Can. J. Math., Vol. XXVIII, No. 4, 1976, pp. 836–839

TRANSFORMATIONS WITH DISCRETE SPECTRUM ARE STACKING TRANSFORMATIONS

ANDRÉS DEL JUNCO

Introduction. The stacking method (see [**1**] and [**5**, Section 6]) has been used with great success in ergodic theory to construct a wide variety of examples of ergodic transformations (see, for example, [**1**; **3**; **4**; **5**; **7**]). However very little is known in general about the class $\mathscr{S}$ of transformations which can be constructed by the stacking method using single stacks. In particular there is no simple characterization of the class $\mathscr{S}$. In [**1**], the following question is raised: is every transformation with simple spectrum an $\mathscr{S}$-transformation? (Since the converse is true by [**2**, Theorem 1], this would give a nice characterization of $\mathscr{S}$). The simplest case of simple spectrum is discrete spectrum and the aim of this paper is to prove that any ergodic transformation T with discrete spectrum belongs to $\mathscr{S}$ (Theorem 2.3).

The method of proof consists in finding an increasing sequence $\{\mathscr{G}_n\}$ of T-invariant σ-algebras which generate the full σ-algebra and such that $T/\mathscr{G}_n$ looks like a cartesian product of several rotations and one cyclic permutation. The result is proved for this concrete case which is where the difficulty lies. One then applies a simple lemma which gives the result for T itself.

I would like to thank Professor M. A. Akcoglu for suggesting this problem.

Section 0: Notation and definitions. All measure spaces $(X, \mathscr{F}, \mu)$ will be isomorphic to the unit interval with Borel sets and Lebesgue measure. A *transformation* (*automorphism*) of $(X, \mathscr{F}, \mu)$ is an invertible bimeasurable, measure-preserving mapping of X onto X. A *partition* of X is a finite collection of mutually disjoint sets in $\mathscr{F}$. If $\{P_n\}$ is a sequence of partitions, $P_n \to \epsilon$ means $\mu(A \bigtriangleup P_n(A)) \to 0$ for all $A \in \mathscr{F}$, where $P_n(A)$ denotes any union of atoms of P_n such that $\mu(P_n(A) \bigtriangleup A)$ is minimal. If T is a transformation of X, a *stack for* T (or T-stack) is an ordered partition $S = \{S_1, \ldots, S_n\}$ of X such that $T(S_j) = S_{j+1}$ for $1 \leqq j < n$. S_1 is called the *base* of S, S_i the *i-th level* and n its *height*.

$\mathscr{S}$ is the class of transformations T for which there exists a sequence $\{S_n\}$ of T-stacks such that $S_n \to \epsilon$ and the base of S_n is a union of levels of S_{n+1}. This is just the class of transformations which can be constructed by the stacking method using single stacks. (For the stacking method see [**5**]). The following theorem, due to Baxter ([**2**, Theorem 2.1]), which we shall use implicitly, shows that the requirement that the base of S_n be a union of levels of S_{n+1} is unnecessary.

Received December 4, 1975.

THEOREM (Baxter). *A transformation T belongs to $\mathscr{S}$ if and only if there is a sequence $\{S_n\}$ of T-stacks such that $S_n \to \epsilon$.*

Section 1. If $\alpha \in [0, 1)$ we define the transformation T_α on $[0, 1)$ by $T_\alpha(x) = x + \alpha \pmod 1$. Let $\alpha(1), \ldots, \alpha(n) \in [0, 1)$ and let π be a cyclic permutation of $S = \{1, \ldots, m\}$. Let $T = T_{\alpha(1)} \times \ldots \times T_{\alpha(m)} \times \pi$ and assume T is ergodic. Denote by $(\Omega, \mathscr{F}, \mu)$ the measure space on which T acts ($\Omega = [0, 1)^n \times S$, $\mathscr{F}$ the product Borel structure, μ the product of Lebesgue measures and normalized counting measure).

For each $\alpha(i)$ choose a sequence $p(i,j)/q(i,j)$ of irreducible fractions such that $q(i,j)$ increases to ∞ as $j \to \infty$ and

$$\left|\alpha(i) - \frac{p(i,j)}{q(i,j)}\right| \leqq \left(\frac{1}{q(i,j)}\right)^2.$$

(It is elementary and well known that this can be done. See, for example, [**6**, Section 11.3]). Denote by T_j the transformation

$$T_{p(1,j)/q(1,j)} \times \ldots \times T_{p(n,j)/q(n,j)} \times \pi.$$

Consider also the partition Q_{ij} of $[0, 1]$ into sets

$$\left[\frac{r}{q(i,j)}, \frac{r+1}{q(i,j)}\right), \quad 0 \leqq r < q(i,j)$$

and the partition $Q_j = Q_{ij} \times \ldots \times Q_{nj} \times \eta$ of Ω where η denotes the partition of S into points. Note that T_j permutes the atoms of Q_j. For $\epsilon > 0$ let $E_\epsilon = [0, \epsilon)^n \times \{1\} \subset \Omega$.

