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AGGREGATION–DECOMPOSITION AND EQUI-ULTIMATE BOUNDEDNESS

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Abstract

The aggregation–decomposition method is used to derive a sufficient condition for the equi-ultimate boundedness of large-scale systems governed by nonlinear ordinary differential equations.

1. Introduction

The aggregation–decomposition method is an effective way of determining stability properties of certain classes of dynamical systems with nonlinearities and high dimensions (e.g. Grujić [1], [2]; Grujić and Šiljak [3], Michel [4]). Basically, it involves the decomposition of a complicated system into several simpler subsystems, each a function of different components of the state vector, with interconnections between them. These subsystems may have some physical meaning or may be just mathematical artifices. The sum of their more easily found Lyapunov functions is tried as a Lyapunov function for some desired stability property of the overall system. Its suitability is however not tested directly, but rather, is determined by the negative definiteness of an aggregation matrix, the elements of which are determined by the interconnections and the Lyapunov functions of the subsystems.

So far this approach has been used to derive sufficient conditions for the asymptotic, exponential and finite-time stabilities of the overall system. In many situations, however, it is required only that the trajectories ultimately satisfy some predetermined bound, rather than approach some equilibrium state (e.g. Yoshizawa [6]). In this paper the aggregation–decomposition approach is used to establish sufficient conditions for the equi-ultimate boundedness of systems described by vector valued nonlinear ordinary differential equations. The simpler subsystems considered are equi-ultimately

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bounded, exponentially stable or unbounded. Three examples illustrate the application of these conditions and it is shown why similar conditions cannot in general be derived for other types of boundedness properties.

2. System description

A system S to be considered is described by an n-dimensional ordinary differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

and can be decomposed into s interconnected subsystems S_i described by n_i-dimensional $(\Sigma_{i=1}^{s} n_i = n)$ ordinary differential equations

$$\frac{dx_i}{dt} = g_i(t, x_i) + h_i(t, x) \qquad i = 1, 2, \cdots, s. \tag{2}$$

The functions $h_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ in (2) are called *interconnections* and the state vectors $x_i \in \mathbb{R}^{n_i}$ of the s subsystems S_i partition the state vector $x \in \mathbb{R}^n$ of system S, that is, $x = (x_1^T, x_2^T, \cdots, x_s^T)$. Also, it is assumed that $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and the $g_i : \mathbb{R}^+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are smooth enough to ensure global existence of solutions $x(t; t_0, x_0)$ of system S and $x_i(t; t_0, x_{i0})$ of the s free subsystems S_i^* described by the n_i-dimensional ordinary differential equations

$$\frac{dx_i}{dt} = g_i(t, x_i) \qquad i = 1, 2, \cdots, s. \tag{3}$$

Further, when a free subsystem S_i^* (3) is regarded as being exponentially stable, it is assumed that $x_i = 0$ is its unique equilibrium state.

For $i = 1, 2, \cdots, s$ an arbitrary norm $\|x_i\|_i$ will be used on $\mathbb{R}^{n_i}$, but for convenience the max norm

$$\|x\| = \max\{\|x_i\|_i ; i = 1, 2, \cdots, s\} \tag{4}$$

will be used on $\mathbb{R}^n$.

Following Yoshizawa [6], a free subsystem S_i^* is called *equi-ultimately bounded with bound* $B_i > 0$ if for every $\alpha > 0$ and $t_0 \in \mathbb{R}^+$ there is a $T_i = T_i(\alpha, t_0) \geqq 0$ such that

$$\|x_i(t; t_0, x_{i0})\|_i \leqq B_i$$

for all $t \geqq t_0 + T_i$ and all $\|x_{i0}\|_i \leqq \alpha$. An analogous statement holds for the overall system S (1).

