A NOTE ON CERTAIN SUBALGEBRAS OF C($\mathfrak{X}$)

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1. Let $\mathfrak{X}$ be a completely regular Hausdorff space and $C(\mathfrak{X})$ the algebra of continuous real-valued functions on $\mathfrak{X}$. In attempts to characterize abstractly those algebras that are isomorphic to $C(\mathfrak{X})$ for some $\mathfrak{X}$, one produces subalgebras of $C(\mathfrak{X})$ which: (a) contain the constant functions, (b) separate points and closed sets in $\mathfrak{X}$, (c) are closed under uniform convergence, and (d) are closed under inversion in $C(\mathfrak{X})$ (see, for example, **(2; 5)**). We call such a subalgebra of $C(\mathfrak{X})$ an *Algebra on* $\mathfrak{X}$. (Since these are the only algebras to be discussed here, no misunderstanding should arise.)

In general, an Algebra on $\mathfrak{X}$ need not be all of $C(\mathfrak{X})$; both **(2)** and **(5)** contain examples of Algebras on the discrete space of cardinal c that are not isomorphic to $C(\mathfrak{Y})$ for any space $\mathfrak{Y}$. The example appearing in **(2)** is easily described: it is the algebra of Baire functions on the real line, viewed as an Algebra on the real line with the discrete topology.

In **(5)**, Isbell proved that the only Algebra on a σ-compact, locally compact space $\mathfrak{X}$ is $C(\mathfrak{X})$. In **(6)**, Mrówka announced that this conclusion holds, in fact, for all Lindelöf spaces. (This result is also readily derived from **(2**, Theorem 5.4**)**.)

In **(4)**, Hewitt proved that $\mathfrak{X}$ is almost compact (i.e., card $(\beta\mathfrak{X} - \mathfrak{X}) \leqslant 1$; see also 6*J* of **(1)**) if and only if $C^*(\mathfrak{X})$ (the algebra of bounded functions in $C(\mathfrak{X})$) contains no proper subalgebra which: (a) contains the constant functions (b) separates points and closed sets in $\mathfrak{X}$, and (c) is closed under uniform convergence. An almost compact space is pseudocompact (i.e., $C^* = C$), and for pseudocompact $\mathfrak{X}$, Hewitt's subalgebras of $C(\mathfrak{X})$ are precisely the Algebras on $\mathfrak{X}$.

Thus, if $\mathfrak{X}$ is either Lindelöf or almost compact, then the only Algebra on $\mathfrak{X}$ is $C(\mathfrak{X})$. In this note, we prove the converse of this statement and the equivalence of "$C(\mathfrak{X})$ is the only Algebra on $\mathfrak{X}$" to each of several other conditions.

Certain aspects of this work overlap with some unpublished work of R. L. Blair. In particular, condition (5) in the theorem of §3 has been considered by him; see also §4.3. We are indebted to Blair for interesting discussions on these matters.

2. We collect here the background information that will be needed. Most of the notation and terminology is as in **(1)**. All topological spaces are assumed to be completely regular Hausdorff.

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2.1. For $f \in C(\mathfrak{X})$, $\mathfrak{Z}(f)$ denotes $\{x \in \mathfrak{X}: f(x) = 0\}$, and coz f denotes $\mathfrak{X} - \mathfrak{Z}(f)$. A zero set (A cozero set) in $\mathfrak{X}$ is a set $\mathfrak{Z}(f)$ (a set coz f) for some $f \in C(\mathfrak{X})$.

Let A be a subfamily of $C(\mathfrak{X})$. Then A is said to separate points and closed sets in $\mathfrak{X}$ if for any closed set $\mathfrak{F}$ and $p \in \mathfrak{X} - \mathfrak{F}$, there is $f \in A$ with $\mathfrak{F} \subseteq \mathfrak{Z}(f)$ and $p \in \operatorname{coz} f$. We say that A is closed under inversion in $C(\mathfrak{X})$ if $f \in A$ and $\mathfrak{Z}(f) = \emptyset$ imply $1/f \in A$.

