# CONSTRUCTIONS IN HYPERBOLIC GEOMETRY

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**Introduction.** In hyperbolic geometry we have three compasses, namely an ordinary compass for drawing ordinary circles with a given centre and a given radius, a hypercompass for drawing hypercycles with a given axis and a given radius, and a horocompass for drawing horocycles with a given diameter and passing through a given point.

Nestorovič has proved that everything that can be constructed by means of one of the compasses and a ruler, can be constructed by means of either of the other compasses and a ruler (**6**; **7**; **8**; **9**). Another important result we want to use in the following is a theorem by Schur concerning ruler constructions. Schur proved that even if we are only able to perform constructions in a finite part $\Omega'$ of the projective plane, we are also able to carry out constructions in the entire plane. A point is then said to be constructed if it is determined as the intersection between two lines in $\Omega'$. A line is said to be constructed if there are constructed two points on the line (**11**, pp. 15–22; see also **13**). Another theorem we shall use is: To a given right-angled triangle $\{a, b, c, A, B\}$ (i.e., a right-angled triangle with hypotenuse $c$, catheti $a$ and $b$, and opposite angles $A$ and $B$) there corresponds a second right-angled triangle[1] $\{\Delta(\frac{1}{2}\pi - A), a, \Delta(B), \Pi(c), \frac{1}{2}\pi - \Pi(b)\}$ and using the same transformation on this triangle we obtain a third right-angled triangle and so on. Triangle number six is identical with triangle number one. This sequence of five triangles is called the Engel Chain (**5**, pp. 40–41).

In this paper, we consider the following instruments: parallel-ruler, ruler, compass with fixed adjustment, and hypercompass with fixed adjustment.

**1. The parallel-ruler.** A parallel-ruler is, as in Euclidean geometry, an instrument for drawing a line through a given point and parallel to a given line. We shall also, as in Euclidean geometry, use the parallel-ruler as an ordinary ruler.

THEOREM 1. *Any construction in hyperbolic geometry that can be performed by means of a ruler and any of the three compasses, can be performed by means of a parallel-ruler.*

Let the hyperbolic plane be the interior of the "absolute" conic $\Omega$ situated in the real projective plane. If $U$ and $V'$ are two points[2] on $\Omega$ determined by

[1]The angle $\pi(p)$ is the angle of parallelism for the segment of length $p$. If $A = \pi(p)$ then $p = \Delta(A)$.

[2]In the following, $U$ and $V(U_1, V_1, U', V'$ and so on) will always be points on $\Omega$. If a line intersects $\Omega$ at $U$ (or $U_1, U_1', \ldots$) then its other end is called $V$ (or $V_1, V_1', \ldots$), unless otherwise indicated.

the lines $u$ and $u'$, we can always, by means of the parallel-ruler, draw the lines $AV'$ and $BU$ where $A$ and $B$ are two arbitrary points on $u$ and $u'$, respectively, neither of them being the point of intersection $u \cdot u'$. The points $U$ and $V'$ are now determined by pairs of lines. This means, according to the result of Schur, that we are able to join two points on $\Omega$ and to perform ruler constructions in the entire projective plane, operating only inside a finite part $\Omega'$ of the hyperbolic plane. Of course we have to choose $\Omega'$ so that it contains parts of the lines determining the points on $\Omega$. Consequently it is possible to make the following constructions:

1.01. *Given a segment $OA$ on one arm of an angle $V'OV$, construct $OB = OA$ where $B$ is on the other arm of the angle.*

Draw $UU'$ and $VV'$. Through their intersection draw a line through $A$. It will meet $U'V'$ at $B$.

*Proof.* $UOAV \barwedge U'OBV'$ and since $U \to U'$ and $V \to V'$, the perspectivity is a congruent transformation that takes $OA$ to $OB$.