In the sequel it is supposed that associated with each free subsystem S_i^* $(i = 1, 2, \cdots, s)$ is a nonnegative, differentiable Lyapunov function $V_i(t, x_i)$ which is defined for all $(t, x_i) \in \mathbb{R}^+ \times \mathbb{R}^{n_i}$ and satisfies

$$a_i(\|x_i\|_i) \leqq V_i(t, x_i) \tag{5}$$

for all $(t, x_i) \in \mathbb{R}^+ \times \mathbb{R}^{n_i}$ with $\|x_i\|_i \geqq B_i \geqq 0$, where $a_i(r)$ is a continuous, increasing, positive function of $r \geqq B_i$ with $a_i(r) \to \infty$ as $r \to \infty$; and

$$\begin{aligned} \frac{d}{dt} V_i(t, x_i) &= \frac{\partial}{\partial t} V_i(t, x_i) + [\text{grad } V_i(t, x_i)]^{\mathrm{T}} g_i(t, x_i) \\ &\leqq \mu_i c_i V_i(t, x_i) \end{aligned} \tag{6}$$

for all $(t, x_i) \in \mathbb{R}^+ \times \mathbb{R}^{n_i}$ with constants $c_i > 0$ and $\mu_i = +1$ or -1.

When $B_i > 0$ and $\mu_i = -1$, the above conditions are sufficient for the equi-ultimate boundedness with bound B_i of the free subsystem S_i^*. They are also necessary provided the differentiability of V_i is dropped and the derivative in (6) replaced by the upper right-hand derivative (Yoshizawa [6], theorem 11). With $B_i = 0$ and $\mu_i = -1$, they are satisfied by an exponentially stable subsystem and with $\mu_i = +1$ by certain unstable or unbounded subsystems.

Finally, it is assumed that the s interconnection functions $h_i(t, x)$ satisfy

$$\sum_{i=1}^{s} [\text{grad } V_i(t, x_i)]^{\mathrm{T}} h_i(t, x) \leqq \sum_{i,j=1}^{s} \alpha_{ij} \sqrt{\{V_i(t, x_i) V_j(t, x_j)\}} \tag{7}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$. Here the α_{ij} are real numbers with $\alpha_{ij} = \alpha_{ji}$. They may be positive, negative or zero.

3. Main result

A constant, symmetric $s \times s$ aggregation matrix $A = (a_{ij})$ and a bound $\bar{B} \geqq 1$ are defined for system S (1) as

$$a_{ij} = \mu_i c_i \delta_{ij} + \alpha_{ij} \qquad i, j = 1, 2, \cdots, s \tag{8}$$

and

$$\bar{B} = \max\{1, B_1, B_2, \cdots, B_s\} \tag{9}$$

where δ_{ij} is the Kronecker delta symbol.

The negative definiteness of this aggregation matrix A provides an easily tested sufficient condition for the equi-ultimate boundedness with bound $\bar{B}$ of system S (1) in terms of the equi-ultimate boundedness, exponential stability or unboundedness of the free subsystems S_i^* (3) and the interconnection functions $h_i(t, x)$, $i = 1, 2, \cdots, s$.

THEOREM. *Suppose that*

(a) *the Lyapunov functions* $V_i(t, x_i)$ *of the free subsystems* S_i^* (3) *satisfy* (5) *and* (6) *for* $i = 1, 2, \cdots, s$;

(b) *the interconnection functions* $h_i(t, x)$, $i = 1, 2, \cdots, s$, *satisfy* (7);

(c) *the aggregation matrix* $A = (a_{ij})$ *defined by* (8) *is negative definite.*

Then system S (1) *is equi-ultimately bounded with bound* $\bar{B}$.

PROOF. The function $V(t, x) = \sum_{i=1}^{s} V_i(t, x_i)$ is defined, nonnegative and differentiable for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$. From the definition of $\bar{B}$ (9) and the properties of the s functions $a_i(r)$, $i = 1, 2, \cdots, s$, the function

$$a(r) = \min\{a_i(r); i = 1, 2, \cdots, s\}$$

is a continuous, positive increasing function of $r \geqq \bar{B}$ with $a(r) \to \infty$ as $r \to \infty$.

