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Vanishing Theorems in Colombeau Algebras of Generalized Functions

V. Valmorin

Abstract. Using a canonical linear embedding of the algebra $\mathcal{G}^\infty(\Omega)$ of Colombeau generalized functions in the space of $\overline{\mathbb{C}}$-valued $\mathbb{C}$-linear maps on the space $\mathcal{D}(\Omega)$ of smooth functions with compact support, we give vanishing conditions for functions and linear integral operators of class $\mathcal{G}^\infty$. These results are then applied to the zeros of holomorphic generalized functions in dimension greater than one.

1 Introduction

Differential algebras of generalized functions display a few differences from the familiar case of C^∞ or holomorphic functions: the fact that a Colombeau generalized function may vanish at every classical point without being null is a well-known structural property of Colombeau algebras, *i.e.,* these generalized functions are not a pointwise concept. Thereby, mathematicians working in this field have been naturally led to seek characteristic conditions for nullity in such algebras. A characterization is given by Oberguggenberger and Kunzinger [11], giving at the same time a positive answer to [10, Problem 27.4] by introducing the new concept of a compactly supported point.

Thus, the recent result by Khelif and Scarpalezos [9] stating that a holomorphic generalized function which vanishes at all classical points of an open set of $\mathbb{C}$ is the zero function has been surprising enough. Other results involve the geometric nature of the set of zeros to conclude the nullity of holomorphic generalized functions. It has been shown [12] that holomorphic generalized functions have global holomorphic representatives, whereas in [3, 5] only local existence of such representatives was obtained. These results and their proofs show the difference between classical holomorphic functions and generalized holomorphic functions (also a holomorphic generalized function may vanish with all its derivatives at a point without being null).

We notice that holomorphic, as well as real analytic, generalized functions are elements of class $\mathcal{G}^\infty$ [10, p. 274]. With different techniques from those in this paper, a canonical embedding of $\mathcal{G}^\infty(\Omega)$ in the space of $\overline{\mathbb{C}}$-valued $\mathbb{C}$-linear maps on the space $\mathcal{D}(\Omega)$ of smooth functions with compact support is given in [13]. This result may be seen as a vanishing one concerning generalized functions in $\mathcal{G}^\infty$.

The main purpose of this paper is to give vanishing theorems related to this canonical embedding, and then vanishing theorems in the frame of $\mathcal{G}^\infty$ classes, covering the

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above mentioned topics. The second section of this paper is devoted to results on injectivity of linear maps leading to vanishing conditions. In the third section we deal with $\mathcal{G}^\infty$ kernels and vanishing theorems. Results of [9] on zeros of holomorphic generalized functions are extended to higher dimensions in the last section.

2 Basic Definitions and Notations

Let Ω be an open set in $\mathbb{R}^d$ and $\mathcal{E}(\Omega)$ the space of smooth functions on Ω with its usual topology. We note $K \Subset \Omega$ to mean that K is a compact set in Ω. Then the set $\mathcal{E}_M(\Omega)$ (resp. $\mathcal{N}(\Omega)$) of moderate sequences (resp. null sequences) consists of sequences $(f_n)_n \in \mathcal{E}(\Omega)^{\mathbb{N}}$ with the properties

$$\forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^d,\ \exists r \in \mathbb{R},\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le C n^r, \quad n \ge 1$$

$$(\ \forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^d,\ \forall q \in \mathbb{R},\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le C n^{-q}, \quad n \ge 1)$$

respectively. These spaces are both algebras and moreover $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_M(\Omega)$. The simplified Colombeau algebra $\mathcal{G}(\Omega)$ is defined as the quotient

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

(see [8, p. 10]). If sequences $(f_n)_n$ consist of constant functions on Ω, then one obtains the corresponding algebras $\mathcal{E}_M$ and $\mathcal{N}_0$. The Colombeau algebra of generalized complex numbers is defined as $\overline{\mathbb{C}} = \mathcal{E}_M/\mathcal{N}_0$. We notice that $\overline{\mathbb{C}}$ is a ring but not a field. The subset of $\mathcal{G}(\Omega)$ consisting of elements for which any representative $(f_n)_n$ satisfies

$$\forall K \Subset \Omega,\ \exists r \in \mathbb{R},\ \forall \alpha \in \mathbb{N}^d,\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le C n^r, \quad n \ge 1,$$

