A GENERALIZATION OF HERMITE'S INTERPOLATION FORMULA IN TWO VARIABLES

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1. Introduction

Spitzbart [1] has considered a generalization of Hermite's interpolation formula in one variable and has obtained a polynomial $p(x)$ of degree $n + \sum_{j=0}^{n} r_j$ in x which interpolates to the values of a function and its derivatives up to order r_j at x_j, $j = 0, 1, \cdots, n$. Ahlin [2] has considered a bivariate generalization of Hermite's interpolation formula. He has developed a bivariate osculatory interpolation polynomial which agrees with $f(x, y)$ and its partial and mixed partial derivatives up to a specified order at each of the nodes of a Cartesian grid. However, the above interpolation problem considered by Ahlin assumes that the values of partial and mixed partial derivatives of the same fixed order $k - 1$ are available at every point of the rectangular grid. It may also be observed that Ahlin's formula is essentially a Cartesian product of a special case of Spitzbart's formula in one variable.

In the present paper, we consider a bivariate generalization of Spitzbart's formula. We discuss the bivariate interpolation problem in which at any point (x_i, y_j), $i = 0, 1, \cdots, m$ and $j = 0, 1, \cdots, n$ of a Cartesian grid, the maximum order of the partial derivative with respect to x depends only on i and the maximum order of the partial derivative with respect to y depends only on j. In other words, we consider interpolation to the data

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} f(x_i, y_j) \qquad \begin{array}{l} i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n; \\ k = 0, 1, \cdots, r_i, \ l = 0, 1, \cdots, s_j. \end{array}$$

The resulting interpolation formula might also be applicable in a situation where only the function values might be available but no partial derivatives with respect to x along some $x = x_i$ or no partial derivatives with respect to y along some $y = y_j$. Ahlin's interpolation formula becomes a particular case of our formula when $r_i = s_j = k - 1$ (fixed) for all i and j.

2. The interpolation formula

We first state the generalized Hermite's interpolation formula considered by Spitzbart [1].

THEOREM 1. *Let there be given* x_i, r_i, $f_i^{(k)}$, $i = 0, 1, \cdots, m$; $k = 0, 1, \cdots, r_i$. *Let* $p_i(x)$ *and* $g_i(x)$ *be defined by*

$$(1)\qquad p_i(x) = (x-x_0)^{r_0+1}\cdots(x-x_{i-1})^{r_{i-1}+1}(x-x_{i+1})^{r_{i+1}+1}\cdots(x-x_m)^{r_m+1}$$

$$(2)\qquad g_i(x) = [p_i(x)]^{-1}.$$

Then the unique polynomial $H_M(x)$ *of degree* $M = m + \sum_{i=0}^{m} r_i$ *such that*

$$(3)\qquad H_M^{(k)}(x_i) = f_i^{(k)}, \qquad i = 0, 1, \cdots, m;\ k = 0, 1, \cdots, r_i,$$

(here, $H^{(k)}(x) = (d^k/dx^k)H(x)$*) is given by*

$$(4)\qquad H_M(x) = \sum_{i=0}^{m} \sum_{k=0}^{r_i} A_{ik}(x) f_i^{(k)}$$

where

$$(5)\qquad A_{ik}(x) = p_i(x)\frac{(x-x_i)^k}{k!}\sum_{t=0}^{r_i-k}\frac{1}{t!}\, g_i^{(t)}(x_i)(x-x_i)^t.$$

The fundamental polynomials $A_{ik}(x)$ *satisfy*

$$(6)\qquad A_{i_1k}^{(u)}(x_i) = \delta_{i_1i}\delta_{uk}, \qquad i_1, i = 0, 1, \cdots, m;\quad k = 0, 1, \cdots, r_{i_1},\ u = 0, 1, \cdots, r_i,$$

where δ_{ir} *is the Kronecker delta function.*

Our main result is the following bivariate generalization of Theorem 1.

