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On an infinite integral linear group

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Let A be a free abelian group of countably infinite rank and Γ the automorphism group of A. This group Γ is found to differ radically from the integral linear group of finite dimension in the "density" of the set of normal subgroups it contains.

The congruence subgroups $\Gamma(m)$ consisting of all automorphisms γ such that $a\gamma - a \in mA$ for all $a \in A$ are, of course, normal in Γ for each positive integer m. Also, the finitary automorphisms, that is, those which act non-trivially only on some direct summand of A of finite rank, form a normal subgroup Φ of Γ. Every normal subgroup of Γ intersects Φ non-trivially, but the intersections $\Phi \cap \Gamma(m)$ are, essentially, the only normal subgroups of Φ.

Other types of normal subgroups of Γ arise from infinite descending chains of subgroups of A. Let $f : N \to N$ be a function defined on the set of positive integers such that $f(i)$ properly divides $f(i+1)$ for all i and let F be the set of all such functions. We define a subset $\Sigma(f)$ of Γ as follows: σ is an element of $\Sigma(f)$ if and only if there exists a descending chain of subgroups of A, namely, $A = H_1 \geq H_2 \geq \ldots \geq H_i \geq \ldots$ such that for all i the rank $r(i)$ of H_i/H_{i+1} is finite and $h_i\sigma - h_i \in f(i)A$ for $h_i \in H_i$. $\Sigma(f)$ is a normal subgroup of Γ and there are uncountably many of these.

Moreover, each $\Sigma(f)$ contains many normal subgroup chains of the order type of the positive reals obtained as follows. Let $f^* : N \to R$ take non-negative real values. An element σ of $\Sigma(f)$ belongs to $\Sigma(f, f^*)$ if and only if there is a positive real constant $c(\sigma)$ such

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that $r(i) \leq c(\sigma)f^*(i)$ for all i . Then $\Sigma(f, f^*)$ is a normal subgroup of Γ , and if f^*, g^* and the quotient f^*/g^* are monotonically increasing and unbounded, then $\Sigma(f, f^*)$ properly contains $\Sigma(f, g^*)$. Now take for example for each real $\alpha > 0$ the functions given by $f^*_\alpha(i) = i^\alpha$ to confirm the claim made above.

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