Let $\mathfrak{X}$ be a subspace of $\mathfrak{Y}$. Then, $\mathfrak{X}$ is said to be C-embedded (respectively, C^*-embedded) in $\mathfrak{Y}$ if, given $f \in C(\mathfrak{X})$ ($f \in C^*(\mathfrak{X})$), there is $g \in C(\mathfrak{Y})$ whose restriction to $\mathfrak{X}$ is f. $\mathfrak{X}$ is C^*-embedded in $\mathfrak{Y}$ if and only if any two disjoint zero sets in $\mathfrak{X}$ have disjoint closures in $\mathfrak{Y}$. We say that $\mathfrak{X}$ is *z-embedded* in $\mathfrak{Y}$ if, given a zero set $\mathfrak{Z}$ in $\mathfrak{X}$, there is a zero set $\mathfrak{Z}'$ in $\mathfrak{Y}$ with $\mathfrak{Z}' \cap \mathfrak{X} = \mathfrak{Z}$. Evidently, a C^*-embedding is a z-embedding.

2.2. A space that contains $\mathfrak{X}$ densely will be called merely an extension of $\mathfrak{X}$. Two extensions of $\mathfrak{X}$ are said to be equivalent if they are homeomorphic via a map that leaves $\mathfrak{X}$ pointwise fixed. Throughout, all statements about the uniqueness of extensions will be in this sense (i.e., up to equivalence). Similarly, two extensions will be called different if they are not equivalent.

If $\mathfrak{Y}$ is an extension of $\mathfrak{X}$, then $C(\mathfrak{Y})$ may be regarded as a subalgebra of $C(\mathfrak{X})$: those $f \in C(\mathfrak{X})$ with continuous extensions over $\mathfrak{Y}$. We shall not distinguish notationally between $C(\mathfrak{Y})$ and the associated subalgebra of $C(\mathfrak{X})$. It is easy to see that $C(\mathfrak{Y})$ is closed under inversion in $C(\mathfrak{X})$ if and only if every non-void G_δ-set in $\mathfrak{Y}$ meets $\mathfrak{X}$—as we shall say, $\mathfrak{X}$ is *G_δ-dense* in $\mathfrak{Y}$. Thus, $C(\mathfrak{Y})$ is an Algebra on $\mathfrak{X}$ if and only if $\mathfrak{X}$ is G_δ-dense in $\mathfrak{Y}$ (the other conditions prevailing automatically).

Among the extensions of a given space $\mathfrak{X}$ are two of special interest: the Stone–Čech compactification $\beta\mathfrak{X}$, characterized as that compact extension in which $\mathfrak{X}$ is C^*-embedded (i.e., $C^*(\mathfrak{X}) = C(\beta\mathfrak{X})$) and the Hewitt real-compactification $\upsilon\mathfrak{X}$, characterized as that realcompact extension in which $\mathfrak{X}$ is C-embedded (i.e., $C(\mathfrak{X}) = C(\upsilon\mathfrak{X})$). Of course, $\mathfrak{X}$ is G_δ-dense in $\upsilon\mathfrak{X}$. (See **(1**, Chapters 6 and 8**)**.

2.3. We now summarize the pertinent information from **(5)**.

If A is an Algebra on $\mathfrak{X}$, then A determines two extensions of $\mathfrak{X}$: a compact one, $\mathfrak{H}(A^*)$, characterized among compact extensions by the property $C(\mathfrak{H}(A^*)) = A^*$, and a realcompact extension, $\mathfrak{H}(A)$. Let $\operatorname{coz}(\mathfrak{H}(A^*), \mathfrak{X})$ denote the collection of cozero sets in $\mathfrak{H}(A^*)$ that contain $\mathfrak{X}$. Then, for each $\mathfrak{S} \in \operatorname{coz}(\mathfrak{H}(A^*), \mathfrak{X})$, $\beta\mathfrak{S} = \mathfrak{H}(A^*)$; moreover,

$$A = \bigcup\{C(\mathfrak{S}): \mathfrak{S} \in \operatorname{coz}(\mathfrak{H}(A^*), \mathfrak{X})\}$$

and

$$\mathfrak{H}(A) = \bigcap\{\mathfrak{S}: \mathfrak{S} \in \operatorname{coz}(\mathfrak{H}(A^*), \mathfrak{X})\}.$$