is a subalgebra of $\mathcal{G}(\Omega)$ denoted by $\mathcal{G}^\infty(\Omega)$ (see [10, p. 274]. It is seen that $\mathcal{G}(\Omega)$ and $\mathcal{G}^\infty(\Omega)$ are sheaves over $\mathbb{R}^d$. The embedding of the Schwartz distribution space $\mathcal{E}'(\Omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\Omega) \ni f \mapsto \mathrm{cl}(f * \phi_n|_\Omega) \in \mathcal{G}(\Omega)$, (cl standing for the class modulo $\mathcal{N}(\Omega)$) where a fixed sequence $(\phi_n)_n$ is defined on $\mathbb{R}^d$ by $\phi_n(x) = n^d\phi(nx),\ n \ge 1$, where ϕ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and satisfies

$$\int \phi(x)\,dx = 1,\ \int x^\alpha \phi(x)\,dx = 0, \alpha \in \mathbb{N}^d, |\alpha| \ne 0.$$

We use the notations $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. This sheaf homomorphism is extended as an embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$.

The integral of $f \in \mathcal{G}(\Omega)$ over $L \Subset \Omega$ is defined as the generalized complex number $\mathrm{cl}(\int_L f_n(x)\,dx)$ and does not depend on the chosen representative $(f_n)_n$. If f has compact support, one defines the integral $\int_\Omega f$ as $\int_L f$ where L is an arbitrary compact set in Ω which contains supp f in its interior.

3 Embeddings and Vanishing Theorems

The following classical result (Poincaré lemma) will be used throughout the paper.

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Lemma 3.1 *Let Ω be an open set in $\mathbb{R}^d$ and $\varphi \in \mathcal{D}_L(\Omega)$. Then*

$$\|\varphi\|_{L^\infty(\Omega)} \leq \sqrt{\lambda(L)}\|D\varphi\|_{L^2(\Omega)}$$

where $D = (\partial/\partial x_1)\cdots(\partial/\partial x_d)$, λ denoting the Lebesgue measure.

Let $\mathcal{G}_c(\Omega)$ denote the algebra of generalized functions $\in \mathcal{G}(\Omega)$ that have compact support. Let Ω denote an open set of $\mathbb{R}^d$. We consider the $\overline{\mathbb{C}}$-linear maps

$$\Lambda\colon \mathcal{G}(\Omega) \to \mathcal{L}(\mathcal{G}_c(\Omega); \overline{\mathbb{C}}) \quad \text{and} \quad \Lambda_f\colon \mathcal{G}_c(\Omega) \longrightarrow \overline{\mathbb{C}}$$
$$f \longmapsto \Lambda_f \qquad\qquad u \longmapsto \int_\Omega fu.$$

The following result is proved in [7] from a study of invertible elements in $\widetilde{\mathbb{R}}_c$. Here we give a direct proof.

Proposition 3.2 *The linear map Λ is injective.*

Proof Let $f \in \mathcal{G}(\Omega)$ be such that $\Lambda_f = 0$. Let $(f_n)_n$ denote a representative of f and let L be a compact set in Ω. Choose a positive function $\phi \in \mathcal{D}(\Omega)$ such that $\phi|U = 1$ on a bounded open neighborhood U of L and set $u_n = D(\phi\overline{Df_n})$ where $D = (\partial/\partial x_1)\cdots(\partial/\partial x_d)$. We have $u = \mathrm{cl}(u_n) \in \mathcal{G}_c(\Omega)$ and consequently $\Lambda_f(u) = 0$. Using integration by parts we find $\int_\Omega f_n(x)u_n(x)\,dx = (-1)^d\int_\Omega \phi(x)|Df_n(x)|^2\,dx$. From Lemma 3.1, it then follows that

$$\|f_n\|_{L^\infty(L)} \leq \sqrt{\lambda(L)}\|Df_n\|_{L^2(U)} \leq \sqrt{\lambda(L)}\|\sqrt{\phi}Df_n\|_{L^2(\Omega)}.$$

Consequently, for all $q > 0$, $\|f_n\|_{L^\infty(K)} = O(n^{-q})$ as $n \to \infty$, and then, $f = 0$ (see [8, Theorem 1.2.3]). ∎

It is well known [4, p. 60] that $\Lambda\colon \mathcal{G}(\Omega) \to \mathcal{L}\left(\mathcal{D}(\Omega); \overline{\mathbb{C}}\right)$ is not injective. Nevertheless, the following theorem was obtained with a different proof in [13].

