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A NOTE ON INHOMOGENEOUS DIOPHANTINE APPROXIMATION

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Abstract. Let α be an irrational number. We determine the Hausdorff dimension of sets of real numbers which are close to infinitely many elements of the sequence $(\{n\alpha\})_{n\geq 1}$.

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1. Introduction. Let α be an irrational real number. It is a well-known fact that, if $\{\cdot\}$ denotes the fractional part, the sequence $(\{n\alpha\})_{n\geq1}$ is uniformly distributed in [0, 1]. In their book, Bernik & Dodson [**4**, p. 105] consider sets of real numbers which are close to infinitely many elements of it. They prove that, for any $v>1$, the Hausdorff dimension of the set

$$\mathcal{V}_v(\alpha) := \left\{\xi \in \mathbf{R} : \|n\alpha - \xi\| < \frac{1}{n^v} \quad \text{holds for infinitely many } n \in \mathbf{N}\right\}$$

satisfies

$$\frac{1}{wv} \leq \dim \mathcal{V}_v(\alpha) \leq \frac{1}{v}, \tag{1}$$

where $w \geq 1$ is any real number for which we have

$$\|n\alpha\| \geq \frac{1}{n^w} \tag{2}$$

for all sufficiently large integers n. Here, and in the sequel, $\|\cdot\|$ denotes the distance to the nearest integer.

The upper bound in (1) is a straightforward consequence of the Borel–Cantelli lemma, while the lower bound depends on precise estimates for the discrepancy of the sequence $(\{n\alpha\})_{n\geq1}$. A metrical result of Khintchine asserts that for almost all real numbers α (in the sense of the Lebesgue measure) inequality (2) is satisfied for any $w>1$ and any n sufficiently large, in terms of α and w. Thus, it immediately follows from (1) that $\dim \mathcal{V}_v(\alpha) = 1/v$ for almost all real numbers α. However, (1) provides no non-trivial lower bound when α is a Liouville number, and is also weak when α is well approximable by rational numbers.

The purpose of the present note is to prove that the exact value of the Hausdorff dimension of $\mathcal{V}_v(\alpha)$ equals $1/v$, and thus does not depend on α. We also discuss a natural generalization of these sets.

2. Statement of the result. Our main result provides the exact value of the Hausdorff dimension of the sets $\mathcal{V}_v(\alpha)$.

THEOREM 1. *Let α be an irrational real number and let $v > 1$ be real. Then we have*

$$\dim \mathcal{V}_v(\alpha) = \frac{1}{v}.$$

The proof of Theorem 1 depends on the notion of regular system, first introduced by Baker & Schmidt [**2**] (see also [**4**, p. 99]). Here, and in the sequel, we denote by $|I|$ the length of a bounded real interval I.

DEFINITION 1. Let I be a real interval. Let Γ be a countable set of real numbers in I and $\mathcal{N} : \Gamma \to \mathbf{R}$ be a positive function. The pair $(\Gamma, \mathcal{N})$ is called a *regular system* if there exists a positive constant $c_1 = c_1(\Gamma, \mathcal{N})$ with the following property: for any bounded interval $J \subset I$, there exists a positive number $K_0 = K_0(\Gamma, \mathcal{N}, J)$ such that, for any $K \geq K_0$, there are $\gamma_1, \ldots, \gamma_t$ in $\Gamma \cap J$ such that

$$\mathcal{N}(\gamma_j) \leq K, \quad |\gamma_j - \gamma_k| \geq K^{-1} \quad (1 \leq j < k \leq t)$$

and

$$t \geq c_1 |J| K.$$

Baker & Schmidt have shown that, for any integer $n \geq 1$, the set Γ_n of real algebraic numbers of degree less than or equal to n, together with a suitable function $\mathcal{N}_n$, is a regular system. This enabled them to determine the exact Hausdorff dimension of sets of real numbers close to infinitely many algebraic numbers of degree less than or equal to n.

