ON THE VANISHING VISCOSITY IN THE CAUCHY PROBLEM FOR THE EQUATIONS OF A NONHOMOGENEOUS INCOMPRESSIBLE FLUID

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1. Introduction. Let us consider the Cauchy problem

$$\begin{cases} \rho_t + \mathbf{v} \cdot \nabla\rho = 0 \\ \rho[\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}] + \nabla p = \mu\Delta\mathbf{v} + \rho\mathbf{f} \\ \operatorname{div} \mathbf{v} = 0 \\ \rho \,|_{t=0} = \rho_0(\mathbf{x}) \\ \mathbf{v} \,|_{t=0} = \mathbf{v}_0(\mathbf{x}) \end{cases} \tag{1.1:μ}$$

in $Q_T = \mathbb{R}^3 \times [0, T]$, where $\mathbf{f}(\mathbf{x}, t)$, $\rho_0(\mathbf{x})$ and $\mathbf{v}_0(\mathbf{x})$ are given, while the density $\rho(\mathbf{x}, t)$, the velocity vector $\mathbf{v}(\mathbf{x}, t) = (v^1(\mathbf{x}, t), v^2(\mathbf{x}, t), v^3(\mathbf{x}, t))$ and the pressure $p(\mathbf{x}, t)$ are unknowns. The viscosity coefficient μ is assumed to be nonnegative. In these equations, the pressure p is automatically determined (up to a function of t) by ρ and $\mathbf{v}$, namely, by solving the equation

$$\operatorname{div}(\rho^{-1}\nabla p) = \operatorname{div}(\mu\rho^{-1}\,\Delta\mathbf{v} + \mathbf{f} - (\mathbf{v} \cdot \nabla)\mathbf{v}). \tag{1.2}$$

Thus we mention $(\rho, \mathbf{v})$ when we talk about the solution of $(1.1\!:\!\mu)$.

The purpose of this paper is to establish the uniform convergence of the solution of $(1.1\!:\!\mu)$ with $\mu > 0$ to the solution of $(1.1\!:\!0)$ as $\mu \to 0$. We wish to prove

THEOREM. *Assume that*

$$\rho_0(\mathbf{x}) - \bar{\rho} \in H^3(\mathbb{R}^3) \quad \textit{for some positive constant } \bar{\rho}, \tag{1.3}$$

$$\inf \rho_0(\mathbf{x}) \equiv m > 0 \quad \textit{and} \quad \sup \rho_0(\mathbf{x}) \equiv M < \infty, \tag{1.4}$$

$$\mathbf{v}_0(\mathbf{x}) \in H^3(\mathbb{R}^3) \quad \textit{and} \quad \operatorname{div} \mathbf{v}_0 = 0, \tag{1.5}$$

$$\mathbf{f}(\mathbf{x}, t) \in L^2(0, T : H^3(\mathbb{R}^3)) \tag{1.6}$$

and

$$\mu \leqq 1. \tag{1.7}$$

Then there exists $T^* \in (0, T]$ *independent of* μ *such that the problem* $(1.1\!:\!\mu)$ *has a unique solution* $(\rho, \mathbf{v})$ *which satisfies*

$$(\rho - \bar{\rho}, \mathbf{v}) \in L^\infty(0, T^* : H^3(\mathbb{R}^3)) \times L^\infty(0, T^* : H^3(\mathbb{R}^3)) \tag{1.8}$$

and

$$\mathbf{v}_\mathbf{x} \in L^2(0, T^* : H^3(\mathbb{R}^3)) \quad \textit{provided } \mu > 0. \tag{1.9}$$

Moreover, let $(\rho^0, \mathbf{v}^0)$ *be the solution of* $(1.1\!:\!0)$ *and* $(\rho^\mu, \mathbf{v}^\mu)$ *the solution of* $(1.1\!:\!\mu)$ *with* $\mu > 0$. *Then we have*

$$\sup_{0 \leqq t \leqq T^*} [\|(\rho^0 - \rho^\mu)(t)\|_2^2 + \|(\mathbf{v}^0 - \mathbf{v}^\mu)(t)\|_2^2] \to 0 \quad \textit{as} \quad \mu \to 0, \tag{1.10}$$

where $\|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{R}^3)}$.

