leqslant (d+1)\left(\frac{d}{2}+1\right)\leqslant d^2$$

unless $d\leqslant 3$. In these small cases the claim follows from Theorem 3 and Corollary 4. ∎

If $2e>d$ and p is large, then the bound in Theorem 6 is approximately dp^e. The groups of the form $G=D=C_{p^e}\times\cdots\times C_{p^e}$ show that there is no bound of the form $LL(ZB)\leqslant Cp^e$ where C is an absolute constant. A more careful argumentation in the proof above gives the stronger (but opaque) bound

$$LL(ZB)\leqslant \alpha(p^e-1)\left(\frac{e(\alpha-1)}{2}+\frac{1}{p-1}+d-\alpha e\right)+\beta(p^e-1)$$
$$+\frac{p^{d-\alpha e}-1}{p-1}+p^{d-2-\beta e}$$

for nonabelian defect groups where $\alpha:=\lfloor (d-1)/e\rfloor$ and $\beta:=\lfloor (d-2)/e\rfloor$. We omit the details.

In the next result we compute the Loewy length for $d=e+1$.

Proposition 7. *Let B be a block of FG with nonabelian defect group of order p^d and exponent p^{d-1}. Then*

$$LL(ZB)\leqslant \begin{cases} 2^{d-2}+1 & \text{if } p=2,\\ p^{d-2} & \text{if } p>2\end{cases}$$

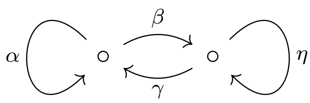
and both bounds are optimal for every $d\geqslant 3$.

Proof. Let D be a defect group of B. If $p>2$, then $D\cong M_{p^d}$ (see [4, Theorem 5.4.4]) and we have shown $LL(ZB)\leqslant p^{d-2}$ in [9, Proposition 10]. Equality holds if and only if B is nilpotent.

Therefore, we may assume $p=2$ in the following. The modular groups M_{2^d} are still handled by [9, Proposition 10]. Hence, it remains to consider the defect groups of maximal nilpotency class, i. e., $D\in\{D_{2^d},Q_{2^d},SD_{2^d}\}$.

By [9, Proposition 10], we may assume that $d \geqslant 4$. The isomorphism type of ZB is uniquely determined by D and the fusion system of B (see [2]). The possible cases are listed in [14, Theorem 8.1]. If B is nilpotent, [9, Proposition 8] gives $LL(ZB) = LL(ZFD) \leqslant LL(FD') = 2^{d-2}$. Moreover, in the case $D \cong D_{2^d}$ and $l(B) = 3$ we have $LL(ZB) \leqslant k(B) - l(B) + 1 = 2^{d-2} + 1$ by [12, Proposition 2.2]. In the remaining cases we present B by quivers with relations which were constructed originally by Erdmann [3]. We refer to [5, Appendix B]. Keep in mind that we need to consider only one quiver for each fusion system.

(i) $D \cong D_{2^d}$, $l(B) = 2$:



$$\beta\eta = \eta\gamma = \gamma\beta = \alpha^2 = 0,\quad \alpha\beta\gamma = \beta\gamma\alpha,\quad \eta^{2^{d-2}} = \gamma\alpha\beta.$$

By [5, Lemma 2.3.3], we have

$$ZB = \operatorname{span}\{1, \beta\gamma, \alpha\beta\gamma, \eta^i : i = 1, \dots, 2^{d-2}\}.$$

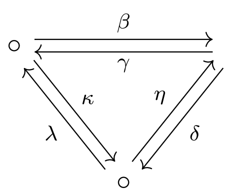
It follows that $JZB^2 = \langle \eta^2 \rangle$ and $LL(ZB) = 2^{d-2} + 1$.

(ii) $D \cong Q_{2^d}$, $l(B) = 2$: Here [16, Lemma 6] gives the isomorphism type of ZB directly as a quotient of a polynomial ring

$$ZB \cong F[U, Y, S, T]/(Y^{2^{d-2}+1}, U^2 - Y^{2^{d-2}}, S^2, T^2, SY, SU, ST, UY, UT, YT).$$

It follows that $JZB^2 = (Y^2)$ and again $LL(ZB) = 2^{d-2} + 1$.