By (4) and (9) for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ with $\|x\| \geqq \bar{B}$, there is a $j = 1, 2, \cdots$, or s, which depends on x, such that $\|x\| = \|x_j\|_j \geqq \bar{B} \geqq B_j$. Consequently

$$a(\|x\|) \leqq a_j(\|x_j\|_j) \leqq V_j(t, x_j) \leqq V(t, x)$$

that is, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ with $\|x\| \geqq \bar{B}$

$$a(\|x\|) \leqq V(t, x).$$

The derivative of $V(t, x)$ along trajectories of system S (1) is for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$

$$\begin{aligned}
\frac{d}{dt} V(t, x)\Big|_1 &= \frac{\partial}{\partial t} V(t, x) + [\text{grad } V(t, x)]^T f(t, x) \\
&= \sum_{i=1}^{s} \left\{ \frac{\partial}{\partial t} V_i(t, x_i) + [\text{grad } V_i(t, x_i)]^T (g_i(t, x_i) + h_i(t, x)) \right\} \\
&= \sum_{i=1}^{s} \frac{d}{dt} V_i(t, x_i)\Big|_3 + \sum_{i=1}^{s} [\text{grad } V_i(t, x_i)]^T h_i(t, x_i) \\
&\leqq \sum_{i=1}^{s} \mu_i c_i V_i(t, x_i) + \sum_{i,j=1}^{s} \alpha_{ij} \sqrt{\{V_i(t, x_i) V_j(t, x_j)\}} \\
&= (v_1, v_2, \cdots, v_s) A (v_1, v_2, \cdots, v_s)^T \\
&\leqq \lambda_{\max}(A) \sum_{i=1}^{s} V_i(t, x_i) \\
&= -|\lambda_{\max}(A)| V(t, x)
\end{aligned}$$

where $v_i = \sqrt{V_i(t, x_i)}$ for $i = 1, 2, \cdots, s$ and $\lambda_{\max}(A) < 0$ is the largest eigenvalue of the negative definite aggregation matrix A.

Thus $V(t, x)$ is a Lyapunov function satisfying sufficient conditions, analogous to (5) and (6), for the equi-ultimate boundedness with bound $\bar{B} > 0$ of the overall system S (1).

This completes the proof of the theorem.

4. Examples

The following three examples illustrate the application of the above theorem. Each consists of two free subsystems which are, respectively, both equi-ultimately bounded, equi-ultimately bounded and exponentially stable, and equi-ultimately bounded and unbounded. The function $T : \mathbb{R} \to \mathbb{R}^+$ is defined as

$$T(x) = \begin{cases} 1 & \text{for} \quad |x| \leqq 1 \\ 1/|x| & \text{for} \quad |x| \geqq 1. \end{cases}$$

Example 1.

System S is composed of two interconnected first order subsystems governed by

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{x_1}{t+1} + h_1(t, x_1, x_2) \\ \frac{dx_2}{dt} &= x_2 - x_2^3 + h_2(t, x_1, x_2) \end{aligned} \tag{10}$$

where

$$h_1(t, x_1, x_2) = \frac{T(x_1)}{t+1}(x_2^2 - 4)\exp\{-\tfrac{1}{2}x_1^2(t+1)\}$$

and

$$h_2(t, x_1, x_2) = T(x_2)\exp\{\tfrac{1}{2}(x_1^2 - 1)(t+1)\}.$$

Free subsystem S_1^*

$$\frac{dx_1}{dt} = -\frac{x_1}{t+1} \tag{11}$$

is equi-ultimately bounded with bound $B_1 = 1$. (It is also asymptotically stable, but not exponentially stable.) A suitable Lyapunov function is

$$V_1(t, x_1) = \exp\{(x_1^2 - 1)(t+1)\}$$

for which

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$$a_1(|x_1|) = |x_1|^2 - 1 \leqq V_1(t, x_1)$$

for all $(t, x_1) \in \mathbb{R}^+ \times \mathbb{R}$ with $|x_1| \geqq 1$; and

$$\frac{d}{dt} V_1(t, x_1)\bigg|_{11} = -(x_1^2 + 1) V_1(t, x_1) \leqq -V_1(t, x_1)$$

for all $(t, x_1) \in \mathbb{R}^+ \times \mathbb{R}$.