Theorem 3.3 *The restriction of Λ to $\mathcal{G}^\infty(\Omega)$ is an injective linear map from $\mathcal{G}^\infty(\Omega)$ to $\mathcal{L}(\mathcal{D}(\Omega); \overline{\mathbb{C}})$.*

Sketch of the proof Let $f \in \mathcal{G}^\infty(\Omega)$ be such that $\Lambda_f = 0$, that is, $\Lambda_f(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ and take $(f_n)_n \in \mathcal{E}_M^\infty(\Omega)$ a representative of f. Let K_0 denote a compact subset of Ω, $\kappa \in \mathcal{D}(\Omega)$ such that $\kappa|K_0 = 1$. There exist a number s such that for all m, $p_{K,m+d}(\kappa f_n) \leq n^s$ for all $n > n_0$ for some $n_0 \in \mathbb{N}$ large enough. Next, fix a positive number q and set $D = (\partial/\partial x_1)\cdots(\partial/\partial x_d)$.

Let $S_n, n \in \mathbb{N}$ defined on $\mathcal{E}(\Omega)$ by $S_n(\varphi) = \int_\Omega D(\kappa f_n)(x)\varphi(x)\,dx$ and let $\mathcal{B}_k = \{n^k S_n; n \in \mathbb{N}\}$, $k > 0$. Using the Banach–Steinhauss theorem leads to the equicontinuity of the $\mathcal{B}_k$ which is equivalent to their boundedness on a neighborhood of zero in $\mathcal{E}(\Omega)$. One may choose a neighborhood U of the form $U = \{\varphi \in \mathcal{E}(\Omega) : p_{L;m}(\varphi) \leq \rho\}$, where L is a compact set containing K and such that

$$|T(\varphi)| \leq 1, \quad T \in \mathcal{B}_k, \varphi \in U.$$

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At this stage one sets

$$\varphi_n = \frac{\rho\overline{D(\kappa f_n)}}{n^{-1} + p_{K;m+d}(\kappa f_n)}.$$

It is shown that $\varphi_n \in U$. Then taking $k = 2q + s$ and using $p_{K;m+d}(\kappa f_n) \le n^s$ for $n > n_0$, a constant C is found such that

$$|n^{2q+s} S_n(\varphi_n)| = \rho n^{2q+s} \int_\Omega \frac{|D(\kappa f_n)(x)|^2}{n^{-1} + p_{K;m+d}(\kappa f_n)}\, dx$$

which leads to

$$|n^{2q+s} S_n(\varphi_n)| \ge C^{-1} n^{2q} \int_\Omega |D(\kappa f_n)(x)|^2\, dx, \quad n > n_0.$$

Now from Lemma 3.1, it follows that

$$\|f_n\|_{L^\infty}(K_0) \le \sqrt{\lambda(K)}\|D(\kappa f_n)\|_{L^2(\Omega)} \le \sqrt{C\lambda(K)}n^{-q}, \quad n > n_0.$$

This proves the theorem. ■

In the sequel, functions with compact support are assumed to be trivially extended to Ω or $\mathbb{R}^d$ if needed.

Corollary 3.4 *Let $f \in \mathcal{G}^\infty(\mathbb{R}^d)$ and Ω denote an open set of $\mathbb{R}^d$. Then we have*

(i) *If $f * \check{\varphi} = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then $f = 0$ in Ω.*
(ii) *If Ω is a convex cone, then $f = 0$ in Ω implies that $f * \check{\varphi} = 0$ for all $\varphi \in \mathcal{D}(\Omega)$.*
(iii) *If Ω is a symmetrical convex cone, then $f = 0$ in Ω if and only if $f * \varphi = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. In particular, $f = 0$ if and only if $f * \varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.*

Proof (i) If $f * \varphi = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then *a fortiori* $(f * \varphi)(0) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. This means that $\int_\Omega f\varphi = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. By Theorem 3.3, we have $f = 0$ in Ω.