(iii) $D \cong Q_{2^d}$, $l(B) = 3$:



$$\beta\delta = (\kappa\lambda)^{2^{d-2}-1}\kappa,\ \eta\gamma = (\lambda\kappa)^{2^{d-2}-1}\lambda,\quad \delta\lambda = \gamma\beta\gamma,\ \kappa\eta = \beta\gamma\beta,\ \lambda\beta = \eta\delta\eta,\quad \gamma\kappa = \delta\eta\delta,\ \gamma\beta\delta = \delta\eta\gamma = \lambda\kappa\eta = 0.$$

By [5, Lemma 2.5.15],

$$ZB = \operatorname{span}\{1, \beta\gamma + \gamma\beta, (\kappa\lambda)^i + (\lambda\kappa)^i, \delta\eta + \eta\delta, (\beta\gamma)^2, (\lambda\kappa)^{2^{d-2}}, (\delta\eta)^2 : i = 1, \dots, 2^{d-2} - 1\}.$$

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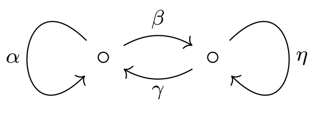
We compute

$$(\beta\gamma+\gamma\beta)^2=(\beta\gamma)^2+(\gamma\beta)^2=(\beta\gamma)^2+\delta\lambda\beta=(\beta\gamma)^2+(\delta\eta)^2,$$
$$(\beta\gamma+\gamma\beta)(\kappa\lambda+\lambda\kappa)=\beta\gamma\kappa\lambda=\beta\delta\eta\delta\lambda=\beta\delta\eta\gamma\beta\gamma=0,$$
$$(\beta\gamma+\gamma\beta)(\delta\eta+\eta\delta)=\gamma\beta\delta\eta=0,$$
$$(\beta\gamma+\gamma\beta)(\beta\gamma)^2=(\beta\gamma)^3=\beta\gamma\beta\delta\lambda=0,$$
$$(\beta\gamma+\gamma\beta)(\lambda\kappa)^{2^{d-2}}=0,$$
$$(\beta\gamma+\gamma\beta)(\delta\eta)^2=\gamma\beta\delta\eta\delta\eta=0,$$
$$((\kappa\lambda)^{2^{d-2}-1}+(\lambda\kappa)^{2^{d-2}-1})(\kappa\lambda+\lambda\kappa)=\kappa\eta\gamma+(\lambda\kappa)^{2^{d-2}}=(\beta\gamma)^2+(\lambda\kappa)^{2^{d-2}},$$
$$(\kappa\lambda+\lambda\kappa)(\delta\eta+\eta\delta)=\lambda\kappa\eta\delta=0,$$
$$(\kappa\lambda+\lambda\kappa)(\beta\gamma)^2=\kappa\lambda\beta\gamma\beta\gamma=\kappa\eta\delta\eta\gamma\beta\gamma=0,$$
$$(\kappa\lambda+\lambda\kappa)(\lambda\kappa)^{2^{d-2}}=\lambda\kappa\eta\gamma\kappa=0,$$
$$(\kappa\lambda+\lambda\kappa)(\delta\eta)^2=0,$$
$$(\delta\eta+\eta\delta)^2=(\delta\eta)^2+(\eta\delta)^2=(\delta\eta)^2+\lambda\beta\delta=(\delta\eta)^2+(\lambda\kappa)^{2^{d-2}},$$
$$(\delta\eta+\eta\delta)(\beta\gamma)^2=0,$$
$$(\delta\eta+\eta\delta)(\lambda\kappa)^{2^{d-2}}=\eta\delta(\lambda\kappa)^{2^{d-2}}=\eta\delta\eta\gamma\kappa=0,$$
$$(\delta\eta+\eta\delta)(\delta\eta)^2=\delta\lambda\beta\delta\eta=\gamma\beta\gamma\beta\delta\eta=0,$$
$$(\beta\gamma)^2(\beta\gamma)^2=(\beta\gamma)^2(\lambda\kappa)^{2^{d-2}}=(\beta\gamma)^2(\delta\eta)^2=0,$$
$$(\lambda\kappa)^{2^{d-2}}(\lambda\kappa)^{2^{d-2}}=(\lambda\kappa)^{2^{d-2}}(\delta\eta)^2=0,$$
$$(\delta\eta)^2(\delta\eta)^2=\gamma\kappa\eta(\delta\eta)^2=\gamma\beta\gamma\beta(\delta\eta)^2=0.$$

Hence, $JZB^2=\langle(\lambda\kappa)^2+(\kappa\lambda)^2,(\beta\gamma)^2+(\delta\eta)^2\rangle$ and $JZB^3=\langle(\lambda\kappa)^3+(\kappa\lambda)^3\rangle$. This implies $LL(ZB)=2^{d-2}+1$.