Free subsystem S_2^*

$$\frac{dx_2}{dt} = x_2 - x_2^3 \tag{12}$$

is equi-ultimately bounded with bound $B_2 = 2$. A suitable Lyapunov function is

$$V_2(t, x_2) = V_2(x_2) = \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ (x_2^2 - 4)^2 & \text{for} \quad |x_2| \geqq 2 \end{cases}$$

for which

$$a_2(|x_2|) = (|x_2|^2 - 4)^2 = V(x_2)$$

for all $x_2 \in \mathbb{R}$ with $|x_2| \geqq 2$; and

$$\begin{aligned} \frac{d}{dt} V_2(x_2)\bigg|_{12} &= \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ -4x_2^2(x_2^2 - 1)(x_2^2 - 4) & \text{for} \quad |x_2| \geqq 2 \end{cases} \\ &\leqq \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ -16(x_2^2 - 4)^2 & \text{for} \quad |x_2| \geqq 2 \end{cases} \\ &= \quad -16 V_2(x_2) \qquad \text{for all} \quad x_2 \in \mathbb{R}. \end{aligned}$$

The bounds (7) for these Lyapunov functions and interconnections are determined as follows

$$\begin{aligned} [\operatorname{grad} V_1(t, x_1)] h_1(t, x) &= 2x_1 T(x_1)(x_2^2 - 4) \exp\{\tfrac{1}{2}(x_2^2 - 2)(t + 1)\} \\ &\leqq 2(x_2^2 - 4) \exp\{\tfrac{1}{2}(x_1^2 - 1)(t + 1)\} \\ &\leqq 2\sqrt{\{V_1(t, x_1) V_2(x_2)\}} \end{aligned}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$; and

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$$[\text{grad } V_2(x_2)]h_2(t,x) = \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ 4x_2 T(x_2)(x_2^2 - 4)\exp\{\frac{1}{2}(x_1^2 - 1)(t+1) & \text{otherwise} \end{cases}$$

$$\leqq \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ 4(x_2^2 - 4)\exp\{\frac{1}{2}(x_1^2 - 1)(t+1)\} & \text{otherwise} \end{cases}$$

$$= \quad 4\sqrt{\{V_1(t,x_1)V_2(x_2)\}}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$. Hence

$$\sum_{i=1}^{2} [\text{grad } V_i(t, x_i)]h_i(t,x) \leqq 6\sqrt{\{V_1(t,x_1)V_2(x_2)\}}$$

so $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = \alpha_{21} = 3$. But $c_1 = 1$, $c_2 = 16$ and $\mu_1 = \mu_2 = -1$, so the aggregation matrix defined by (8) is

$$A = \begin{bmatrix} -1 & 3 \\ 3 & -16 \end{bmatrix}.$$

This is negative definite with eigenvalues $(-17 \pm \sqrt{261})/2 < 0$.

Thus the overall system is equi-ultimately bounded with bound $\bar{B} = \max\{1, 1, 2\} = 2$.

Example 2.

System S is composed of two interconnected subsystems, with $n_1 = 1$ and $n_2 = 2$, governed by

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{x_1}{t+1} + h_1(t,x) \\ \frac{dx_2}{dt} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_2 + h_2(t,x) \end{aligned} \tag{13}$$

where

$$h_1(t,x) = \frac{T(x_1)}{6(t+1)} \operatorname{sech}(x_{21}x_{22})$$

and

$$h_{2i}(t,x) = x_{2i}(1 + \sqrt{\{x_{21}^2 + x_{22}^2\}})^{-1} \exp\{\tfrac{1}{2}(x_1^2 - 1)(t+1)\}$$

for $i = 1$ and 2.

Free subsystem S_1^* is the same as in example 1, namely (11). It is equi-ultimately bounded with $B_1 = 1$ and a suitable Lyapunov function is again

$$V_1(t, x_1) = \exp\{(x_1^2 - 1)(t + 1)\}.$$

Free subsystem S_2^*

$$\frac{d}{dt}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} \tag{14}$$

is exponentially stable. A suitable Lyapunov function is

$$V_2(t, x_2) = V_2(x_2) = x_{21}^2 + x_{22}^2$$

for which $a_2(r) = r^2$ can be used with $B_2 = 0$, and

$$\left.\frac{d}{dt} V_2(x_2)\right|_{14} = -2(x_{21}^2 + x_{22}^2) = -2V_2(x_2).$$

The above Lyapunov functions and interconnections satisfy

$$\sum_{i=1}^{2} [\text{grad } V_i(t, x_i)]^{\mathrm{T}} h_i(t, x) \leqq \tfrac{1}{3} V_1(t, x_1) + 2\sqrt{\{V_1(t, x_1) V_2(x_2)\}}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. Thus $\alpha_{11} = \frac{1}{3}$, $\alpha_{12} = \alpha_{21} = 1$ and $\alpha_{22} = 0$. But $c_1 = 1$, $c_2 = 2$ and $\mu_1 = \mu_2 = -1$, so the aggregation matrix here is

$$A = \begin{bmatrix} -\frac{2}{3} & 1 \\ 1 & -2 \end{bmatrix}.$$

This is negative definite with eigenvalues $(-4 \pm \sqrt{13})/3 < 0$.