THEOREM 2. *Let there be given a set of values*

$$f_{i,j}^{(k,l)}, \qquad i = 0, 1, \cdots, m,\ j = 0, 1, \cdots, n;\quad k = 0, 1, \cdots, r_i,\ l = 0, 1, \cdots, s_j.$$

Then the unique polynomial $H_{M,N}(x, y)$ *of degree* $M = m + \sum_{i=0}^{m} r_i$ *in* x *and of degree* $N = n + \sum_{j=0}^{n} s_j$ *in* y *such that*

$$(7)\qquad \frac{\partial^{k+l}}{\partial x^k \partial y^l} H_{M,N}(x_i, y_j) = f_{i,j}^{(k,l)}, \qquad i = 0, 1, \cdots, m,\ j = 0, 1, \cdots, n;\quad k = 0, 1, \cdots, r_i,\ l = 0, 1, \cdots, s_j$$

is given by

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$$(8) \qquad H_{M,N}(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{r_i} \sum_{l=0}^{s_j} A_{ik}(x) B_{jl}(y) f_{i,j}^{(k,l)}$$

where $A_{ik}(x)$ *are the same as in* (5), *and*

$$(9) \qquad B_{jl}(y) = q_j(y) \frac{(y-y_j)^l}{l!} \sum_{t=0}^{s_j-l} \frac{h_j^{(t)}(y_j)}{t!} (y-y_j)^t$$

where

$$(10) \quad q_j(y) = (y-y_0)^{s_0+1} \cdots (y-y_{j-1})^{s_{j-1}+1} (y-y_{j+1})^{s_{j+1}+1} \cdots (y-y_n)^{s_n+1}$$

and

$$(11) \qquad h_j(y) = [q_j(y)]^{-1}.$$

PROOF. We first observe that the total number of the given data $f_{i,j}^{(k,l)}$ is $\sum_{i=0}^{m} \sum_{j=0}^{n} (r_i + 1)(s_j + 1)$ which is equal to the number of (unknown) coefficients in a polynomial of maximum degree N in x and N in y.

Now, the polynomials $A_{ik}(x)$ in (8) are the same as the fundamental polynomials of Theorem 1 and satisfy (6), therefore, the polynomials $B_{jl}(y)$ which have been defined in an analogous manner satisfy

$$(12) \qquad B_{j_1 l}^{(v)}(y_j) = \delta_{j_1 j} \delta_{vl}, \qquad \begin{array}{l} j_1, j = 0, 1, \cdots, n; \\ l = 0, 1, \cdots, s_{j_1}, \ v = 0, 1, \cdots, s_j. \end{array}$$

We next verify that the polynomial $H_{M,N}(x, y)$ defined in (8) satisfies the interpolation conditions (7). Since

$$\frac{\partial^{u+v}}{\partial x^u \partial y^v} H_{M,N}(x_i, y_j) = \sum_{i_1=0}^{m} \sum_{j_1=0}^{n} \sum_{k=0}^{r_{i_1}} \sum_{l=0}^{s_{j_1}} A_{i_1 k}^{(u)}(x_i) B_{j_1 l}^{(v)}(y_j) f_{i_1,j_1}^{(k,l)}$$

using (6) and (12) it follows that

$$\begin{aligned} \frac{\partial^{u+v}}{\partial x^u \partial y^v} H_{M,N}(x_i, y_j) &= \sum_{i_1=0}^{m} \sum_{j_1=0}^{n} \sum_{k=0}^{r_{i_1}} \sum_{l=0}^{s_{j_1}} \delta_{i_1 i} \delta_{uk} \delta_{j_1 j} \delta_{vl} f_{i_1,j_1}^{(k,l)} \\ &= f_{i,j}^{(u,v)}, \qquad \begin{array}{l} i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n; \\ u = 0, 1, \cdots, r_i, \ v = 0, 1, \cdots, s_j. \end{array} \end{aligned}$$

For the uniqueness of $H_{M,N}(x, y)$, suppose there exists another polynomial $H_{M,N}^*(x, y)$ of maximum degree M in x and of maximum degree N in y which also satisfies the interpolation conditions (7). Then,

$$T(x, y) = H_{M,N}^*(x, y) - H_{M,N}(x, y)$$

is a polynomial of maximum degree M in x and of maximum degree N in y such that

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} T(x_i, y_j) = 0, \qquad i = 0, 1, \cdots, m,\ j = 0, 1, \cdots, n; \quad k = 0, 1, \cdots, r_i, l = 0, 1, \cdots, s_j. \tag{13}$$