However, to get their final result, it would have been sufficient to show that $(\Gamma_n, \mathcal{N}_n)$ is a *weakly* regular system, a concept introduced by Rynne [**9**, Definition 4].

DEFINITION 2. Let I be a real interval. Let Γ be a countable set of real numbers in I and $\mathcal{N} : \Gamma \to \mathbf{R}$ be a positive function. The pair $(\Gamma, \mathcal{N})$ is called a *weakly regular system* if there exist a strictly increasing sequence of positive integers $(K_r)_{r \geq 1}$ and a positive constant $c_2 = c_2(\Gamma, \mathcal{N})$ with the following property: for any bounded interval $J \subset I$, there exists a positive number $r_0 = r_0(\Gamma, \mathcal{N}, J)$ such that, for any $r \geq r_0$, there are $\gamma_1, \ldots, \gamma_t$ in $\Gamma \cap J$ such that

$$\mathcal{N}(\gamma_j) \leq K_r, \quad |\gamma_j - \gamma_k| \geq K_r^{-1} \quad (1 \leq j < k \leq t) \tag{3}$$

and

$$t \geq c_2 |J| K_r.$$

For our purpose, this is a crucial remark. Indeed, we shall establish that, for any irrational real number α, the sequence $(\{n\alpha\})_{n \geq 1}$ together with the function $\mathcal{N}_\lambda$: $\{n\alpha\} \mapsto \max\{(n-5)/3, 1\}$ is a *weakly* regular system.

REMARK 1. It follows from a theorem of Cassels [**6**] that, for any real irrational α and for any real $\xi \notin \mathbf{Z} + \alpha\mathbf{Z}$, the inequality

$$\|n\alpha - \xi\| < \frac{1}{n}$$

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holds for infinitely many $n \in \mathbf{N}$. A trivial consequence is that the set

$$\mathcal{V}_1(\alpha) := \left\{\xi \in \mathbf{R} : \|n\alpha - \xi\| < \frac{1}{n} \quad \text{holds for infinitely many } n \in \mathbf{N}\right\}$$

has full Lebesgue measure. A natural problem is then to replace the function $n \mapsto n^{-v}$ occurring in the definition of the sets $\mathcal{V}_v(\alpha)$ by any decreasing positive function Ψ and to consider the set

$$\mathcal{V}_\Psi(\alpha) := \{\xi \in \mathbf{R} : \|n\alpha - \xi\| < \Psi(n) \quad \text{holds for infinitely many } n \in \mathbf{N}\}.$$

It easily follows from the Borel–Cantelli lemma that the Lebesgue measure of $\mathcal{V}_\Psi(\alpha)$ is zero whenever the series $\sum_{n\geq 1} \Psi(n)$ converges. However, is it possible to provide a nontrivial lower bound for the Lebesgue measure of $\mathcal{V}_\Psi(\alpha) \cap [0, 1]$ when the series $\sum_{n\geq 1} \Psi(n)$ diverges?

It seems to us that the answer to this question heavily depends on α, and, more precisely, on the partial quotients in the continued fraction expansion of α. For instance, when α has bounded partial quotients, one can prove that there exists a constant $c > 0$ such that the infinite sequence $(N_r)_{r\geq 1}$ of integers satisfying (4) below is such that $N_{r+1} \leq cN_r$ for any $r \geq 1$. Arguing then as in the proof of Theorem 2 of Beresnevich [**3**], one can give an affirmative answer to our question, by showing that the set $\mathcal{V}_\Psi(\alpha)$ has full Lebesgue measure whenever the series $\sum_{n\geq 1} \Psi(n)$ diverges. Moreover, Theorem 3 of [**5**] yields the generalized Hausdorff measure of the set $\mathcal{V}_\Psi(\alpha)$ when the series $\sum_{n\geq 1} \Psi(n)$ converges. However, we suspect that these results are no longer true for a general irrational α.