Thus system S (13) is equi-ultimately bounded with bound $\bar{B} = \max\{1, 1, 0\} = 1$.

Example 3.

System S here is composed of two interconnected subsystems, with $n_1 = 2$ and $n_2 = 1$, governed by

$$\begin{aligned} \frac{dx_1}{dt} &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x_1 + h_1(t, x) \\ \frac{dx_2}{dt} &= x_2 - x_2^3 + h_2(t, x) \end{aligned} \tag{15}$$

where

$$h_{11}(t, x) = -x_{11}(1 + t)\cosh(x_{12}x_2)$$
$$h_{12}(t, x) = -x_{12}(1 + |x_2|)\cosh\{T(x_{11})\}$$

and

$$h_2(t, x) = T(x_2)\sqrt{\{x_{11}^2 + x_{12}^2\}}.$$

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Free subsystem S_1^*

$$\frac{d}{dt}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \tag{16}$$

is unbounded. A suitable Lyapunov function is

$$V_1(t, x_1) = V_1(x_1) = x_{11}^2 + x_{12}^2$$

for which

$$\frac{d}{dt} V_1(x_1)\bigg|_{16} = x_{11}^2 + x_{12}^2 = V_1(x_1).$$

Here $a_1(r) = r^2$ and $B_1 = 0$.

Free subsystem S_2^* is the same as in example 1, namely (12). The same Lyapunov function

$$V_2(t, x_2) = V_2(x_2) = \begin{cases} 0 & \text{for} \quad |x_2| \leqq 2 \\ (x_2^2 - 4)^2 & \text{for} \quad |x_2| \geqq 2 \end{cases}$$

can be used.

The Lyapunov functions and interconnections here satisfy

$$\sum_{i=1}^{2} [\text{grad } V_i(x_i)]^{\mathrm{T}} h_i(t, x) \leqq -2V_1(x_1) + 2\sqrt{\{V_1(x_1)V_2(x_2)\}}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. Thus $\alpha_{11} = -2$, $\alpha_{12} = \alpha_{21} = 1$ and $\alpha_{22} = 0$. But $c_1 = 1$, $c_2 = 16$, $\mu_1 = 1$ and $\mu_2 = -1$, so the aggregation matrix here is

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -16 \end{bmatrix}$$

This is negative definite with eigenvalues $(-17 \pm \sqrt{229})/2 < 0$.

Thus system S (15) is equi-ultimately bounded with bound $\bar{B} = \max\{1, 0, 2\} = 2$.

Remark

Existence of solutions of the above composite systems is guaranteed by theorem 3.4 in Yoshizawa [7], using $V(t, x) = \sum_{i=1}^{s} V_i(t, x_i)$ as the required Lyapunov functions. The nonlinear nature of (10), (13) and (15) may result in nonuniqueness of solutions. The theorem proved in section 3 is still valid as analogous necessary and sufficient conditions to (5) and (6) hold in such cases (Stonier [5]).

5. Concluding remarks

The aggregation–decomposition method has been used to derive a simple algebraic criterion guaranteeing the equi-ultimate boundedness of a nonlinear composite system, composed of equi-ultimate bounded, exponentially stable or unbounded subsystems. It cannot in general be used for other types of boundedness properties because of the nature of their necessary and sufficient Lyapunov conditions. Example 8 of Yoshizawa [6] shows that the Lyapunov functions for equi-boundedness need not be continuous, let alone differentiable. For uniform boundedness and uniform-ultimate boundedness the Lyapunov functions need only be defined outside a possibly very large, neighbourhood of the origin (Yoshizawa [6], theorems 4, 5, 7). Consequently, the sum of Lyapunov functions of the free subsystems need not be defined everywhere in the exterior of any neighbourhood of the origin and thus cannot be used as a Lyapunov function for the composite system.

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