Thus, $\mathfrak{X}$ is G_δ-dense in $\mathfrak{H}(A)$, and A may be regarded as an Algebra on $\mathfrak{H}(A)$. While A and $C(\mathfrak{H}(A))$ may differ (as in the examples mentioned in §1), if A is isomorphic to any $C(\mathfrak{Y})$, then $A = C(\mathfrak{H}(A))$.

If A and B are distinct Algebras on $\mathfrak{X}$, then $\mathfrak{H}(A^*)$ and $\mathfrak{H}(B^*)$ are distinct (so that $A^* \neq B^*$, as well). Of course, $\mathfrak{H}(C^*(\mathfrak{X})) = \beta\mathfrak{X}$ and $\mathfrak{H}(C(\mathfrak{X})) = \upsilon\mathfrak{X}$. Hence, in order that the Algebra A on $\mathfrak{X}$ coincide with $C(\mathfrak{X})$, it is necessary and sufficient that $\mathfrak{H}(A^*) = \beta\mathfrak{X}$.

3. THEOREM. *For any (completely regular Hausdorff) space $\mathfrak{X}$, the following conditions are equivalent:*

(1) *The only Algebra on $\mathfrak{X}$ is $C(\mathfrak{X})$.*

(2) *The only realcompact space in which $\mathfrak{X}$ is G_δ-dense is $\upsilon\mathfrak{X}$.*

(3) *$\upsilon\mathfrak{X}$ is Lindelöf,* $\mathrm{card}(\upsilon\mathfrak{X} - \mathfrak{X}) \leqslant 1$, *and $\upsilon\mathfrak{X}$ is the only space in which $\mathfrak{X}$ is G_δ-dense with these two properties.*

(4) *Either $\mathfrak{X}$ is Lindelöf or $\mathfrak{X}$ is almost compact.*

(5) *Every embedding of $\mathfrak{X}$ is a z-embedding.*

Proof. (1) *implies* (2). If $\mathfrak{Y}$ is a realcompact space in which $\mathfrak{X}$ is G_δ-dense and if $\mathfrak{Y} \neq \upsilon\mathfrak{X}$, then $C(\mathfrak{Y})$ is an Algebra on $\mathfrak{X}$ that is different from $C(\mathfrak{X})$.

(2) *implies* (3). Assume (2). In **(7)**, Mrówka proves that a space is Lindelöf if and only if it is G_δ-dense in no proper superspace. Hence, if $\upsilon\mathfrak{X}$ is not Lindelöf, then there is a space $\mathfrak{Y}$, properly containing $\upsilon\mathfrak{X}$, in which $\upsilon\mathfrak{X}$ is G_δ-dense. Evidently, $\mathfrak{X}$ is G_δ-dense in $\upsilon\mathfrak{Y}$ and, moreover, $\upsilon\mathfrak{X}$ and $\upsilon\mathfrak{Y}$ are different extensions of $\mathfrak{X}$ (e.g., $C(\upsilon\mathfrak{Y}) \neq C(\upsilon\mathfrak{X})$), contradicting (2).

Thus, $\upsilon\mathfrak{X}$ is Lindelöf. If $\mathrm{card}(\upsilon\mathfrak{X} - \mathfrak{X}) > 1$, pick two distinct points of $\upsilon\mathfrak{X} - \mathfrak{X}$, and let $\mathfrak{Y}$ denote the quotient of $\upsilon\mathfrak{X}$ obtained by identifying them. The quotient mapping is not one-to-one, so that $\mathfrak{Y}$ and $\upsilon\mathfrak{X}$ are different. $\mathfrak{Y}$ is Lindelöf (being the continuous image of $\upsilon\mathfrak{X}$) and contains $\mathfrak{X}$ G_δ-densely (since $\upsilon\mathfrak{X}$ does). Again, (2) is contradicted, so $\mathrm{card}(\upsilon\mathfrak{X} - \mathfrak{X}) \leqslant 1$.