Along any one of the mesh lines $y = y_j$, $T(x, y_j)$ is a polynomial of maximum degree M in x such that

$$T(x_i, y_j) = \frac{\partial}{\partial x} T(x_i, y_j) = \cdots = \frac{\partial^{r_i}}{\partial x^{r_i}} T(x_i, y_j) = 0.$$

Hence, $(x - x_i)^{r_i + 1}$ must be a factor of $T(x, y_j)$ for $i = 0, 1, \cdots, m$. Thus,

$$T(x, y_j) = K(x - x_0)^{r_0 + 1} \cdots (x - x_m)^{r_m + 1}$$

where K is a constant. Since the right side is a polynomial of degree $M + 1$ in x, and $T(x, y_j)$ is of maximum degree M in x, comparing the coefficient of x^{N+1} on either side, it follows that $K = 0$. Hence,

$$T(x, y_j) \equiv 0, \qquad j = 0, 1, \cdots, n. \tag{14}$$

By a similar reasoning we obtain

$$\frac{\partial^l}{\partial y^l} T(x, y_j) \equiv 0, \qquad l = 1, 2, \cdots, s_j;\ j = 0, 1, \cdots, n. \tag{15}$$

Now choose any arbitrary line $x = \xi$. Using (14) and (15) it follows after a similar argument that

$$T(\xi, y) \equiv 0. \tag{16}$$

Since the choice of ξ is arbitrary, we conclude that $T(x, y) \equiv 0$, proving the uniqueness of $H_{M,N}(x, y)$. This completes the proof of Theorem 2.

The Taylor two-point interpolation formula (Davis [3], page 37) is a particular case of Theorem 1.

COROLLARY 1. *The unique polynomial $H_{2n-1}(x)$ in x of degree $2n - 1$ which interpolates to the data $f_a^{(k)}, f_b^{(k)}$, $k = 0, 1, \cdots, n - 1$, is given by*

$$H_{2n-1}(x) = (x - b)^n \sum_{k=0}^{n-1} \frac{A_k}{k!}(x - a)^k + (x - a)^n \sum_{k=0}^{n-1} \frac{B_k}{k!}(x - b)^k \tag{17}$$

where

$$A_k = \left[\frac{d^k}{dx^k} \frac{f(x)}{(x - b)^n}\right]_{x=a} \tag{18}$$

and

$$B_k = \left[\frac{d^k}{dx^k} \frac{f(x)}{(x - a)^n}\right]_{x=b}. \tag{19}$$

We note that, from Theorem 1,

$$(20)\quad H_{2n-1}(x) = (x-b)^n \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} \sum_{t=0}^{n-1-k} \frac{\{D^t(x-b)^{-n}\}_{x=a}}{t!}(x-a)^t f_a^{(k)}$$
$$+ (x-a)^n \sum_{k=0}^{n-1} \frac{(x-b)^k}{k!} \sum_{t=0}^{n-1-k} \frac{\{D^t(x-a)^{-n}\}_{x=b}}{t!}(x-b)^t f_b^{(k)}$$

where $D \equiv d/dx$. Simplifying the summations in the two terms of the right side of (20) and with the help of Leibnitz theorem, we get

$$H_{2n-1}(x) = (x-b)^n \sum_{k=0}^{n-1} \frac{A_k}{k!}(x-a)^k + (x-a)^n \sum_{k=0}^{n-1} \frac{B_k}{k!}(x-b)^k,$$

where A_k and B_k are given by (18) and (19).

The following result is a particular case of our Theorem 2, and may be regarded as a two-dimensional generalization of the above two-point Taylor interpolation formula.