REMARK 2. Dodson [**7**] and Levesley [**8**] have determined the Hausdorff dimension of two sets which are closely related to $\mathcal{V}_v(\alpha)$. Namely, for any $v \geq 1$, they have, respectively, established that

$$\dim\{(\alpha, \xi) \in \mathbf{R}^2 : \|n\alpha - \xi\| < n^{-v} \quad \text{holds for infinitely many } n \in \mathbf{N}\} = 1 + \frac{2}{v+1}$$

and, for any given real number ξ, that

$$\dim\{\alpha \in \mathbf{R} : \|n\alpha - \xi\| < n^{-v} \quad \text{holds for infinitely many } n \in \mathbf{N}\} = \frac{2}{v+1},$$

which can be viewed as a ‘dual’ form of Theorem 1. The difference in the Hausdorff dimension of $\mathcal{V}_v(\alpha)$ is interesting to notice.

The above-quoted results of Dodson and Levesley are special cases of general theorems for systems of linear forms. In a further work, we plan to study whether the approach followed here can be extended to simultaneous approximation and to linear forms.

3. Proof. Let α be an irrational real number and set $\Gamma := (\{n\alpha\})_{n\geq 1}$. Denote by $\mathcal{N}$ the function defined on Γ by

$$\mathcal{N}(n\alpha) = \max\left\{\frac{n-5}{3}, 1\right\}.$$

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Our aim is to prove that $(\Gamma, \mathcal{N})$ is a *weakly* regular system. Then, in order to get Theorem 1, it suffices to apply Theorem 3.2 of **[9]** (whose proof is similar to that of Lemma 1 of **[2]**), which we quote as Proposition 1.

PROPOSITION 1. *Suppose that the system* $(\Gamma, \mathcal{N})$ *is weakly regular and denote by* $(K_r)_{r\geq 1}$ *the sequence occurring in Definition* 2. *Let* $\Psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ *be a decreasing function with* $\lim_{x\to\infty} \Psi(x) = 0$. *Assume that* $\Psi(K_r) \leq 1/(2K_r)$ *for large r and let* δ *be the supremum of the set of* δ' *for which*

$$\limsup_{r\to\infty} K_r \Psi(K_r)^{\delta'} = \infty$$

holds. Then the Hausdorff dimension of the set

$$\{\xi \in \mathbf{R} : \|\xi - \gamma\| < \Psi(\mathcal{N}(\gamma)) \quad \textit{holds for infinitely many } \gamma \in \Gamma\}$$

is at least equal to δ.

Let us now prove that $(\Gamma, \mathcal{N})$ is a *weakly* regular system. Let $N \geq 12$ be an integer. The well-known *Three Distance Theorem* (see for instance **[11]**, **[10]** or **[1]**(*)) asserts that the points $\{\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}$ divide the interval [0, 1] in $N+1$ intervals, denoted by $I_1, \dots I_{N+1}$, whose lengths take at most three distinct values, one of these being the sum of the two others. Denoting the three values by $d_1(N), d_2(N), d_3(N)$, where $d_1(N) < d_2(N) < d_3(N)$, one has

$$d_3(N+1) = d_3(N), \quad d_2(N+1) = d_2(N), \quad d_1(N+1) = d_1(N),$$

or

$$d_3(N+1) = d_2(N), \quad d_2(N+1) = d_2(N) - d_1(N), \quad d_1(N+1) = d_1(N),$$

or

$$d_3(N+1) = d_2(N), \quad d_2(N+1) = d_1(N), \quad d_1(N+1) = d_2(N) - d_1(N), \tag{4}$$

according as $d_2(N) < 2d_1(N)$ or not. Since α is irrational, (4) holds for infinitely many values of N. For each of these values, we have $d_2(N) < 2d_1(N)$, thus

$$d_3(N) = d_2(N) + d_1(N) < 3d_1(N).$$

Consequently, there are infinitely many integers N such that $d_3(N) < 3d_1(N)$. For such values of N, any closed subinterval of [0, 1] of length $d_3(N)$ contains at least one of the points $\{\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}$. Further, since the sum of the lengths of the intervals I_j, $1 \leq j \leq N+1$, is equal to 1, we get

$$1 \geq (N+1)d_1(N) \geq \frac{N+1}{3} d_3(N),$$

whence

$$d_3(N) \leq \frac{3}{N+1}. \tag{5}$$

(*) There is a misprint in the statement of the *Three Distance Theorem* in **[1]**: η_k is the smallest length, not the largest one.