(ii) Conversely, assume that Ω is convex and $f = 0$ in Ω. Let $\varphi \in \mathcal{D}(\Omega)$ and $K \Subset \Omega$. We have $\operatorname{supp}\check{\varphi} = -\operatorname{supp}\varphi$. If $x \in K$ and $y \in (-\operatorname{supp}\varphi)$, then $x - y \in K + \operatorname{supp}\varphi$. Since Ω is a convex cone, it follows that $K + \operatorname{supp}\varphi \subset \Omega$. Let Ω_1 denote a bounded open neighborhood of $K + \operatorname{supp}\varphi$ in Ω. Since $f = 0$ in Ω, we have $f = 0$ in Ω_1 which is a neighborhood of K. Hence $f * \varphi = 0$ in Ω.

(iii) Since Ω is symmetrical, it suffices to note that $\varphi \mapsto \check{\varphi}$ is an isomorphism from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\Omega)$ and apply Theorem 3.3. ■

As in [7, Theorem 1.1], for $\mathcal{L}t(\mathcal{G}_c(\Omega)\,;\overline{\mathbb{C}})$, it is easily seen that $\mathcal{L}(\mathcal{D}(\Omega)\,;\overline{\mathbb{C}})$ is a sheaf over $\mathbb{R}^d$, which allows us to give the following definition.

Definition 3.1 Let $T \in \mathcal{L}(\mathcal{D}(\Omega)\,;\overline{\mathbb{C}})$. Then the support of T, denoted $\operatorname{supp} T$, is defined by

$$\Omega \setminus \operatorname{supp} T := \{x \in \Omega\,;\exists V_x \text{ open neighborhood of } x : \forall\varphi \in \mathcal{D}(V_x), T(\varphi) = 0\},$$

where $\varphi \in \mathcal{D}(V_x)$ is trivially extended to Ω.

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Proposition 3.5 *For all* $f \in \mathcal{G}^\infty(\Omega)$, $\operatorname{supp} f = \operatorname{supp} \Lambda_f$.

Proof Let $f \in \mathcal{G}^\infty(\Omega)$. Let $x \in \Omega \setminus \operatorname{supp} \Lambda_f$. There exists an open neighborhood V of x such that $\Lambda_f(\varphi) = 0$ for all $\varphi \in \mathcal{D}(V)$. This means that $\forall \varphi \in \mathcal{D}(V)$, $\int_V f\varphi = 0$. It follows that $f|V = 0$. Hence $x \in \Omega \setminus \operatorname{supp} f$ and then $\operatorname{supp} f \subset \operatorname{supp} \Lambda_f$.

Conversely, if $x \in \Omega \setminus \operatorname{supp} f$, there exists an open neighborhood U of x such that $f|U = 0$. Then we have $\int_U f\varphi = 0$ for all $\varphi \in \mathcal{D}(U)$, that is, $\Lambda_f(\varphi) = 0$ for all $\varphi \in \mathcal{D}(U)$; hence $x \in \operatorname{supp} f$ and $\operatorname{supp} \Lambda_f \subset \operatorname{supp} f$. ■

We now determine the image of the restriction of Λ to $\mathcal{G}^\infty(\Omega)$. The following proposition will be needed for this purpose.

Proposition 3.6 *Let E, F, and G denote vector spaces over a field* $\mathbb{K}$ *and* $\pi\colon E \to G$ *a surjective linear map. If* $T\colon F \to G$ *is a linear map, then there exists a linear map* $p\colon F \to E$ *such that* $T = \pi \circ p$ *and* $\ker p = \ker T$.

Proof Let E_1 be a supplementary subspace of $\ker \pi$ in E. Denote by q the projection on E_1 parallel to $\ker \pi$, r the canonical embedding of E_1 in E, and $\pi_1 = \pi|E_1$. It is easily seen that π_1 is bijective and $q \circ r = Id_{E_1}$ and $\pi = \pi_1 \circ q$. We set $p = r \circ \pi_1^{-1} \circ T$. The maps r, π^{-1} and T being linear, the same is true for p. We have

$$\pi \circ p = (\pi_1 \circ q) \circ (r \circ \pi_1^{-1} \circ T) = [\pi_1 \circ (q \circ r) \circ \pi_1^{-1}\pi \circ p] \circ T = T.$$

Since r is injective, $\ker p = T^{-1}(\pi_1(\ker r)) = T^{-1}(\{0_G\}) = \ker T$. ■