1.02. *Given a segment $OA$, construct $C$ on the line $OA$ so that $OA = OC$ $(A \neq C)$.*

Draw any line $U_1V_1$ ($\neq OA$) through $A$. Draw $U_1O$ and $V_1O$ and call their other ends (i.e., intersections with $\Omega$) $U_1'$ and $V_1'$, respectively. Then $U_1'V_1'$ intersects $OA$ at $C$, and $OA = OC$.

1.03 *Given an angle $V'OV$, construct its internal bisector.*

Construct $A$ and $B$ on $OV$ and $OV'$, respectively, so that $OA = OB$ (1.01). $AV'$ and $BV$ will intersect at a point of the angle bisector.

1.04 *Given a line $UV$ and a point $P$ not on the line, construct the perpendicular line to $UV$ through $P$.*

Bisect the angle $UPV$ (1.03). The angle bisector is perpendicular to $UV$.

1.05 *Given an angle $VOV'$, construct an angle $VOV'' = 2 \cdot VOV'$.*

Take a point $P$ on $OV$ and construct the symmetric point $P'$ to $P$ with respect to $OV'$ (1.04 and 1.02). Then $VOP' = 2 \cdot VOV'$.

1.06 *Given a line $l$ and a point $P$ on $l$, construct a line $n$ perpendicular to $l$ through $P$.*

Draw any ray (not contained in $l$) beginning at $P$, and double the angle around $l$ (1.05). Construct the internal bisector $n$ of the supplement of the double angle.

1.07 *Given a segment $AB = p$, construct* $\Pi(p)$.

Construct the perpendicular line $l$ to $AB$ at $A$ (1.06) and draw a parallel line to $l$ through $B$.

1.08 *Given a segment $AB$ and a point $A'$, both on a line $UV$, construct $B'$ on $UV$ such that $AB = A'B'$.*

Take any line $U_1V_1$ ($\neq UV$) through $A$. If $U_2$ is the other end of $U_1B$ and $V_2$ is the other end of $V_1A'$ then $B'$ is $UV \cdot U_2V_2$.

*Proof.* Use Andrianov's theorem (**1**; for a more elegant proof see **2**): Let the four sides of a quadruply asymptotic crossed quadrangle meet an arbitrary transversal in points $A$, $C$, $B$, $D$; then $BC$ and $DA$ are congruent segments.

1.09 *Given a segment $AB$ and a ray $l'$ starting at $A'$, construct $B'$ on $l'$, such that $AB = A'B'$.*

Construct $B_1$ on $AA'$ such that $AB = AB_1$ (1.01) and the point $B_2$ on $AA'$ such that $AB_1 = A'B_2$ (1.08), and finally the point $B'$ on $l'$ such that $A'B_2 = A'B'$ (1.01).

1.10 *Given a segment $AB$, construct the mid-point $M$.*

If $A_1A$ and $BB_1$ are equal and both perpendicular to $AB$ at $A$ and $B$, respectively (1.06 and 1.09), and $A_1$ and $B_1$ are on opposite sides of $AB$, then $A_1B_1$ intersects $AB$ at $M$.

1.11 *Given two segments $a$ and $c$ ($a < c$), construct a right-angled triangle with hypotenuse $c$ and cathetus $a$.*

Construct $\Pi(c)$ (1.07) and use $\Pi(c)$ and $a$ to construct the second triangle of the Engel chain (cathetus $a$ fixed) (1.09 and 1.06), so as to obtain $B$ as the angle of parallelism of the hypotenuse (1.07). Construct then the right-angled triangle containing this angle $B$ and the adjacent cathetus $a$ (1.09 and 1.06) The hypotenuse is $c$ and the required triangle is constructed.

1.12 *Given a point $O$, a segment $r = AB$ and a line $l$ intersecting the circle $O(r)$, construct the points of intersection.*

Construct $OO_1$ perpendicular to $l$ (with $O_1$ on $l$) (1.04) and the right-angled triangle with hypotenuse $AB$ and cathetus $OO_1$ (1.11). The other cathetus $O_1C$ can now be moved to $l$ (1.09). $C_1$ and $C_2$ (where $O_1C = O_1C_1 = O_1C_2$) are the intersections.