(iv) $D\cong SD_{2^d}$, $k(B)=2^{d-2}+3$ and $l(B)=2$:



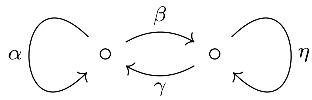
$$\gamma\beta=\eta\gamma=\beta\eta=0,$$
$$\alpha^2=\beta\gamma,\ \alpha\beta\gamma=\beta\gamma\alpha,$$
$$\eta^{2^{d-2}}=\gamma\alpha\beta.$$

By [6, Section 5.1], we have

$$ZB=\operatorname{span}\{1,\beta\gamma,\alpha\beta\gamma,\eta^i:i=1,\dots,2^{d-2}\}.$$

As in (i) we obtain $JZB^2=\langle\eta^2\rangle$ and $LL(ZB)=2^{d-2}+1$.

(v) $D \cong SD_{2^d}$, $k(B) = 2^{d-2} + 4$ and $l(B) = 2$:



$$\beta\eta = \alpha\beta\gamma\alpha\beta,\ \ \gamma\beta = \eta^{2^{d-2}-1},$$
$$\eta\gamma = \gamma\alpha\beta\gamma\alpha,$$
$$\beta\eta^2 = \eta^2\gamma = \alpha^2 = 0.$$

By [6, Section 5.2.2], we have

$$ZB = \operatorname{span}\{1, \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, \beta\gamma\alpha\beta\gamma, (\alpha\beta\gamma)^2, \eta^i,$$
$$\eta + \alpha\beta\gamma\alpha : i = 2, \dots, 2^{d-2}\}.$$

Since $(\alpha\beta\gamma)^2 = \beta\eta\gamma = (\beta\gamma\alpha)^2$ and $(\gamma\alpha\beta)^2 = \eta\gamma\beta = \eta^{2^{d-2}}$, it follows that

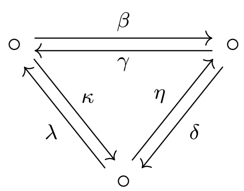
$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)^2 = (\alpha\beta\gamma)^2 + (\beta\gamma\alpha)^2 + (\gamma\alpha\beta)^2 = \eta^{2^{d-2}}.$$

Similarly,

$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)\beta\gamma\alpha\beta\gamma = 0,$$
$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)(\alpha\beta\gamma)^2 = 0,$$
$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)\eta^2 = 0,$$
$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)(\eta + \alpha\beta\gamma\alpha) = 0,$$
$$(\beta\gamma\alpha\beta\gamma)^2 = 0,$$
$$\beta\gamma\alpha\beta\gamma(\alpha\beta\gamma)^2 = 0,$$
$$\beta\gamma\alpha\beta\gamma\eta^2 = \beta\gamma\alpha\beta\eta^2\gamma = 0,$$
$$\beta\gamma\alpha\beta\gamma(\eta + \alpha\beta\gamma\alpha) = \beta\gamma(\alpha\beta\gamma)^2\alpha = 0,$$
$$(\alpha\beta\gamma)^2(\alpha\beta\gamma)^2 = 0,$$
$$(\alpha\beta\gamma)^2\eta^2 = 0,$$
$$(\alpha\beta\gamma)^2(\eta + \alpha\beta\gamma\alpha) = 0,$$
$$\eta^2(\eta + \alpha\beta\gamma\alpha) = \eta^3,$$
$$(\eta + \alpha\beta\gamma\alpha)^2 = \eta^2.$$

Consequently, $JZB^2 = \langle\eta^2\rangle$ and $LL(ZB) = 2^{d-2} + 1$.

(vi) $D \cong SD_{2^d}$, $l(B) = 3$:



$$\kappa\eta = \eta\gamma = \gamma\kappa = 0,\ \ \delta\lambda = (\gamma\beta)^{2^{d-2}-1}\gamma,$$
$$\beta\delta = \kappa\lambda\kappa,\ \ \lambda\beta = \eta.$$

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From [5, Lemma 2.4.16] we get

$$ZB = \operatorname{span}\{1, (\beta\gamma)^i + (\gamma\beta)^i, \kappa\lambda + \lambda\kappa, (\beta\gamma)^{2^{d-2}}, (\lambda\kappa)^2, \\ \delta\eta : i = 1, \ldots, 2^{d-2} - 1\}.$$