LEMMA 1.1. *Given $\epsilon > 0$, there exists a K such that if $x \in \Omega$ then for some k, $0 \leqq k < K$, $T^k x \in E_\epsilon$.*

Proof. This follows easily from the fact that the T-orbit of any point is dense in Ω, which in turn follows easily from the ergodicity of T.

LEMMA 1.2. *Given $\epsilon > 0$ there exist K and J such that if $j > J$ and ξ is an atom of Q_j then for some k, $0 \leqq k < K$, $T_j{}^k \xi \subset E_\epsilon$.*

Proof. By Lemma 1.1 we can choose a K such that for all $x \in \Omega$ there is a k, $0 \leqq k < K$ such that $T^k x \in E_{\epsilon/4}$. Then choose J so large that

$$K\left|\alpha(i) - \frac{p(i,j)}{q(i,J)}\right| < \frac{\epsilon}{4} \quad \text{and} \quad \frac{1}{q(i,J)} < \frac{\epsilon}{4}$$

for all i. One checks easily that K and J satisfy the desired condition.

PROPOSITION 1.3. $T \in \mathscr{S}$.

Proof. It will suffice to show that for each $\epsilon > 0$ we can find a T-stack whose base is contained in E_ϵ and which covers a part of the space of measure more than $1 - \epsilon$. Given ϵ, then, choose K and J as in Lemma 1.2 and such that $2/K < \epsilon/2$.

Now for any j, Q_j breaks up into a disjoint union of T_j-stacks. Let us call these stacks $\xi(j, 1), \ldots, \xi(j, n_j)$. It is easy to see that they all have the same height, say h_j, and that h_j is at least as large as $\max_i q(i, j)$, so that $h_j \to \infty$. Fix for the moment a j such that $j > J$, $h(j) > K^3$ and $1/q(i, j) < \epsilon$ for all i. Let $h_j = K^2D + r$, $0 \leqq r < K^2$.

For each n, $1 \leqq n \leqq n_j$, and for each d, $1 \leqq d \leqq D$, choose a level of $\xi(j, n)$ between the $(dK^2 + 1)$-th level and the $(dK^2 + K)$-th level which is contained in E_ϵ (this can be done by our choice of J and K) and let B be the union of all these levels. Each of these levels we have chosen has at least $K(K - 1)$ images disjoint from any other chosen level, so B is the base for a T_j-stack of height $K(K - 1) + 1$ which will cover Ω except for a set of measure less than $2K^2/h(j) \leqq 2/K < \epsilon/2$.

We now get a T-stack from this T_j-stack by shrinking B by a small fraction of its measure. This is done as follows. Each atom γ of Q_j is a product of intervals I_{ij}^γ of length $1/q(i, j)$, $1 \leqq i \leqq n$, and a single point in S. By chopping off from each end of I_{ij}^γ an interval of length $K(K - 1)|\alpha_i - p(i, j)/q(i, j)|$ one gets an interval $\bar{I}_{ij}^\gamma$ such that

$$T^l_{\alpha(i)}\bar{I}_{ij}^\gamma \subset T_{p(i,j)/q(i,j)}I_{ij}^\gamma \quad \text{for } 0 \leqq l \leqq K(K - 1).$$

(Note for future use that since $|\alpha(i) - p(i, j)/q(i, j)| \leqq (1/q(i, j))^2$ we can make the amount chopped off from I_{ij}^γ as small a fraction of its length as we like by choosing j large). It follows that if we set $\bar{\gamma} = \prod_i \bar{I}_{ij}^\gamma \times \{1\}$ then $T^l\bar{\gamma} \subset T_j{}^l\gamma$ for $0 \leqq l \leqq K(K - 1)$. Finally, if $B = \bigcup_{\gamma \in \Gamma} \gamma$ for $\Gamma \subset Q_j$, we set $\bar{B} = \bigcup_{\gamma \in \Gamma} \bar{\gamma}$ and we have again $T^l\bar{B} \subset T_j{}^lB$ for $0 \leqq l \leqq K(K - 1)$. Since $\bar{I}_{ij}^\gamma$ can be made as large a portion of I_{ij}^γ as we wish, the same is true of $\bar{B}$ and B so that our T-stack can be made to cover a part of Ω of measure more than $1 - \epsilon$. Of course $\bar{B} \subset B \subset E_\epsilon$ so this finishes the proof.