1.13 *Given two points $O_1$ and $O_2$ and two segments of length $r_1$ and $r_2$, construct the intersections of the circles $O_1(r_1)$ and $O_2(r_2)$.*

Let $d$ denote the distance $O_1O_2$, and $b$ the distance from $O_1$ to the intersection of $O_1O_2$ and the radical axis; then

$$\tanh b = \frac{\cosh r_1 \cosh d - \cosh r_2}{\cosh r_1 \sinh d}.$$

The segment $b$ can be constructed in the following way: Construct $O_1O_1' = r_1$ and $O_2O_2' = r_2$, both perpendicular to $O_1O_2$ (1.06 and 1.09), with $O_1'$ and $O_2'$

on the same side of $O_1O_2$. Construct the mid-point $M$ of $O_1'O_2'$ (1.10) and construct the line $m$ perpendicular to $O_1'O_2'$ at $M$ (1.06). Let $m$ meet $O_1O_2$ at $A$; then $AO_2 = b$.

*Proof.* $O_1'A = AO_2'$. If $AO_2 = x$, so that $O_1A = d - x$, then

$$\tanh x = \frac{\cosh r_1 \cosh d - \cosh r_2}{\cosh r_1 \sinh d}.$$

Since tanh is a single-valued function, we have $x = b$ and 1.13 reduces to 1.12 **(8)**.

Any construction that can be performed by means of a compass and a ruler can then be performed by means of a parallel-ruler, and this result, along with the theorem of Nestorovič, proves Theorem 1.

## 2. Analogues of Steiner's construction

THEOREM 2. *Any construction that can be performed by means of any of the three compasses and ruler, can be carried out with the ruler alone if there is drawn somewhere in the plane* (i) *a circle with its centre and two parallel lines, or* (ii) *a hypercycle with its axis and two parallel lines with their common end not on the axis, or* (iii) *a horocycle with one diameter and two parallel lines with their common end not at the centre of the horocycle* **(12)**.

(i) Let $\Omega$ again be the absolute conic, $\omega$ the given circle with centre $A$, and $P$ the common end of the two given parallel lines. We want to prove that if $O$ is any given ordinary point and $l$ is any given line, we are able to construct the parallels from $O$ to $l$. When this is proved, Theorem 1 will give us Theorem 2(i). Let $\Omega'$ be a finite part of the hyperbolic plane containing $\omega$, $O$, a part of $l$, and a part of the two lines that define $P$. By means of two harmonic constructions, we can obtain the polar $a$ of $A$ with respect to $\omega$. This is also the absolute polar of $A$ (i.e., the polar with respect to $\Omega$). The construction can be carried out by using the ruler only inside $\Omega'$. Join $P$ and $A$ and let $Q$ be one of its intersections with $\omega$. The homology $H$, with axis $a$, centre $A$, taking $Q$ to $P$, will take $\omega$ to $\Omega$ **(4**, pp. 173–174**)**. $H^{-1}$ will then take $\Omega$ to $\omega$.

Construct now the images $O'$ and $l'$ of $O$ and $l$ in the homology $H^{-1}$. Join $O'$ to the intersections, $P_1$ and $P_2$, of $\omega$ and $l'$, and construct the images of $O'P_1$ and $O'P_2$ in the homology $H$. These lines are the parallel lines desired.

(ii). Given a hypercycle $\omega$, with axis $a$, and two parallel lines with end $P$ ($P$ not on $a$), we can again use two harmonic constructions to obtain the pole $A$ of $a$ with respect to $\omega$. The constructions can be carried out by using the ruler inside a suitable finite part $\Omega'$ of the hyperbolic plane. The point $A$ is also the absolute pole of $a$. Let $AP$ intersect $\omega$ at $Q$, as before. The homology $H$, with axis $a$, centre $A$, taking $Q$ to $P$, will take $\omega$ to $\Omega$. Using the same principle as above, we are able to construct a line through a given point parallel to a given line. This proves Theorem 2(ii).

