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ON A CLASS OF ANALYTIC FUNCTIONS OF SMIRNOV

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1. Introduction. The class S of functions under study in this paper was introduced by V. I. Smirnov in 1932. This class was subsequently investigated by various authors, a pertinent paper to the present work being that of Tumarkin and Havinson [**2**], who showed that a plane compact set of logarithmic capacity zero is S-removable. Another important development, due to Yamashita [**3**], was that the class S could be characterized as those analytic functions f for which $\log^+ |f|$ has a quasi-bounded harmonic majorant.

In what follows, we discuss the Smirnov class in the context of planar surfaces, exploiting some ideas in the work of Hejhal [**1**] to establish that a closed, bounded, totally disconnected set is S-removable if and only if its complement belongs to the null class O_S.

This result would be elementary in the event that $O_G = O_S$. However, at present, it is not known if the inclusion $O_G \subset O_S$ is strict or not, nor for that matter whether or not the related inclusion $O_G \subset O_{H^p}$ is strict for $0 < p < 1$. In fact, a resolution of the former question in the affirmative (i.e. the inclusion is strict), would resolve the latter question likewise, since $O_G \subset O_S \subset O_{H^p}$.

2. Preliminaries. In the sequel, we make use of the following notation.

$\overline{\mathbf{C}}$: Riemann sphere

R: open Riemann surface

$A(R)$: the class of analytic functions on R

$\phi^\wedge$: the least harmonic majorant of ϕ

$S(R)$: $\{f \in A(R) : \log^+ (|f|/\mu)$ has a harmonic majorant on R for some $\mu > 0$ (and hence for all $\mu > 0$) and $(\log^+ (|f|/\mu))^\wedge(z_0) \rightarrow 0$ as $\mu \rightarrow \infty$, for some $z_0 \in R\}$

O_S: the class of Riemann surfaces R which carry no nonconstant functions belonging to $S(R)$.

3. Planar Surfaces. We turn now to the problem of establishing a necessary and sufficient condition for a planar set to be S-removable.

Let E be a bounded closed totally disconnected subset of $\overline{\mathbf{C}}$.

THEOREM 1. *Let $U \subset \overline{\mathbf{C}}$ be a hyperbolic domain containing E. If $f \in S(U - E) \cap A(U)$, then $f \in S(U)$.*

Proof. Let $\chi = (\log^+ |f|)^\wedge$ on $U - E$. We may assume $\infty \notin U$, and let $\{U_n\}$ be an exhaustion of U by smoothly bounded subregions, with $E \subset U_1$, as

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in Hejhal [1]. Set $\chi_n = (\log^+ |f|)^\wedge$ on $U_n - E$ and fix $z_0 \in U_1 - E$. We take an exhaustion $U_n - G_m \nearrow U_n - E$, where the G_m are finite unions of disjoint Jordan regions such that

$$\bigcap_{m=1}^{\infty} G_m = E.$$

Then

$$\chi(z_0) \geqq \chi_n(z_0) = \lim_{m\to\infty} \int_{\partial U_n + \partial G_m} \log^+|f(\zeta)| \frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n - G_m)|d\zeta|$$
$$\geqq \lim_{m\to\infty} \int_{\partial U_n} \log^+|f(\zeta)| \frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n - G_m)|d\zeta|$$
$$= \int_{\partial U_n} \log^+|f(\zeta)| \frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n - E)|d\zeta|,$$

where the last equality follows from the fact that on ∂U_n we have

$$\frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n - G_m) \nearrow \frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n - E).$$

Furthermore, as in [1], there exists a λ with $0 < \lambda \leqq 1$ such that

$$g(\zeta; z_0; U_n - E) \geqq \lambda g(\zeta; z_0; U_n)$$

for all $n \geqq 1$ and $\zeta \in C$, a simple closed curve in U_1 with $\{z_0\} \cup E \subset \operatorname{int} C \subset U_1$. Hence

$$(*) \qquad \chi(z_0) \geqq \chi_n(z_0) \geqq \frac{\lambda}{2\pi} \int_{\partial U_n} \log^+|f(\zeta)| \frac{\partial g}{\partial n_\zeta}(\zeta; z_0; U_n)|d\zeta|,$$

for all $n \geqq 1$. Since $f \in A(U)$, $\log^+ |f|$ is subharmonic on U. Thus (*) implies $\log^+ |f|$ has a least harmonic majorant on U, say h, and

$$(**) \qquad \chi(z_0) \geqq \lambda h(z_0).$$

As λ is independent of f, and $f/\mu (\mu > 0)$ belongs to $S(U - E) \cap A(U)$ whenever f does, (**) holds for f replaced by f/μ. Consequently,

$$\lim_{\mu\to\infty} [\log^+(|f|/\mu)]_U^\wedge (z_0) \leqq \frac{1}{\lambda} \lim_{\mu\to\infty} [\log^+(|f|/\mu)]_{U-E}^\wedge (z_0) = 0,$$

and thus $f \in S(U)$ which proves the theorem.