COROLLARY 2. *The unique polynomial $H_{2n-1,2n-1}(x,y)$ of degree $2n-1$ in x and of degree $2n-1$ in y which satisfies the interpolation conditions*:

$$(21)\quad \frac{\partial^{k+l}}{\partial x^k \partial y^l} H_{2n-1,2n-1}(x_i, y_j) = f_{i,j}^{(k,l)}, \quad i = 0,1,\ j = 0,1;\ k,l = 0,1,\cdots,n-1;$$

$x_0 = a,\ x_1 = b;\ y_0 = c,\ y_1 = d$, *is given by*

$$(22)\quad H_{2n-1,2n-1}(x,y) = (x-a)^n(y-c)^n \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} A_{kl} \frac{(x-b)^k}{k!}\frac{(y-d)^l}{l!}$$
$$+ (x-a)^n(y-d)^n \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} B_{kl} \frac{(x-b)^k}{k!}\frac{(y-c)^l}{l!}$$
$$+ (x-b)^n(y-c)^n \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} C_{kl} \frac{(x-a)^k}{k!}\frac{(y-d)^l}{l!}$$
$$+ (x-b)^n(y-d)^n \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} D_{kl} \frac{(x-a)^k}{k!}\frac{(y-c)^l}{l!},$$

where

$$(23)\quad A_{kl} = \left[\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x,y)}{(x-a)^n(y-c)^n}\right]_{x=b,y=d}$$

with similar expressions for B_{kl}, C_{kl}, and D_{kl}.

The above result can be established from Theorem 2 by proceeding the same way as in Corollary 1.

3. Illustration

Suppose that the values of a function $f(x, y)$ as well as the values of $\partial f/\partial x$ are known at the four corners $(\pm 1, \pm 1)$ of a square. Then the unique polynomial of degree 3 in x and of degree 1 in y which interpolates to these values is given by

$$\begin{aligned} H_{3,1}(x,y) = (1/8)[&(x-1)^2(x+2)(1-y)f(-1,-1) \\ &+ (x-1)^2(x+2)(y+1)f(-1,1) + (x+1)^2(2-x)(1-y)f(1,-1) \\ &+ (x+1)^2(2-x)(y+1)f(1,1) \\ &+ (x-1)^2(x+1)(1-y)f_x(-1,-1) \\ &+ (x-1)^2(x+1)(y+1)f_x(-1,1) \\ &+ (x+1)^2(x-1)(1-y)f_x(1,-1) + (x+1)^2(x-1)(y+1)f_x(1,1)] \end{aligned}$$

where $f_x = \partial f/\partial x$.

For the particular function $f(x, y) = 1/(16 + x^2 + y)$, the interpolation polynomial $H_{3,1}(x, y)$ becomes

$$\begin{aligned} H_{3,1}(x,y) = (1/8)[&(1/16)(x-1)^2(x+2)(1-y) + (1/18)(x-1)^2(x+2)(y+1) \\ &+ (1/16)(x+1)^2(2-x)(1-y) + (1/18)(x+1)^2(2-x)(1+y) \\ &+ (1/128)(x-1)^2(x+1)(1-y) \\ &+ (1/162)(x-1)^2(x+1)(y+1) \\ &- (1/128)(x+1)^2(x-1)(1-y) \\ &- (1/162)(x+1)^2(x-1)(y+1)]. \end{aligned}$$

Computing the values at the origin, we obtain $f(0,0) = 1/16$, while $H_{3,1}(0,0) = 2593/41472$. Thus, the approximation $f(0,0) \approx H_{3,1}(0,0)$ has an error of 1/41472 !

Next, consider the function $f(x, y) = 1/(16 + x^2 + \sqrt{y})$ over $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. The partial derivatives of f with respect to y do not exist along $y = 0$. However, from Theorem 2 we can obtain an osculatory interpolation polynomial interpolating to the values of f at $(0,0), (0,1), (1,0), (1,1)$, to the values of the partial derivatives with respect to x (only) at $(0,0)$ and $(1,0)$, and to the values of the partial and mixed partial derivatives up to *any* order at the points $(0,1)$ and $(1,1)$.

This example distinguishes our interpolation formula (Theorem 2) from that of Ahlin [2].