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In the case that $\rho \equiv 1$, we refer to Ebin and Marsden [2] and Ladyzhenskaya [4].

2. Preliminaries. In this section we obtain an *a priori* estimate for solutions of $(1.1:\mu)$. Let $(\rho, \mathbf{v})$ be a sufficiently regular solution.

LEMMA 2.1. *If we put*

$$\tilde{\rho} = \rho - \bar{\rho} \tag{2.1}$$

and

$$\Psi(t) = \int_0^t [1 + \|\tilde{\rho}(s)\|_3^2 + \|\mathbf{v}(s)\|_3^2]^2\, ds, \tag{2.2}$$

then

$$\sup_{0 \leqq s \leqq t} \|\tilde{\rho}(s)\|_3^2 \leqq \|\tilde{\rho}_0\|_3^2 + \tilde{c}\Psi(t), \tag{2.3}$$

where $\tilde{\rho}_0 = \rho_0 - \bar{\rho}$ and $\tilde{c}$ is a positive constant depending only on imbedding theorems.

Proof. It follows from $(1.1:\mu)_1$ and $(1.1:\mu)_4$ that $\tilde{\rho}$ satisfies the equation

$$\begin{cases} \tilde{\rho}_t + \mathbf{v} \,.\, \nabla\tilde{\rho} = 0 \\ \tilde{\rho}\,|_{t=0} = \tilde{\rho}_0(\mathbf{x}). \end{cases} \tag{2.4}$$

Applying the operator $D^\alpha(=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3})$ to $(2.4)_1$, multiplying the result by $D^\alpha\tilde{\rho}$, integrating over $\mathbb{R}^3$ and adding in α with $|\alpha|(=\alpha_1+\alpha_2+\alpha_3) \leqq 3$, then we have

$$\frac{d}{dt}\|\tilde{\rho}(t)\|_3^2 \leqq \tilde{c}\|\mathbf{v}(t)\|_3\|\tilde{\rho}(t)\|_3^2. \tag{2.5}$$

Hence, by Young's inequality, it is easy to see that (2.3) holds.

LEMMA 2.2. *Put*

$$A = 1 + \|\tilde{\rho}_0\|_3^2 \tag{2.6}$$

and

$$B = \|\mathbf{v}_0\|_3^2 + \int_0^T \|\mathbf{f}(t)\|_3^2\, dt. \tag{2.7}$$

Then we have

$$\|\mathbf{v}(t)\|_3^2 + \int_0^t \|\mathbf{v}_t(s)\|_2^2\, ds + \mu\int_0^t \|\mathbf{v}_{\mathbf{x}}(s)\|_3^2\, ds \leqq \hat{c}[A^2B + A(A+B)\Psi(t) + (A+B)\Psi(t)^2 + \Psi(t)^3], \tag{2.8}$$

where $\hat{c}$ is a positive constant depending only on m, M and imbedding theorems.

Proof. We first note that

$$m \leqq \rho(\mathbf{x}, t) \leqq M, \tag{2.9}$$

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since we have the representation

$$\rho(\mathbf{x}, t) = \rho_0(\mathbf{y}(\tau, \mathbf{x}, t)\,|_{\tau=0}), \tag{2.10}$$

where $\mathbf{y}(\tau, \mathbf{x}, t)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\mathbf{y}}{d\tau} = \mathbf{v}(\mathbf{y}, \tau) \\ \mathbf{y}\,|_{\tau=t} = \mathbf{x}. \end{cases} \tag{2.11}$$

(i) We multiply $(1.1:\mu)_2$ by $\mathbf{v}$ and integrate over $\mathbb{R}^3$. Taking $(1.1:\mu)_1$, $(1.1:\mu)_3$ and (2.9) into account, we get