If $\mathfrak{X}$ is Lindelöf, then the remaining statement of (3) follows by Mrówka's theorem quoted above. If $\mathfrak{X}$ is not Lindelöf, it is G_δ-dense in any Lindelöf superspace $\mathfrak{Y}$ with $\mathrm{card}(\mathfrak{Y} - \mathfrak{X}) = 1$. (Otherwise, $\mathfrak{X}$ would be an F_σ in $\mathfrak{Y}$, and hence Lindelöf.) By (2), $\mathfrak{Y} = \upsilon\mathfrak{X}$.

(3) *implies* (4). Assume (3), and suppose that $\mathfrak{X}$ is not Lindelöf. Since $\upsilon\mathfrak{X} - X$ is a singleton, say $\{p_0\}$, $\beta\mathfrak{X} = \upsilon\mathfrak{X}$ if and only if $\mathfrak{X}$ is almost compact. Suppose, therefore, that there is $p_1 \in \beta\mathfrak{X} - \upsilon\mathfrak{X}$. Consider the subspace $\mathfrak{X} \cup \{p_0, p_1\}$ of $\beta\mathfrak{X}$, and let $\mathfrak{Y}$ denote the quotient of this space obtained by identifying p_0 and p_1. Now, $\mathfrak{Y}$ is Lindelöf, since it is the continuous image of a Lindelöf space; since X is not Lindelöf, it is G_δ-dense in $\mathfrak{Y}$. Finally, $\upsilon\mathfrak{X}$ and $\mathfrak{Y}$ are different extensions of $\mathfrak{X}$: for example, $\mathfrak{X}$ is clearly not C-embedded in $\mathfrak{Y}$. Thus, (3) is contradicted.

(4) *implies* (5). By Lemma 5.3 of **(2)**, every embedding of a Lindelöf space is a z-embedding. Also, every embedding of an almost compact space is a C^*-embedding **(1**, 6J**)**, hence a z-embedding.

(5) *implies* (1). Assume (5), and let A be an Algebra on $\mathfrak{X}$. We show that $\mathfrak{X}$ is C^*-embedded in $\mathfrak{H}(A^*)$. Thus, $\mathfrak{H}(A^*) = \beta\mathfrak{X}$, and $A = C(\mathfrak{X})$ follows (by 2.3).

It suffices to show that if $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are disjoint zero sets in $\mathfrak{X}$, then $\overline{\mathfrak{Z}}_1 \cap \overline{\mathfrak{Z}}_2 = \emptyset$, where the bar denotes closure in $\mathfrak{H}(A^*)$. By (5), choose $f_i \in C(\mathfrak{H}(A^*)) = A^*$ with $\mathfrak{Z}(f_i) \cap \mathfrak{X} = \mathfrak{Z}_i$, for $i = 1, 2$. Now $\mathfrak{Z}(f_1^2 + f_2^2) = \mathfrak{Z}(f_1) \cap \mathfrak{Z}(f_2)$ is disjoint from $\mathfrak{X}$, so $\operatorname{coz}(f_1^2 + f_2^2)$ is C^*-embedded in $\mathfrak{H}(A^*)$. (See 2.3.) Let $\mathfrak{Z}'_i$ denote the zero set (in $\operatorname{coz}(f_1^2 + f_2^2)$) of $f_i^2/(f_1^2 + f_2^2)$, for $i = 1, 2$. Then $\mathfrak{Z}'_1 \cap \mathfrak{Z}'_2 = \emptyset$, so that $\overline{\mathfrak{Z}}'_1 \cap \overline{\mathfrak{Z}}'_2 = \emptyset$. For $i = 1, 2$, $\mathfrak{Z}_i \subseteq \mathfrak{Z}'_i$, so $\overline{\mathfrak{Z}}_1 \cap \overline{\mathfrak{Z}}_2 = \emptyset$ as well.