4. The error of interpolation

We next derive expressions for the error of the interpolation formula given in Theorem 2 for the following two classes of functions:

(i) $$\frac{\partial^{M+N+2}}{\partial x^{M+1}\partial y^{N+1}} f(x,y) \text{ is continuous,}$$

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(ii) $f(x, y)$ can be continued analytically as a single valued and regular function $f(z, w)$ of two complex variables in a certain cross-product region $D_z \times D_w$.

5. Error in terms of partial derivatives

The error of the interpolation formula of Theorem 1 is given (Davis [3], page 67) by

$$(24)\qquad \begin{aligned} R(x) &= f(x) - H_M(x) \\ &= (x-x_0)^{r_0+1}\cdots(x-x_m)^{r_m+1}\frac{f^{(M+1)}(\xi)}{(M+1)!} \end{aligned}$$

where $\min(x; x_0, \cdots, x_m) \leqq \xi \leqq \max(x; x_0, \cdots, x_m)$; that is,

$$f(x) = H_M(x) + R(x).$$

In the case of two variables, if we keep y fixed, we can write

$$(25)\qquad f(x, y) = \sum_{i=0}^{m}\sum_{k=0}^{r_i} A_{ik}(x)\frac{\partial^k}{\partial x^k}f(x_i, y) + \frac{\alpha(x)}{(M+1)!}\frac{\partial^{M+1}}{\partial x^{M+1}}f(\xi, y)$$

where

$$(26)\qquad \alpha(x) = (x-x_0)^{r_0+1}\cdots(x-x_m)^{r_m+1}$$

and $\min(x; x_0, \cdots, x_m) \leqq \xi \leqq \max(x; x_0, \cdots, x_m)$. Similarly, if x is kept fixed,

$$(27)\qquad f(x, y) = \sum_{j=0}^{n}\sum_{l=0}^{s_j} B_{jl}(y)\frac{\partial^l}{\partial y^l}f(x, y_j) + \frac{\beta(y)}{(N+1)!}\frac{\partial^{N+1}}{\partial y^{N+1}}f(x, \eta)$$

where

$$(28)\qquad \beta(y) = (y-y_0)^{s_0+1}\cdots(y-y_n)^{s_n+1}$$

and $\min(y; y_0, \cdots, y_n) \leqq \eta \leqq \max(y; y_0, \cdots, y_n)$. From (25) and (27), it follows that

$$(29)\qquad \begin{aligned} f(x, y) = &\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} A_{ik}(x)B_{jl}(y)\frac{\partial^{k+l}}{\partial x^k \partial y^l}f(x_i, y_j) + \\ &\frac{\alpha(x)}{(M+1)!}\sum_{j=0}^{n}\sum_{l=0}^{s_j} B_{jl}(y)\frac{\partial^{M+1+l}}{\partial x^{M+1}\partial y^l}f(\xi, y_j) \\ &+ \frac{\beta(y)}{(N+1)!}\sum_{i=0}^{m}\sum_{k=0}^{r_i} A_{ik}(x)\frac{\partial^{k+N+1}}{\partial x^k \partial y^{N+1}}f(x_i, \eta) \\ &+ \frac{\alpha(x)\beta(y)}{(M+1)!(N+1)!}\frac{\partial^{M+N+2}}{\partial x^{M+1}\partial y^{N+1}}f(\xi, \eta); \end{aligned}$$

that is,

$$f(x, y) = H_{M,N}(x, y) + R(x, y).$$

Now, from (25),

(30)
$$\sum_{i=0}^{m} \sum_{k=0}^{r_i} A_{ik}(x) \frac{\partial^{k+N+1}}{\partial x^k \partial y^{N+1}} f(x_i, \eta) = \frac{\partial^{N+1}}{\partial y^{N+1}} \left[f(x, y) - \frac{\alpha(x)}{(M+1)!} \frac{\partial^{M+1}}{\partial x^{M+1}} f(\xi, y) \right]_{y=\eta}$$

and, from (27),

(31)
$$\sum_{j=0}^{n} \sum_{l=0}^{s_j} B_{jl}(y) \frac{\partial^{M+1+l}}{\partial x^{M+1} \partial y^l} f(\xi, y_j) = \frac{\partial^{M+1}}{\partial x^{M+1}} \left[f(x, y) - \frac{\beta(y)}{(N+1)!} \frac{\partial^{N+1}}{\partial y^{N+1}} f(x, \eta) \right]_{x=\xi}.$$

