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MOST POWER SERIES HAVE RADIUS OF CONVERGENCE 0 OR 1*

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Consider a random power series $\sum_0^\infty c_n z^n$, that is, with coefficients $\{c_n\}_0^\infty$ chosen independently at random from the complex plane. What is the radius of convergence of such a series likely to be?

One approach to this question is to let the $\{c_n\}_0^\infty$ be independent random variables on some probability space. It turns out that, with probability one, the radius of convergence is constant. Moreover, if the c_n are symmetric and have the same distribution, then the circle of convergence is almost surely a natural boundary for the analytic function given by the power series (See [1, Ch. IV, Section 3]). Our treatment of the question will be elementary and will not use these facts.

Another approach is to think of the sequence $\{c_n\}_0^\infty$ as a member of the topological product $\pi_0^\infty C_n$, of copies of the complex plane. This is a complete metric space with the metric:

$$\rho(c, d) = \sum_0^\infty \frac{1}{2^n} \cdot \frac{|c_n - d_n|}{1+|c_n - d_n|}.$$

It is appropriate to say that a property is possessed by "most" power series if the set of sequences $\{c_n\}_0^\infty$ for which the power series does not possess the property is of first (Baire) category in $\pi_0^\infty C_n$.

This note is based on a discussion which took place at a Canadian Mathematical Congress Summer Institute.

Let us begin with the probability viewpoint. We need a probability measure on the complex plane C. A natural way to get such a measure is to normalize surface area on the Riemann sphere and transfer this to C by stereographic projection. The result is that

$$\text{prob}(|z| > R) = \frac{1}{R^2+1}.$$

Suppose that the $\{c_n\}_0^\infty$ are chosen independently from C, each with the above distribution. Let r denote the radius of convergence of $\sum_0^\infty c_n z^n$.

First if $r > 1$, then $\sum_0^\infty c_n$ converges so that, for some integer N, $|c_n| \leq N$ for

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all n. But prob$\{|c_n|\leq N$ for all $n\}=$

$$\prod_{n=0}^{\infty}\left(1-\frac{1}{N^2+1}\right) = 0.$$

Letting $N\to\infty$, we obtain prob$(r>1)=0$.

Again let N be a positive integer and consider prob$\{|c_n|\leq n$ for all $n\geq N\}=$

$$\prod_{n=N}^{\infty}\left(1-\frac{1}{n^2+1}\right) > 0.$$

Note that this probability tends to 1 as N tends to ∞. But if $|c_n|\leq n$ for all $n\geq N$ we have $r\geq 1$. Letting $N\to\infty$, we obtain prob$(r\geq 1)=1$. Hence prob$(r=1)=1$. In the sense of probability most power series have radius of convergence 1.

Now we take the Baire category viewpoint. Fix an integer $N>0$. If $r>1/N$, then $\sum_0^\infty c_n N^{-n}$ converges so that, for some integer M, we have $|c_n|\leq MN^n$ for all n. But $S_{M,N}=\{c\in\pi_0^\infty C_n \mid |c_n|\leq MN^n$ for all $n\}$ is closed and its complement is dense in $\pi_0^\infty C_n$. (To see this suppose that c in $\pi_0^\infty C_n$ and $\varepsilon>0$ are given. Choose m so that $1/2^m<\varepsilon$. Define d in $\pi^\infty C_n$ by:

$$d_n = \begin{cases} c_n & \text{if} \quad n\neq m \\ 2MN^m & \text{if} \quad n=m \end{cases}$$

Then $\rho(c,d)<1/2^m<\varepsilon$ and $d\notin S_{M,N}$).

Clearly $\{c \mid r>0\}\subset\bigcup_{M,N=1}^\infty S_{M,N}$ and is of first category. Hence in the sense of category most power series have radius of convergence 0.

REMARK 1. In the probability discussion above, instead of viewing the $\{c_n\}_0^\infty$ as independent random variables with respect to a probability measure on C, we can consider the product probability measure on $\pi_0^\infty C_n$. The set $\{c \mid r=1\}$ has full measure 1 but is of first category in $\pi^\infty C_n$. The fact that a set in R^n can have full Lebesgue measure but be of first category is of course well known. Our example may be surprising, however, since it arises in such a natural way.

REMARK 2. We note that the probability result does depend a bit on the measure that we use. But the same conclusion holds for many reasonable probability distributions on C. Unless the point 0 has mass 1 we can show that prob$(r>1)=0$. If $\sum_{n=0}^\infty$ prob$(|z|>n^k)<\infty$ for some k, then prob$(r\geq 1)=1$.

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