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{v}\|_0^2 + \mu\|D\mathbf{v}\|_0^2 \leqq M\|\mathbf{f}\|_0\|\mathbf{v}\|_0, \tag{2.12}$$

where we use the notation $D^k\mathbf{u} = \Sigma_{|\alpha|=k}D^\alpha\mathbf{u}$. Multiplying by $\mathbf{v}_t$ and integrating over $\mathbb{R}^3$ then gives

$$\begin{aligned} m\|\mathbf{v}_t\|_0^2 + \frac{\mu}{2}\frac{d}{dt}\|D\mathbf{v}\|_0^2 &\leqq M(\|\mathbf{v}\|_1\|D\mathbf{v}\|_1\|\mathbf{v}_t\|_0 + \|\mathbf{f}\|_0\|\mathbf{v}_t\|_0) \\ &\leqq c_1(\|\mathbf{v}\|_2^4 + \|\mathbf{f}\|_0^2) + \frac{m}{2}\|\mathbf{v}_t\|_0^2. \end{aligned} \tag{2.13}$$

Thus

$$m\|\mathbf{v}_t\|_0^2 + \mu\frac{d}{dt}\|D\mathbf{v}\|_0^2 \leqq c_2(\|\mathbf{v}\|_2^4 + \|\mathbf{f}\|_0^2). \tag{2.14}$$

Here and hereafter c_j are positive constants depending only on m, M and imbedding theorems.

(ii) Apply the operator D^α with $|\alpha| = 1$ on each side of $(1.1:\mu)_2$, multiply the result by $D^\alpha\mathbf{v}$ and integrate over $\mathbb{R}^3$. Then, similarly to (i), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}D\mathbf{v}\|_0^2 + \mu\|D^2\mathbf{v}\|_0^2 &\leqq c_3(\|D\rho\|_2\|\mathbf{v}_t\|_0\|D\mathbf{v}\|_0 + \|D\mathbf{v}\|_1^3 \\ &\quad + \|D\rho\|_2\|\mathbf{v}\|_2\|D\mathbf{v}\|_0^2 + \|D\rho\|_2\|\mathbf{f}\|_0\|D\mathbf{v}\|_0 + \|D\mathbf{f}\|_0\|D\mathbf{v}\|_0) \\ &\leqq c_4(\|D\rho\|_2^4 + \|\mathbf{v}\|_2^2 + \|\mathbf{v}\|_2^4 + \|\mathbf{f}\|_1^2) + \frac{m}{2}\|\mathbf{v}_t\|_0^2. \end{aligned} \tag{2.15}$$

If we multiply by $D^\alpha\mathbf{v}_t$ and integrate over $\mathbb{R}^3$, then we have

$$\begin{aligned} m\|D\mathbf{v}_t\|_0^2 + \frac{\mu}{2}\frac{d}{dt}\|D^2\mathbf{v}\|_0^2 &\leqq c_5(\|D\rho\|_2\|\mathbf{v}_t\|_0\|D\mathbf{v}_t\|_0 + \|\mathbf{v}\|_2\|D^2\mathbf{v}\|_0\|D\mathbf{v}_t\|_0 \\ &\quad + \|D\mathbf{v}\|_1^2\|D\mathbf{v}_t\|_0 + \|D\rho\|_2\|\mathbf{v}\|_2\|D\mathbf{v}\|_0\|D\mathbf{v}_t\|_0 + \|D\rho\|_2\|\mathbf{f}\|_0\|D\mathbf{v}_t\|_0 + \|D\mathbf{f}\|_0\|D\mathbf{v}_t\|_0) \\ &\leqq c_6(\|D\rho\|_2^2\|\mathbf{v}_t\|_0^2 + \|\mathbf{v}\|_2^4 + \|D\rho\|_2^2\|\mathbf{v}\|_2^4 + \|D\rho\|_2^2\|\mathbf{f}\|_0^2 + \|D\mathbf{f}\|_0^2) + \frac{m}{2}\|D\mathbf{v}_t\|_0^2. \end{aligned} \tag{2.16}$$

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Therefore

$$m\|D\mathbf{v}_t\|_0^2+\mu\frac{d}{dt}\|D^2\mathbf{v}\|_0^2\leqq c_7[\|D\rho\|_2^2(\|\mathbf{v}_t\|_0^2+\|\mathbf{v}\|_2^4+\|\mathbf{f}\|_0^2)+\|\mathbf{v}\|_2^4+\|\mathbf{f}\|_1^2]. \tag{2.17}$$