Substituting (31) and (30) into the second and third terms, respectively, of the right side of (29), we obtain

(32)
$$R(x, y) = \frac{\alpha(x)}{(M+1)!} \frac{\partial^{M+1}}{\partial x^{M+1}} f(\xi, y) + \frac{\beta(y)}{(N+1)!} \frac{\partial^{N+1}}{\partial y^{N+1}} f(x, \eta) - \frac{\alpha(x)\beta(y)}{(M+1)!(N+1)!} \frac{\partial^{M+N+2}}{\partial x^{M+1} \partial y^{N+1}} f(\xi, \eta),$$

which gives the error of interpolation.

6. Error in terms of contour integrals

In the case of a single variable x, let C be a closed contour in the region D_z of analytic continuation of $f(x)$ containing the points $x_0, \cdots, x_m$ in its interior. By applying the residue theorem to the contour integral

$$\frac{1}{2\pi i} \int_C \frac{f(z)dz}{\alpha(z)(z-x)}$$

with $\alpha(z)$ defined in (26), we obtain

(33)
$$f(x) = H_M(x) + \frac{\alpha(x)}{2\pi i} \int_C \frac{f(z)dz}{\alpha(z)(z-x)}$$

where $H_M(x)$ is the interpolation polynomial of Theorem 1.

Applying (33) to a function $f(x, y)$ of two variables, and keeping y fixed, we obtain

(34)
$$f(x, y) = \sum_{i=0}^{m} \sum_{k=0}^{r_i} A_{ik}(x) \frac{\partial^k}{\partial x^k} f(x_i, y) + \frac{\alpha(x)}{2\pi i} \int_{C_1} \frac{f(z, y)dz}{\alpha(z)(z-x)}$$

where C_1 is a simple closed contour in the region D_z of analyticity of $f(z, y)$ (y fixed) and containing the points $x_0, \cdots, x_m$ in its interior. Similarly, if x is fixed, we can write

$$(35)\qquad f(x,y)=\sum_{j=0}^{n}\sum_{l=0}^{s_j} B_{jl}(y)\frac{\partial^l}{\partial y^l}f(x,y_j)+\frac{\beta(y)}{2\pi i}\int_{C_2}\frac{f(x,w)dw}{\beta(w)(w-y)}$$

where $\beta(w)$ is defined as in (28) and C_2 is a simple closed contour in the region D_w of analyticity of $f(x,w)$ (x fixed) and containing the points $y_0,\cdots,y_m$ in its interior.

Now assume that $f(z,w)$ is simultaneously analytic in $D_z \times D_w$. From (34) and (35) we obtain

$$(36)\qquad f(x,y)=H_{M,N}(x,y)+R(x,y),$$

where $H_{M,N}(x,y)$ is the interpolation polynomial of Theorem 2, and $R(x,y)$ is the error of interpolation. Simplifying the expression for the error, we obtain

$$(37)\qquad R(x,y)=\frac{\alpha(x)}{2\pi i}\int_{C_1}\frac{f(z,y)dz}{\alpha(z)(z-x)}+\frac{\beta(y)}{2\pi i}\int_{C_2}\frac{f(x,w)dw}{\beta(w)(w-y)} -\frac{\alpha(x)\beta(y)}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{f(z,w)dzdw}{\alpha(z)\beta(w)(z-x)(w-y)}.$$

Again, with the help of Cauchy's integral formula for $f(x,w)$ and $f(z,y)$, we can write (37) as

$$(38)\qquad R(x,y)=(2\pi i)^{-2}\int_{C_1}\int_{C_2}\frac{\alpha(x)\beta(w)+\alpha(z)\beta(y)-\alpha(x)\beta(y)}{\alpha(z)\beta(w)(z-x)(w-y)}f(z,w)dzdw.$$

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