Let $\varphi \in \mathcal{G}^\infty(\Omega)$ and let $(\varphi_n)_n$ denote a representative of φ. We denote by Ψ_n the linear map defined on $\mathcal{D}(\Omega)$ by $\Psi_n(\psi) = \int_\Omega \varphi_n\psi$, $\psi \in \mathcal{D}(\Omega)$. It is easily seen that the Ψ_n's satisfy the following property:

$$\begin{aligned} &\forall K \Subset \Omega, \exists r \in \mathbb{R}, \forall \alpha \in \mathbb{N}^d, \exists C > 0 \text{ such that} \\ &\qquad |\Psi_n(\partial^\alpha \psi)| \le Cn^r \|\psi\|_{L^1(K)}, \quad \text{for all } \psi \in \mathcal{D}_K(\Omega), n \ge 1. \end{aligned} \tag{3.1}$$

Let T be a linear map from $\mathcal{D}(\Omega)$ to $\overline{\mathbb{C}}$. From Proposition 3.6, there exists a linear map $\Phi = (\Phi_n)_n$ associated with T, such that $\pi \circ \Phi = T$ and $\ker \Phi = \ker T$ where π denotes the canonical map from $\mathcal{C}_M$ to $\overline{\mathbb{C}}$. The map Φ will be called a *representative* of T.

Definition 3.2 A linear map $\Phi = (\Phi_n)_n \in \mathcal{L}(\mathcal{D}(\Omega, \mathbb{C}^{\mathbb{N}}))$ is said to be $\mathcal{G}^\infty$ *of type* L^1, if it satisfies (3.1). A linear map in $\mathcal{L}(\mathcal{D}(\Omega, \overline{\mathbb{C}}))$ is $\mathcal{G}^\infty$ of type L^1 if it admits a representative which is $\mathcal{G}^\infty$ of type L^1.

The subspace of $\mathcal{L}(\mathcal{D}(\Omega, \overline{\mathbb{C}}))$ of all linear maps $\mathcal{G}^\infty$ of type L^1 will be denoted by $\mathcal{L}_0^\infty(\mathcal{D}(\Omega, \overline{\mathbb{C}}))$. It follows straightforwardly from the definitions that $\Lambda(\mathcal{G}^\infty(\Omega)) \subset \mathcal{L}_0^\infty(\mathcal{D}(\Omega, \overline{\mathbb{C}}))$. Actually we have the following.

Theorem 3.7 *The restriction of* Λ *to* $\mathcal{G}^\infty(\Omega)$ *is a bijective linear map from* $\mathcal{G}^\infty(\Omega)$ *to* $\mathcal{L}_0^\infty\left(\mathcal{D}(\Omega)\,;\overline{\mathbb{C}}\right)$.

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Proof We already know from Theorem 3.3 that Λ is injective; it remains to show that it is surjective. Let $S \in \mathcal{L}_0^\infty(\mathcal{D}(\Omega, \overline{\mathbb{C}}))$ and let $\Psi = (\Psi_n)_n$ denote a representative of S which is $\mathcal{G}^\infty$ of type L^1. Using the inequality $\|\cdot\|_{L^1(K)} \le \operatorname{mes}(K)\|\cdot\|_{L^\infty(K)}$, it is seen from (3.1) that the Ψ_n and all their derivatives are measures, that is, distributions of order 0. Hence each Ψ_n, $n \ge 1$ is represented by a smooth function φ_n on Ω. It follows that $\Psi_n(\partial^\alpha \psi) = (-1)^{|\alpha|} \int_K \partial^\alpha \varphi_n \psi$ for every $\psi \in \mathcal{D}_K(\Omega)$. Set $\partial^\alpha \varphi_n = g_n$. We note that for any compact set N in Ω we have

$$\sup_{\substack{f \in L^1(N) \\ \|f\|_{L^1(N)} \le 1}} \Big| \int_K g_n f \Big| = \|g_n\|_{L^\infty(N)}. \tag{3.2}$$

Let $f \in L^1(K)$ be such that $\|f\|_{L^1(K)} \le 1$ and let $\varepsilon > 0$. Let Ω_1 denote a relatively compact open neighborhood of K in Ω and set $\overline{\Omega_1} = K_1$. Choose $\psi \in \mathcal{D}_{K_1}(\Omega)$ such that $\|f - \psi\|_{L^1(K)} \le \varepsilon$. Hence we have $\|\psi\|_{L^1(K)} \le \|f\|_{L^1(K)} + \varepsilon \le 1 + \varepsilon$. Let $\tilde{f}$ denote the trivial extension of f to K_1. Writing