(iii) Adding (2.14) to (2.15), we get

$$m\|\mathbf{v}_t\|_0^2+2\mu\frac{d}{dt}\|D\mathbf{v}\|_0^2+\frac{d}{dt}\|\sqrt{\rho}D\mathbf{v}\|_0^2+2\mu\|D^2\mathbf{v}\|_0^2\leqq c_8(\|D\rho\|_2^4+\|\mathbf{v}\|_2^2+\|\mathbf{v}\|_2^4+\|\mathbf{f}\|_1^2). \tag{2.18}$$

Thus, noting that $\mu\leqq 1$, we get

$$\int_0^t\|\mathbf{v}_t\|_0^2\,ds+\mu\|D\mathbf{v}\|_0^2+\|D\mathbf{v}\|_0^2+\mu\int_0^t\|D^2\mathbf{v}\|_0^2\,ds\leqq c_9[B+\Psi(t)]. \tag{2.19}$$

(iv) Making use of the operator D^α with $|\boldsymbol{\alpha}|=2$ in place of the operator D^α with $|\boldsymbol{\alpha}|=1$ and repeating the argument in (ii), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}D^2\mathbf{v}\|_0^2+\mu\|D^3\mathbf{v}\|_0^2&\leqq c_{10}(\|D\rho\|_2\|D\mathbf{v}_t\|_0\|D^2\mathbf{v}\|_0\\&\quad+\|D^2\rho\|_1\|\mathbf{v}_t\|_1\|D^2\mathbf{v}\|_0+\|D\rho\|_2\|\mathbf{v}\|_2^2\|D^2\mathbf{v}\|_0+\|D\rho\|_2\|\mathbf{v}\|_2\|D^2\mathbf{v}\|_0^2\\&\quad+\|\mathbf{v}\|_3\|D^2\mathbf{v}\|_0^2+\|D\rho\|_2\|\mathbf{f}\|_1\|D^2\mathbf{v}\|_0+\|\mathbf{f}\|_2\|D^2\mathbf{v}\|_0)\\&\leqq c_{11}(\|D\rho\|_2^4+\|\mathbf{v}\|_3^4+\|\mathbf{v}\|_2^2+\|\mathbf{f}\|_2^2+\|\mathbf{v}_t\|_0^2)+\frac{m}{2}\|D\mathbf{v}_t\|_0^2\end{aligned} \tag{2.20}$$

and

$$m\|D^2\mathbf{v}_t\|_0^2+\mu\frac{d}{dt}\|D^2\mathbf{v}\|_0^2\leqq c_{12}[\|D\rho\|_2^2(\|D\mathbf{v}_t\|_0^2+\|\mathbf{v}\|_3^4+\|\mathbf{f}\|_1^2)+\|\mathbf{v}\|_3^4+\|\mathbf{f}\|_2^2]. \tag{2.21}$$

(v) If we add (2.17) to (2.20), then we obtain

$$\begin{aligned}m\|D\mathbf{v}_t\|_0^2+2\mu\frac{d}{dt}\|D^2\mathbf{v}\|_0^2+\frac{d}{dt}\|\sqrt{\rho}D^2\mathbf{v}\|_0^2+2\mu\|D^3\mathbf{v}\|_0^2\\\leqq c_{13}[\|D\rho\|_2^2(\|\mathbf{v}_t\|_0^2+\|\mathbf{v}\|_2^4+\|\mathbf{f}\|_0^2)\\+\|D\rho\|_2^4+\|\mathbf{v}\|_3^4+\|\mathbf{v}\|_2^2+\|\mathbf{f}\|_2^2+\|\mathbf{v}_t\|_0^2].\end{aligned} \tag{2.22}$$

Hence, due to (2.3) and (2.19),

$$\begin{aligned}\int_0^t\|D\mathbf{v}_t\|_0^2\,ds+\mu\|D^2\mathbf{v}\|_0^2+\|D^2\mathbf{v}\|_0^2+\mu\int_0^t\|D^3\mathbf{v}\|_0^2\,ds\\\leqq c_{14}[AB+(A+B)\Psi(t)+\Psi(t)^2].\end{aligned} \tag{2.23}$$