4. *Remarks.*

4.1. The proof that (5) implies (1) is essentially the method employed in the proof of Lemma 5.2 of **(2)**. Virtually the same argument reappears in the proof of Lemma 2.4 of **(3)**.

4.2 Consider the following condition on $\mathfrak{X}$.

(3′) $\upsilon\mathfrak{X}$ *is Lindelöf and* $\operatorname{card}(\upsilon\mathfrak{X} - \mathfrak{X}) \leqslant 1$. Condition (3′) does not imply (3), as is shown by the following example.

Let ω_1 denote the first uncountable ordinal, W^* the space of all ordinal numbers $\leqslant \omega_1$ in the order topology (see **(1**, 5.11**)**), and let $\mathfrak{Y}$ denote the discrete union of countably many copies of W^*. Let $\mathfrak{X}$ denote the subspace of $\mathfrak{Y}$ obtained by deleting ω_1 from one of the copies of W^*. Evidently, $\upsilon\mathfrak{X} = \mathfrak{Y}$, but $\mathfrak{X}$ is neither Lindelöf nor almost compact.

4.3. It can be shown that (3′) is equivalent to the following condition: *Of any pair of disjoint zero sets in* $\mathfrak{X}$, *at least one is Lindelöf.* (This has also been observed by R. L. Blair.) Upon replacing "Lindelöf" by "compact" in this statement, Hewitt's definition of "almost compact" is obtained.

4.4. *If* $\upsilon\mathfrak{X}$ *is Lindelöf, then each Algebra on* $\mathfrak{X}$ *is a* $C(\mathfrak{Y})$. To prove this assertion, we note the

LEMMA. $\upsilon\mathfrak{X}$ *is Lindelöf if and only if each realcompact space in which* $\mathfrak{X}$ *is* G_δ*-dense is the continuous image of* $\upsilon\mathfrak{X}$ *by a mapping that leaves* $\mathfrak{X}$ *pointwise fixed.*

Proof. If $\upsilon\mathfrak{X}$ is not Lindelöf, then we may pick a realcompact space $\mathfrak{Y}$ which contains $\upsilon\mathfrak{X}$ as a proper, G_δ-dense subspace, as in the proof that (2) implies (3). It is easily seen that $\mathfrak{Y}$ is not a continuous image of $\upsilon\mathfrak{X}$ as prescribed. Conversely, if Y is a realcompact space in which $\mathfrak{X}$ is G_δ-dense, let c denote the Stone extension of the identity map on $\mathfrak{X}$ over $\beta\mathfrak{X}$ onto $\beta\mathfrak{Y}$. (See **(1**, Theorem 6.5**)**). Then, as is easily checked, $c[\upsilon\mathfrak{X}] \subseteq \mathfrak{Y}$. But, if $\upsilon\mathfrak{X}$ is Lindelöf, then $c[\upsilon\mathfrak{X}]$ is Lindelöf, and can be G_δ-dense in no proper superspace. Hence, $c[\upsilon\mathfrak{X}] = \mathfrak{Y}$. This completes the proof of the lemma.

To prove 4.4, note that if $\upsilon\mathfrak{X}$ is Lindelöf and A is an Algebra on $\mathfrak{X}$, then A is an Algebra on $\mathfrak{H}(A)$ and $\mathfrak{H}(A)$ is Lindelöf, by the lemma. But then $A = C(\mathfrak{H}(A))$, by the theorem.

It seems likely that the converse of 4.4 holds, and even that the following apparently stronger statement is true: if $\mathfrak{X}$ is realcompact but not Lindelöf, then there is an Algebra A on $\mathfrak{X}$, different from $C(\mathfrak{X})$, with $\mathfrak{H}(A) = \mathfrak{X}$. But we have been unable to prove this.

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