(iii). In the third case, where we have a horocycle $\omega$ with centre $A$, the homology is an elation. But here, the centre $A$ is not a given point. To determine $A$, we have to construct a second diameter of the horocycle. This can be done as follows: Let $B$ be the ordinary end-point of the given diameter $d$, and let $F$ be any other given point on $\omega$ (neither $B$ nor $A$). Choose on the conic three distinct points $C$, $D$, $E$, none of them coincident with $B$ or $F$, and let $l$ be the join of the intersections $d \cdot DE$ and $BC \cdot EF$. Join $F$ to the intersection $l \cdot CD$. Since this is a line passing through $A$, it is a diameter. For, $l$ is the Pascal line of the hexagon $ABCDEF$.

The centre $A$ is now determined by two parallel lines. The tangent $a$ to $\omega$ can be constructed as the Pascal line of the hexagon $AABCDE$. This is also the tangent at $A$ to $\Omega$. All the above constructions can be performed by ruler inside a suitable finite part $\Omega'$ of the hyperbolic plane. The elation, with centre $A$ and axis $a$, taking $Q$ to $P$ (where $P$ is the given end and $Q$ an intersection of $AP$ and $\omega$), plays now the same role as the homology $H$ in (i) and (ii).

As shown by Obláth in connection with Steiner constructions (**10**; for a more elegant proof see **3**), it is sufficient if we are given only an arc, however small, of the circle, hypercycle, or horocycle. Hüttemann's proof, being projective, is valid here.

## 3. Compasses with fixed adjustment

THEOREM 3. *Every construction that can be performed by any one of the three compasses and ruler can be performed by either* (i) *a compass with fixed adjustment and a ruler or* (ii) *a hypercompass with fixed adjustment and a ruler.*

If we can prove that by means of our instruments we are able to construct a pair of parallel lines, then Theorem 3 will follow from Theorem 2.

(i) Draw a circle $\omega$ with centre $A$ and a diameter $l$. Construct (by means of two harmonic constructions) the pole $L$ of $l$ with respect to $\omega$. This is also the absolute pole of $l$. Given a point $P$ either on $l$ or outside $l$, $PL$ is then perpendicular to $l$. All the constructions can be performed inside a suitable part $\Omega'$ of the hyperbolic plane.

The usual parallel construction can now be carried out, taking the arbitrary radius to be the radius given by the adjustment.

(ii). Perpendicular lines can be constructed in the same way as in (i), using a hypercycle instead of a circle.

Two parallel lines can be constructed in the following way: Draw an acute angle $AOB$ and construct on $OA$ a point $A_1$ so that $OA_1$ is equal to the adjustment of the hypercompass. Construct $l$ perpendicular to $OB$ at $O$ and $l_1$ perpendicular to $l$ through $A_1$. Let the hypercycle with axis $OB$ intersect $l_1$ at $S$. The line $m$ perpendicular to $OB$ through $S$ is parallel to $OA$. As a matter of fact, this is only the usual parallel construction here performed by a hypercompass instead of the ordinary compass.

**4. The common perpendicular to two skew lines.** Finally we wish construct (e.g., by means of ruler and compass) the common perpendicular to two skew lines in hyperbolic 3-space.

Let the given lines be $g$ and $g'$. Take an arbitrary point $A$ on $g$ and construct the two lines $AE_1$ and $AE_2$, where $E_1$ and $E_2$ are the ends of $g'$. On $g$, construct points $M_1$, $M_2$, such that $M_iE_i$ is parallel to $AE_i$ and perpendicular to $g(i = 1 \text{ or } 2)$. If $M$ is the mid-point of $M_1M_2$ and $MN$ is perpendicular to $g'$, then $MN$ is the required common perpendicular.

*Proof.* Project the whole figure on the plane $Ng$. If the projections of $E_1$ and $E_2$ are $F_1$ and $F_2$, respectively, then $MM_1F_1N \equiv MM_2F_2N$ and therefore $\angle M_1MN = \angle M_2MN = \frac{1}{2}\pi$.

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