(vi) Applying the operator D^α with $|\boldsymbol{\alpha}|=3$ to $(1.1:\mu)_2$, multiplying by $D^\alpha\mathbf{v}$ and integrating over $\mathbb{R}^3$, then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}D^3\mathbf{v}\|_0^2+\mu\|D^4\mathbf{v}\|_0^2\\\leqq c_{16}(\|D\rho\|_2^4+\|\mathbf{v}\|_3^4+\|\mathbf{v}\|_3^2+\|\mathbf{f}\|_3^2+\|\mathbf{v}_t\|_0^2+\|D\mathbf{v}_t\|_0^2)+\frac{m}{2}\|D^2\mathbf{v}_t\|_0^2.\end{aligned} \tag{2.24}$$

(vii) Add (2.21) to (2.24). Then, due to (2.3), (2.19) and (2.23), we get

$$\int_0^t \|D^2\mathbf{v}_t\|_0^2\,ds + \mu\|D^3\mathbf{v}\|_0^2 + \|D^3\mathbf{v}\|_0^2 + \mu\int_0^t \|D^4\mathbf{v}\|_0^2\,ds$$
$$\leqq c_{17}[A^2B + A(A+B)\Psi(t) + (A+B)\Psi(t)^2 + \Psi(t)^3]. \tag{2.25}$$

Consequently, it follows from (2.12), (2.19), (2.23) and (2.25) that (2.8) holds.

LEMMA 2.3. *There exists* $T^* = T^*(\bar{c}, \hat{c}, A, B) \in (0, T]$ *such that*

$$\Psi(t) \leqq 1 \quad \textit{for} \quad t \leqq T^*. \tag{2.26}$$

Proof. From Lemma 2.1 and Lemma 2.2, we have a differential inequality

$$\frac{d}{dt}y(t) \leqq Ly(t)^6, \tag{2.27}$$

where $y(t) = 1 + \Psi(t)$ and $L = [\bar{c}\hat{c}A^2(1+B)]^2$. We conclude that

$$y(t) \leqq (1 - 5Lt)^{-1/5} \text{ provided } t < (5L)^{-1}, \tag{2.28}$$

and thus

$$y(t) \leqq 2 \quad \text{for} \quad t \leqq T^* = 31/160L. \tag{2.29}$$

Because of the above lemmas, the following is easily proved.

PROPOSITION 2.4. *There exists a positive constant* $c = c(\bar{c}, \hat{c}, A, B)$ *such that*

$$\sup_{0 \leqq t \leqq T^*} [\|\tilde{\rho}(t)\|_3^2 + \|\mathbf{v}(t)\|_3^2] + \int_0^{T^*} \|\mathbf{v}_t\|_2^2\,dt + \mu\int_0^{T^*} \|\mathbf{v}_\mathbf{x}\|_3^2\,dt \leqq c. \tag{2.30}$$

3. Proof of Theorem. We first prove the unique solvability of $(1.1:\mu)$. We apply the semi Galerkin method with the basis in $H^4(\mathbb{R}^3) \cap J$ provided $\mu = 0$ and $H^5(\mathbb{R}^3) \cap J$ provided $\mu > 0$, where $J = \{\mathbf{u} \in \{C_0^\infty(\mathbb{R}^3)\}^3 \colon \operatorname{div} \mathbf{u} = 0\}$. Our approach is completely parallel with that of [**1**, Chapter 3] without any specific difficulty. To be brief, estimates of the type (2.9) and (2.30) are true for the semi Galerkin approximations and these are sufficient in order to pass to the limit. Hence we can verify the existence of a unique solution of the problem $(1.1:\mu)$ as well as the applicability of the inequalities (2.9) and (2.30) to it. For the detail we refer to [**1**].