$$\int_K g_n f = \int_{K_1} g_n \tilde{f} = \varepsilon \int_{K_1} g_n \Big(\frac{\tilde{f} - \psi}{\varepsilon} \Big) + \int_{K_1} g_n \psi$$

and using (3.2) and (3.1), we then find

$$\Big| \int_K g_n f \Big| \le \varepsilon \|g_n\|_{L^\infty(K_1)} + C(1+\varepsilon) n^r$$

for some constants r and C. Now if we let $\varepsilon \to 0$ and use again (3.2), we finally get

$$\forall K \text{ compact set } \subset \Omega, \exists r \in \mathbb{R}, \forall \alpha \in \mathbb{N}^d, \exists C > 0, \|\partial^\alpha \varphi_n\|_{L^\infty(K)} \le C n^r, \ n \ge 1.$$

This means that $(\varphi_n)_n \in \mathcal{E}_M^\infty(\Omega)$. It follows that $\varphi = \operatorname{cl}(\varphi_n) \in \mathcal{G}^\infty(\Omega)$ satisfies $S = \Lambda_\varphi$, thus proving the theorem ■

4 $\mathcal{G}^\infty$ Kernels and Vanishing Theorems

Let $K \in \mathcal{G}(Y \times X)$ where Y and X denote two open sets of $\mathbb{R}^p$ and $\mathbb{R}^m$. We define linear integral operators

$$\widetilde{K}\colon \mathcal{G}_c(Y) \longrightarrow \mathcal{G}(X) \quad \text{and} \quad {}^t\widetilde{K}\colon \mathcal{G}_c(X) \longrightarrow \mathcal{G}(Y)$$
$$u \longmapsto \widetilde{K} \cdot u \qquad\qquad v \longmapsto {}^t\widetilde{K} \cdot v,$$

where $(\widetilde{K} \cdot u) = \operatorname{cl}\big(\int_Y K_n(y, \cdot) u_n(y)\, dy\big)$ and $({}^t\widetilde{K} \cdot v) = \operatorname{cl}\big(\int_Y K_n(\cdot, x) v_n(x)\, dx\big)$; $(K_n)_n$, $(u_n)_n$, and $(v_n)_n$ being representatives of K, u, and v respectively. If $u \in \mathcal{G}_c(Y)$ and $v \in \mathcal{G}_c(X)$, we set

$$\Lambda_{\widetilde{K}\cdot u}\colon \mathcal{G}_c(X) \longrightarrow \overline{\mathbb{C}} \quad \text{and} \quad \Lambda_{{}^t\widetilde{K}\cdot v}\colon \mathcal{G}_c(Y) \longrightarrow \overline{\mathbb{C}}$$
$$v \longmapsto \int_X (\widetilde{K} \cdot u) v \qquad\qquad u \longmapsto \int_Y ({}^t\widetilde{K} \cdot v) u.$$

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It is easily seen that if $K \in \mathcal{G}^\infty(Y \times X)$, then $\widetilde{K} \cdot \mathcal{G}_c(Y) \subset \mathcal{G}^\infty(X)$ and ${}^t\widetilde{K} \cdot \mathcal{G}_c(X) \subset \mathcal{G}^\infty(Y)$.

We give a proof of [2, Theorem 21] without the compactness hypothesis on the support of K.

Theorem 4.1 *The linear maps $K \mapsto \widetilde{K}$ and $K \mapsto {}^t\widetilde{K}$ are injective.*

Proof It is clear that it suffices to prove the result for the first map. We set $D_y = (\partial/\partial y_1)\cdots(\partial/\partial y_p)$ and $D_x = (\partial/\partial x_1)\cdots(\partial/\partial x_m)$. Note that if $\widetilde{K} = 0$, it follows that $\widetilde{D_x K} \cdot u = D_x(\widetilde{K} \cdot u) = 0$ for every $u \in \mathcal{G}_c(Y)$. Let $(K_n)_n$ be a representative of K, M, and L compact subsets of Y and X respectively. We choose $V \subset Y$ and $U \subset X$ relatively compact open neighborhoods of M and L respectively, $\varphi \in \mathcal{D}(Y)$ and $\psi \in \mathcal{D}(X)$ positive functions such that $\varphi|M = 1$, $\operatorname{supp}\varphi \subset V$ and $\psi|L = 1$, $\operatorname{supp}\psi \subset U$. For each n there exists $x_n \in U$ such that

$$\int_Y |D_y D_x((\varphi \otimes \psi)K_n)(\,\cdot\,, x_n)|^2 = \sup_{x \in U} \int_Y |D_y D_x((\varphi \otimes \psi)K_n)(\,\cdot\,, x)|^2.$$