We compute

$$(\beta\gamma + \gamma\beta)((\beta\gamma)^{2^{d-2}-1} + (\gamma\beta)^{2^{d-2}-1}) = (\beta\gamma)^{2^{d-2}} + \delta\lambda\beta = (\beta\gamma)^{2^{d-2}} + \delta\eta,$$
$$(\beta\gamma + \gamma\beta)(\kappa\lambda + \lambda\kappa) = \beta\gamma\kappa\lambda = 0,$$
$$(\beta\gamma + \gamma\beta)(\beta\gamma)^{2^{d-2}} = \beta\delta\lambda\beta\gamma = \kappa\lambda\kappa\eta\gamma = 0,$$
$$(\beta\gamma + \gamma\beta)(\lambda\kappa)^2 = 0,$$
$$(\beta\gamma + \gamma\beta)\delta\eta = \gamma\beta\delta\eta = \gamma\kappa\lambda\kappa\eta = 0,$$
$$(\kappa\lambda + \lambda\kappa)^2 = \beta\delta\lambda + (\lambda\kappa)^2 = (\beta\gamma)^{2^{d-2}} + (\lambda\kappa)^2,$$
$$\begin{aligned}(\kappa\lambda + \lambda\kappa)(\beta\gamma)^{2^{d-2}} &= \kappa\lambda\beta\gamma(\beta\gamma)^{2^{d-2}-1} \\ &= \kappa\eta\gamma(\beta\gamma)^{2^{d-2}-1} = 0,\end{aligned}$$
$$\begin{aligned}(\kappa\lambda + \lambda\kappa)(\lambda\kappa)^2 &= \lambda(\beta\gamma)^{2^{d-2}}\kappa \\ &= \eta\gamma(\beta\gamma)^{2^{d-2}-1}\kappa = 0,\end{aligned}$$
$$(\kappa\lambda + \lambda\kappa)\delta\eta = 0,$$
$$\begin{aligned}(\beta\gamma)^{2^{d-2}}(\beta\gamma)^{2^{d-2}} &= (\beta\gamma)^{2^{d-2}}(\lambda\kappa)^2 \\ &= (\beta\gamma)^{2^{d-2}}\delta\eta = 0,\end{aligned}$$
$$(\lambda\kappa)^2(\lambda\kappa)^2 = (\lambda\kappa)^2\delta\eta = 0,$$
$$(\delta\eta)^2 = \delta\lambda\beta\delta\eta = \delta\lambda\kappa\lambda\kappa\eta = 0.$$

Hence, $JZB^2 = \langle(\beta\gamma)^2 + (\gamma\beta)^2, (\kappa\lambda)^2 + \delta\eta\rangle$ and $JZB^3 = \langle(\beta\gamma)^3 + (\gamma\beta)^3\rangle$. This implies $LL(ZB) = 2^{d-2} + 1$. ∎

It is interesting to note the difference between even and odd primes in Proposition 7. For $p = 2$, non-nilpotent blocks give larger Loewy lengths while for $p > 2$ the maximal Loewy length is only attained for nilpotent blocks.

Recall that a *lower defect group* of a block B of FG is a p-subgroup $Q \leqslant G$ such that

$$I_{<Q}(G)1_B \neq I_{\leqslant Q}(G)1_B.$$

In this case Q is conjugate to a subgroup of a defect group D of B and conversely D is also a lower defect group since $1_B \in I_{\leqslant D}(G) \setminus I_{<D}(G)$. It is clear that in the proofs of Theorems 3 and 6 it suffices to sum over the lower defect groups of B. In particular there exists a chain of lower defect groups $Q_1 < \cdots < Q_n = D$ such that $LL(ZB) \leqslant \sum_{i=1}^{n} LL(F\,\mathrm{Z}(Q_i))$. Unfortunately, it is hard to compute the lower defect groups of a given block.

The following proposition generalizes [15, Theorem 1.5].

PROPOSITION 8. *Let B be a block of FG. Then ZB is uniserial if and only if B is nilpotent with cyclic defect groups.*

Proof. Suppose first that ZB is uniserial. Then $ZB \cong F[X]/(X^n)$ for some $n \in \mathbb{N}$; in particular, ZB is a symmetric F-algebra. Then [11, Theorems 3 and 5] implies that B is nilpotent with abelian defect group D. Thus, by a result of Broué and Puig [1] (see also [8]), B is Morita equivalent to FD; in particular, FD is also uniserial. Thus D is cyclic.

Conversely, suppose that B is nilpotent with cyclic defect group D. Then the Broué–Puig result mentioned above implies that B is Morita equivalent of FD. Thus $ZB \cong ZFD = FD$. Since FD is uniserial, the result follows. ∎

A similar proof shows that ZB is isomorphic to the group algebra of the Klein four group over an algebraically closed field of characteristic 2 if and only if B is nilpotent with Klein four defect groups.

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