Next we prove (1.10), which is the main result in this paper. If we subtract $(1.1:\mu)$ with $\mu > 0$ from $(1.1:0)$, then we get the following linear system for $\tau = \rho^0 - \rho^\mu$, $\mathbf{w} = \mathbf{v}^0 - \mathbf{v}^\mu$ and $q = p^0 - p^\mu$:

$$\begin{cases} \tau_t + \mathbf{v}^0 \cdot \nabla\tau = -\mathbf{w} \cdot \nabla\rho^\mu, \\ \rho^\mu[\mathbf{w}_t + (\mathbf{v}^\mu \cdot \nabla)\mathbf{w}] + \nabla q = -\rho^\mu(\mathbf{w} \cdot \nabla)\mathbf{v}^0 + (\nabla p^0/\rho^0)\tau - \mu\Delta\mathbf{v}^\mu \equiv F, \\ \operatorname{div} \mathbf{w} = 0, \\ \tau\,|_{t=0} = 0, \\ \mathbf{w}\,|_{t=0} = \mathbf{0}. \end{cases} \tag{3.1}$$

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From this, by proceeding in the same way used for getting *a priori* estimates, we have

$$\|\tau(t)\|_2^2 \leqq K_1 \int_0^t \|\mathbf{w}(s)\|_2^2\, ds \tag{3.2}$$

and

$$\|\mathbf{w}(t)\|_2^2 \leqq K_2 \int_0^t \|F(s)\|_2^2\, ds, \tag{3.3}$$

where K_1 and K_2 are positive constants depending only on $\sup_{0\leqq t\leqq T^*} \|\tilde{\rho}^\mu(t)\|_3^2$, $\sup_{0\leqq t\leqq T^*} \|\mathbf{v}^\mu(t)\|_3^2$, T^*, m, M and imbedding theorems.

Let us estimate for the right hand side of (3.3). To begin with, by the usual calculation, we get

$$\begin{aligned} \|(\rho^\mu(\mathbf{w}\,.\,\nabla)\mathbf{v}^0)(t)\|_2^2 &\leqq \|(\rho^\mu(\mathbf{w}\,.\,\nabla)\mathbf{v}^0)(t)\|_0^2 + \|D(\rho^\mu(\mathbf{w}\,.\,\nabla)\mathbf{v}^0)(t)\|_0^2 \\ &+ \|D^2(\rho^\mu(\mathbf{w}\,.\,\nabla)\mathbf{v}^0)(t)\|_0^2 \leqq K_3(M + \|\tilde{\rho}(t)\|_3)^2 \|\mathbf{v}^0(t)\|_3^2 \|\mathbf{w}(t)\|_2^2, \end{aligned} \tag{3.4}$$

where K_3 is the constant of the theorems of imbedding.

Next, from $(1.1:0)_2$ and (3.2), we obtain

$$\begin{aligned} \|((\nabla p^0/\rho^0)\tau)(t)\|_2^2 &\leqq K_3 \|(\nabla p^0/\rho^0)(t)\|_2^2 \|\tau(t)\|_2^2 \\ &\leqq K_2 K_3 (\|f(t)\|_2^2 + \|\mathbf{v}_t^0(t)\|_2^2 + \|\mathbf{v}^0(t)\|_2^2 \|D\mathbf{v}^0(t)\|_2^2) \int_0^t \|\mathbf{w}(s)\|_2^2\, ds, \end{aligned} \tag{3.5}$$

and thus it follows from Proposition 2.4 that

$$\int_0^t \|F(s)\|_2^2\, ds \leqq K_4 \left(\mu + \int_0^t \|\mathbf{w}(s)\|_2^2\, ds\right), \tag{3.6}$$

where $K_4 = K_4(K_2, K_3, T^*, M, c)$.

Hence, if we put $K = K_1 K_4$, then

$$\|\mathbf{w}(t)\|_2^2 \leqq K \left(\mu + \int_0^t \|\mathbf{w}(s)\|_2^2\, ds\right), \tag{3.7}$$

and, by Gronwall's inequality,

$$\|\mathbf{w}(t)\|_2^2 \leqq K\mu(K \exp(KT^*) - 1). \tag{3.8}$$

Now, because of Lemma 2.3 and Proposition 2.4, we find that K and T^* are independent of μ, which completes the proof of the theorem.

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