Set

$$u_n = D_y \overline{[D_y D_x((\varphi \otimes \psi)K_n)]}(\,\cdot\,, x_n) \text{ and } v_n = D_x[(\widetilde{\varphi \otimes \psi})K_n] \cdot u_n.$$

Using partial integrations, we find

$$v_n(x) = (-1)^p \int_Y D_y D_x((\varphi \otimes \psi)K_n)(\,\cdot\,, x)\overline{D_y D_x((\varphi \otimes \psi)K_n)}(\,\cdot\,, x_n),$$

which gives

$$|v_n(x_n)| = \int_Y |D_y D_x((\varphi \otimes \psi)K_n)(\,\cdot\,, x_n)|^2.$$

From the definition of x_n, we then have for every $x \in U$,

$$|v_n(x_n)| \geq \int_Y |D_y D_x((\varphi \otimes \psi)K_n)(\,\cdot\,, x)|^2.$$

Since $\operatorname{supp}\varphi \subset V$, integrating over U and using Fubini's theorem yields

$$\lambda(U)|v_n(x_n)| \geq \int_{Y \times U} |D_y D_x((\varphi \otimes \psi)K_n)|^2.$$

Taking into account $\varphi|M = 1$, $\psi|L = 1$, and positiveness, Lemma 3.1 gives

$$\lambda(V)\lambda^2(U)|v_n(x_n)| \geq \|(\varphi \otimes \psi)K_n\|^2_{L^\infty(V \times U)} \geq \|K_n\|^2_{L^\infty(M \times L)}.$$

Now it is easily seen that the above left-hand side is the general term of an element of $\mathcal{N}_0$. Because $(x_n)_n$ is compactly supported, $\psi\widetilde{K}.(\varphi u) = 0$ and $[(\widetilde{\varphi \otimes \psi})K] \cdot u = \psi\widetilde{K}\cdot(\varphi u)$. Since every compact set of $Y \times X$ has a finite covering consisting of compact sets of the form $M \times L$, it follows that $K = 0$. ∎

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We now consider the following $\overline{\mathbb{C}}-$linear map:

$$T_K \colon \mathcal{G}_c(Y \times X) \longrightarrow \overline{\mathbb{C}}$$
$$w \longmapsto \int_{Y \times X} Kw.$$

We note that $\forall u \in \mathcal{D}(Y), \forall v \in \mathcal{D}(X)$, $T_K(u \otimes v) = \Lambda_{\widetilde{K} \cdot u}(v) = \Lambda_{{}^t\widetilde{K} \cdot v}(u)$. Then we have the following.

Theorem 4.2 *Let $K \in \mathcal{G}^\infty(Y \times X)$. The following conditions are equivalent:*

(i) $K = 0$;
(ii) $T_K | \mathcal{D}(Y) \otimes \mathcal{D}(X) = 0$;
(iii) $\widetilde{K} | \mathcal{D}(Y) = 0$;
(iv) ${}^t\widetilde{K} | \mathcal{D}(X) = 0$.

Proof Clearly (i) implies (ii) and (iii) is equivalent to (iv). Assume that (ii) is satisfied, that is, $T_K(\varphi \otimes \psi) = 0$ for all $\varphi \otimes \psi \in \mathcal{D}(Y) \otimes \mathcal{D}(X)$. Then we have $\Lambda_{\widetilde{K} \cdot \varphi}(\psi) = 0$ for all $\psi \in \mathcal{D}(X)$. Since $\widetilde{K} \cdot \varphi \in \mathcal{G}^\infty(X)$, by Theorem 3.3, $\widetilde{K} \cdot \varphi = 0$ for all $\varphi \in \mathcal{D}(Y)$, proving (iii). Assume that (iii) is satisfied. Hence, $\Lambda_{\widetilde{K} \cdot \varphi}(v) = 0$ for all $v \in \mathcal{G}_c(X)$ and all $\varphi \in \mathcal{D}(Y)$. By Fubini's theorem, this means that $\Lambda_{{}^t\widetilde{K} \cdot v}(\varphi) = 0$ for all $\varphi \in \mathcal{D}(Y)$. Since ${}^t\widetilde{K} \cdot v \in \mathcal{G}^\infty(Y)$, Theorem 3.3 implies that ${}^t\widetilde{K} \cdot v = 0$ for all $v \in \mathcal{G}_c(X)$, that is, ${}^t\widetilde{K} = 0$. Now from Theorem 4.1, it follows that ${}^t\widetilde{K} = 0$ implies (i). ■