Section 2. Our aim in this section is to extend Proposition 1.3 to the case of ergodic T with discrete spectrum. We begin with a simple general lemma.

LEMMA 2.1. *Suppose T is a transformation of $(X, \mathscr{F}, \mu)$ and $\{\mathscr{G}_n\}$ is an increasing sequence of T-invariant σ-algebras which generate $\mathscr{F}$ such that $T|_{\mathscr{G}_n} \in \mathscr{S}$. Then $T \in \mathscr{S}$.*

Proof. If Σ is a σ-algebra and $\{E_n\}$ is a sequence of sets in Σ we'll say $\{E_n\}$ is an *approximating sequence for* Σ if for each $E \in \Sigma$ and $\epsilon > 0$ there is an E_n such that $\mu(E_n \triangle E) < \epsilon$. Let $\{E_n\}$ be a sequence of sets in $\bigcup_n \mathscr{G}_n$ which contains an approximating sequence for each $\mathscr{G}_n$. Since any set in $\mathscr{F}$ can be approximated arbitrarily well by sets in $\bigcup_n \mathscr{G}_n$ it follows that E_n is an approximating sequence for $\mathscr{F}$. Now for each n, $\{A_1, \ldots, A_n\} \subset G_m$ for some m and since $T|_{\mathscr{G}_m} \in \mathscr{S}$ we can find a ($\mathscr{G}_m$-measurable) T-stack S_n such that $\mu(A_i \triangle S_n(A)) < 1/n$ for $1 \leqq i \leqq n$. Then it is clear that $\mu(A \triangle S_n(A)) \to 0$ for every $A \in \mathscr{F}$.

Now let T be an ergodic transformation with discrete spectrum (see [**8**, p. 46] for the definition). Let $\{\lambda_i\}$ be an enumeration of the eigenvalues of the induced unitary operator and suppose f_i is an eigenvector with eigenvalue λ_i. Let $\mathscr{A}_n$ denote the complex algebra of functions generated by $\{f_i, \bar{f}_i : i = 1, \ldots, n\}$. Note that $\mathscr{A}_n \subset \mathscr{L}_\infty \subset \mathscr{L}_2$. Denote by $\mathscr{H}_n$ the $\mathscr{L}_2$ closure of $\mathscr{A}_n$. Let $\mathscr{G}_n$ denote the σ-algebra of sets generated by $f_1, \ldots, f_n$ (that is, the σ-algebra generated by $\{f_i^{-1}(B) : i = 1, \ldots, n, B \text{ a borel set}\}$). Note that $\mathscr{G}_n$ is T-invariant.

LEMMA 2.2. $\mathscr{H}_n = \mathscr{L}_2(X, \mathscr{G}_n, \mu)$.

Proof. This can be shown using the Stone-Weierstrass theorem together with some straightforward measure-theoretic arguments.

THEOREM 2.3. $T \in \mathscr{S}$.

Proof. Lemma 2.2 implies that $T|_{\mathscr{G}_n}$ has discrete spectrum and that its set of eigenvalues is the multiplicative group generated by $\{\lambda_1, \ldots, \lambda_n\}$. This group can be generated by a set $\{e^{2\pi i\alpha(j)} : j = 1, \ldots, r\}$ where $\{\alpha(1), \ldots, \alpha(r)\}$ is independent over the rationals. Supposing for convenience that $\alpha(r) = 1/m$ is the sole rational member of this set, we have by the discrete spectrum theorem ([**8**, p. 46]) that $T|_{\mathscr{G}_n}$ is isomorphic to $T_{\alpha(1)} \times \ldots \times T_{\alpha(r-1)} \times \pi$ where π is a cyclic permutation of $\{1, \ldots, m\}$. Thus $T|_{\mathscr{G}_n} \in \mathscr{S}$ by Proposition 1.3. In view of Lemma 2.1 we need only show that $\mathscr{G}_n \uparrow \mathscr{F}$ to complete the proof. But this follows immediately from the fact that $\mathscr{L}_2(X, \mathscr{G}_n, \mu) \uparrow \mathscr{L}_2(X, \mathscr{F}, \mu)$.

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University of British Columbia,
Vancouver, British Columbia