If the set E is sufficiently "small", it turns out that a function in $S(U - E)$ will have an analytic extension to U, and consequently by the preceding theorem belong to $S(U)$.

THEOREM 2. *Let U be a hyperbolic domain containing E. If $\overline{\mathbf{C}} - E \in O_S$, then $S(U - E) \subset A(U)$.*

Proof. It suffices to prove the result for U a Jordan domain with smooth boundary. Let $\chi = (\log^+ |f|)^\wedge$ on $U - E$ for $f \in S(U - E)$. As in [1], let

$\Omega_n = U - E_n$ be an exhaustion of $U - E$ towards E. Then $f = g + h$ on $U - E$, where $g \in A(U)$ and $h \in A(\bar{\mathbf{C}} - E)$ with $h(\infty) = 0$. Also, $|g| \leqq M$ and hence $|h| \leqq M + |f|$. Since $\log^+ (a + b) \leqq a + \log^+ b$, for a and b nonnegative, we obtain

$$\log^+ |h| \leqq \log^+ (M + |f|) \leqq M + \log^+ |f| \leqq M + \chi$$

on $U - E$.

Let $T_n = (\log^+ |h|)\hat{}$ on $\bar{\mathbf{C}} - E_n$. Then, as before, there is a constant K independent of f, g, or h such that $1 \leqq K < \infty$ and (cf. Hejhal [**1**], p. 11)

$$T_n(z_0) \leqq K \cdot \frac{1}{2\pi} \int_{\partial\Omega_n} (M + \chi(\zeta)) \frac{\partial g}{\partial n_\zeta} (\zeta; z_0; \Omega_n)|d\zeta| = K(M + \chi(z_0))$$

for z_0 fixed in $U - E \cap \{\Omega_n\}$. Since $\{T_n\}$ is an increasing sequence bounded at z_0, $T_n \nearrow T = (\log^+ |h|)\hat{}_{\bar{\mathbf{C}}-E}$. Moreover

$$T(z_0) \leqq K(M + \chi(z_0)).$$

Replacing f, g, h, by f/μ, g/μ, h/μ respectively, it follows that

$$[\log^+ (|h|/\mu)]\hat{}_{\bar{\mathbf{C}}-E}(z_0) \leqq K(M/\mu + [\log^+ (|f|/\mu)]\hat{}_{U-E}(z_0)).$$

Since the $RHS \to 0$ as $\mu \to \infty$, we have $h \in S(\bar{\mathbf{C}} - E)$. However, $\bar{\mathbf{C}} - E \in O_S$, and $h(\infty) = 0$, implying $h \equiv 0$. Therefore $f = g \in A(U)$.

Definition. $E \in N_S$ if and only if $S(U - E) = S(U)$ for every subdomain U of $\bar{\mathbf{C}}$ containing E.

The set N_S is characterized by the following:

THEOREM 3. *$E \in N_S$ if and only if $\bar{\mathbf{C}} - E \in O_S$.*

Proof. That $E \in N_S$ implies $\bar{\mathbf{C}} - E \in O_S$ is trivial. Assume $\bar{\mathbf{C}} - E \in O_S$. If $U \in O_G$ (parabolic), we may take $\infty \in U$, and set $U = \bar{\mathbf{C}} - F$, where F is compact, cap $(F) = 0$, $E \cap F = \emptyset$. By Theorem 3 of Tumarkin and Havinson [**2**], $F \in N_S$. Thus $S(U - E) = S(\bar{\mathbf{C}} - F - E) = S(\bar{\mathbf{C}} - E)$, and $\bar{\mathbf{C}} - E \in O_S$ yields $S(\bar{\mathbf{C}} - E) = \{\text{constants}\} = S(U)$.

For $U \notin O_G$, Theorems 1 and 2 imply $S(U - E) = S(U)$, i.e. $E \in N_S$.

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