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Denote by $(N_r)_{r\geq 1}$ the strictly increasing sequence of integers $N \geq 12$ satisfying (4), and thus (5). For any $r \geq 1$, let K_r be the greatest even integer smaller than $(N_r + 1)/3$. We divide the interval [0,1] into the K_r intervals $J_m := [m/K_r, (m+1)/K_r]$, for $0 \leq m < K_r$. By (5) and the definition of K_r, for any integer $1 \leq m \leq K_r/2$, there exist $\beta_m \in J_{2m-1}$ and $1 \leq j \leq N_r$ such that $\beta_m = \{j\alpha\}$. Now, we see that the $K_r/2$ points $\beta_1, \dots \beta_{K_r/2}$ satisfy $|\beta_j - \beta_\ell| \geq 1/K_r$ whenever $1 \leq j < \ell \leq K_r/2$. Recall that we have defined the function $\mathcal{N}$ on Γ by

$$\mathcal{N}(\{n\alpha\}) = \max\left\{\frac{n-5}{3}, 1\right\}.$$

We observe that $\mathcal{N}(\beta_m) \leq K_r$ for any $1 \leq m \leq K_r/2$.

Further, let J be a subinterval of [0, 1]. Let r be a positive integer such that

$$K_r \geq 12|J|^{-1}, \tag{6}$$

and denote by $r_0(J)$ the smallest positive integer with this property. At least $A := |J|K_r/2 - 2$ intervals J_{2m-1} with $1 \leq m \leq K_r/2$ are completely included in J. By (6), we get

$$A \geq \frac{1}{3}|J|K_r,$$

whence at least $|J|K_r/3$ numbers among $\{\alpha\}, \dots, \{N_r\alpha\}$ belong to J. Denoting these numbers by $\gamma_1, \dots, \gamma_t$, we have $\mathcal{N}(\gamma_j) \leq K_r$ and $|\gamma_j - \gamma_\ell| \geq 1/K_r$ whenever $1 \leq j < \ell \leq t$, as was already noticed above. Consequently, for any interval J in [0, 1] and any integer $r \geq r_0(J)$, there are at least $|J|K_r/3$ elements of $\Gamma \cap J$ satisfying (3). Since $(K_r)_{r\geq 1}$ contains a strictly increasing subsequence, this proves that $(\Gamma, \mathcal{N})$ is a weakly regular system.

Let $v > 1$ and $\varepsilon > 0$. By applying Proposition 1 to $(\Gamma, \mathcal{N})$ with the function $\Psi : x \mapsto x^{-v-\varepsilon}$, we obtain that dim $\mathcal{V}_v(\alpha) \geq 1/(v+\varepsilon)$. Since ε can be arbitrarily small, we then have dim $\mathcal{V}_v(\alpha) \geq 1/v$, as claimed.

As was kindly pointed out to me by the referee, the original form of ubiquity (see **[9]** for the definition) gives the lower bound for the Hausdorff dimension of $\mathcal{V}_v(\alpha)$ straightforwardly, once the inequality (5) for infinitely many positive integers is established. Namely, the $\{n\alpha\}$, $n \geq 1$, are the resonant points, and, for each integer $N \geq N_1$, the 'ubiquity' function λ may be defined by $\lambda(N) = 3/(N_r + 1)$, where $N_r \leq N < N_{r+1}$. The desired result follows then from [**9**, Theorem 2.1].

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