5 Zeros of Holomorphic Generalized Functions

We consider holomorphic generalized functions in an open set Ω of $\mathbb{C}^d$. A generalized function $f \in \mathcal{G}(\Omega)$ is said to be holomorphic if it satisfies the Cauchy–Riemann equation $\overline{\partial} f = 0$. The set of holomorphic generalized functions is a subalgebra of $\mathcal{G}(\Omega)$ denoted by $\mathcal{G}_H(\Omega)$. For a general account of this topic we refer to [1, 3, 5, 6, 12]. In [12], it was proved that any $f \in \mathcal{G}_H(\Omega)$ admits a representative $(f_n)_n$ such that the f_n's are holomorphic in Ω.

From the Cauchy formula, it is immediately seen that $\mathcal{G}_H(\Omega) \subset \mathcal{G}^\infty(\Omega)$. Contrary to the general situation for generalized functions, it is proved in [9] that if a holomorphic generalized function vanishes at every point of a connected open set of $\mathbb{C}$, then it must be the zero function in this open set. We extend this result to higher dimension. For the sake of simplicity we work in dimension $d = 2$.

Theorem 5.1 *Let Ω denote a connected open set in $\mathbb{C}^2$, Y a nonvoid open set in $\mathbb{C}$, and Γ a nonvoid open interval in $\mathbb{C}$ such that $Y \times \Gamma \subset \Omega$. If $F \in \mathcal{G}_H(\Omega)$ satisfies $F(\xi, \zeta) = 0$ for all $(\xi, \zeta) \in Y \times \Gamma$, then $F = 0$ in Ω.*

Proof Let X denote an open set in $\mathbb{C}$ such that $Y \times X \subset \Omega$ and $X \cap \Gamma \neq \varnothing$. Let $(F_n)_n$ be a holomorphic representative of F (see [12]). For every fixed $\zeta \in X$, $F_n(\,\cdot\,, \zeta) \in \mathcal{H}(Y)$, and the corresponding sequence is moderate because this is true for $(F_n)_n$. We denote by g_ζ its class in $\mathcal{G}_H(Y)$. If $\zeta \in X \cap \Gamma$, it follows from the hypothesis that $g_\zeta(\xi) = 0$ for every $\xi \in Y$. Using the result of [9], we get $g_\zeta = 0$ for every

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$\zeta \in X \cap \Gamma$. Consequently, for every $\varphi \in \mathcal{D}(Y)$, $(\Lambda_{\widetilde{F}}(\varphi))(\zeta) = 0$ for ζ in a nonvoid interval non reduced to a point. Since $\Lambda_{\widetilde{F}}(\varphi) \in \mathcal{G}_H(X)$, it follows, using a result of [9], that $\Lambda_{\widetilde{F}}(\varphi) = 0$. Hence $\Lambda_{\widetilde{F}} = 0$, and from Theorem 4.2, $F = 0$ in $Y \times X$. Since Ω is connected, using the analytic continuation property of holomorphic generalized functions [6, 12], we get $F = 0$ in Ω ■

A straightforward induction gives the following.

Corollary 5.2 *Let Ω denote a connected open set in $\mathbb{C}^d$, $d \geq 2$, Y a nonvoid open set in $\mathbb{C}$ and Γ_i, $1 \leq i \leq d-1$ nonvoid open intervals in $\mathbb{C}$ such that $Y \times \Gamma_1 \times \cdots \times \Gamma_{d-1} \subset \Omega$. If $F \in \mathcal{G}_H(\Omega)$ satisfies $F(\xi, \zeta_1, \ldots, \zeta_{d-1}) = 0$ for all $(\xi, \zeta_1, \ldots, \zeta_{d-1}) \in Y \times \Gamma_1 \times \cdots \times \Gamma_{d-1}$, then $F = 0$ in Ω.*

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Département Math-Info, Université des Antilles et de la Guyane, Campus de Fouillole: 97159 Pointe à Pitre Cedex, France
e-mail: vincent.valmorin@univ-ag.fr