ON THE C-PROJECTIVITY OF IDEALS IN BANACH ALGEBRAS[†]

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The notion of projective Banach module was defined by Helemskii in **[1]**—the paper which properly founded the homological theory of Banach algebras. The same author introduced the definition of the (relatively) flat Banach module in **[2]**. Recently M. C. White **[3]** modified both of those definitions, introducing so called C-projective and C-flat Banach modules.

For a given constant $C > 0$ the Banach module X over a Banach algebra A, (abbreviated below as "module"), is called *C-projective* **[3]** if for arbitrary modules Y, Z and morphism $\phi : X \to Z$, epimorphism $\sigma : Y \to Z$, and bounded linear operator $j : Z \to Y$ such that $\sigma j = 1$, there exists a morphism $\psi : Z \to Y$ such that $\sigma\psi = \phi$ and $\|\psi\| \leq C\|\phi\|\|j\|$. As well as **[1]**, the paper **[3]** gives us the more useful equivalent definition of C-projectivity. Namely, a module X is *C-projective* if and only if the morphism of external multiplication $\pi : A \hat{\otimes} X \to X$, defined by the formula $\pi(a \otimes x) = ax$, has a right inverse morphism ρ such that $\|\rho\| \leq C$. Here the symbol $\hat{\otimes}$ denotes the projective tensor product of Banach spaces **[4]**. If $\|\rho\| = C$ and there is no right inverse with a norm smaller than C, then it is natural to say that X is *exactly C-projective*. In this paper we give answers to two questions that (directly or not) were put in **[3]**. First, for arbitrary $C > 1$, we give an example of an exactly C-projective Banach A-module. (Moreover, it is a maximal ideal in a uniform algebra A.) Note that C-projectivity is impossible for $C < 1$ and for $C = 1$ there exist trivial examples: consider for example any maximal ideal in the disc-algebra, corresponding to an inner point of the disc. Second, we shall show that C-projectivity does not possess the same "continuity property" as C-flatness **[3]**: that is, there exists a module (again a maximal ideal in the uniform algebra) that is $(C + \epsilon)$-projective for all $\epsilon > 0$ but not C-projective.

As usual, we denote by $A(E)$ the uniform algebra of functions that are continuous on the given compact subset $E \subset \mathbf{C}$ and analytic in its interior.

EXAMPLE 1. Consider the compact subset $K = D \cup E$ *of* $\mathbf{C} \times \mathbf{R}$, where $D = \{(z, 0) : |z| \leq 1\}$ is the closed disc and $E = \{(z, t) : |C|^{-1} \leq |z| \leq 1, 0 < t \leq 1\}$ is the cylindric annulus. (We denote by E_t its section, where t is constant.) Consider the uniform algebra

$$A = \{f \in C(K), f(z, 0) \in A(D), f(z, t) \in A(E_t) \quad \text{for} \quad t \in (0, 1]\}$$

PROPOSITION 1. *For the maximal ideal $M \subset A$, corresponding to the point $O = (0, 0)$,*
(1) *M is a C-projective Banach A-module,*
(2) *M is not k-projective Banach A-module for any $k < C$.*

Proof. (1) Consider two functions: $h \in M$ such that $h(z, t) = z$ for $(z, t) \in K$ and $f(z, t) = 1/z$ for $(z, t) \in K \setminus O$. Note that $\|h\| = 1$ and, for each $m \in M$, fm is defined on $K \setminus O$ and we extend the definition to K by continuity. We have $\|fm\| \leq C\|m\|$, because $|f| \leq C$ on

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E_t and multiplication by z preserves the uniform norm in $A(D)$. Now define the morphism $\rho : M \to A \hat{\otimes} M$ by the formula $\rho(m) - mf \otimes h$. Obviously $\pi(\rho(m)) = mfh = m$ and $\|\rho(m)\| = \|mf\|\|h\| \leq C\|m\|$. Hence part (1) is proved.

(2) Assume the contrary; then for the epimorphism $\omega : A \to \mathbf{C}$ of evaluation at the point O (which has a right inverse operator j given by the natural injection) and morphism $\phi : M \to \mathbf{C}$ given by the formula $\phi(f) = \frac{\partial f}{\partial z}(O)$ there exists a morphism $\psi : M \to A$ such that $\omega\psi = \phi$ and $\|\psi\| \leq k\|\phi\|\|j\| = k$. But each morphism $\psi : M \to A$ is a multiplication by some function $g \in C(K \backslash O)$. See, for example, the standard argument in [5]. Since $\|gm\| \leq k\|m\|$ we can conclude that $|g| \leq k$ on each annulus E_t. Since $\omega\psi = \phi$ it is evident that $g = \frac{1}{z} + a$ on $D \backslash O$, where $a \in A(D)$. By continuity $|\frac{1}{z} + a| \leq k$ on E_0 and hence on the circle $T = \{(z, 0) : |z| = 1/C\}$. Therefore $|1 + az| = |gz| \leq k|z| = k/C < 1$ on T. This contradicts the maximum modulus principle, and so M is an exactly C-projective A-module.

Example 2. Consider the compact subset $K = D \cup (\cup_{n>C+1} E_n)$ of $\mathbf{C} \times \mathbf{R}$, where D is the same disc and $E_n = \left\{(z, \frac{1}{n}) : \frac{1}{C} - \frac{1}{n} \leq |z| \leq 1 + \frac{1}{n}\right\}$ is the closed annulus. Then

$$A = \left\{ f \in C(K) : f(z, 0) \in A(D), f\left(z, \frac{1}{n}\right) \in A(E_n), f\left(\frac{1}{C}, \frac{1}{n}\right) = f\left(\frac{1}{C}, 0\right), \forall n > C + 1 \right\}$$

is a uniform algebra. Let M be the maximal ideal, corresponding to the point $O = (0, 0)$.

Proposition 2. *For the maximal ideal $M \subset A$, corresponding to the point $O = (0, 0)$,*
(1) *M is a $(C + \epsilon)$-projective Banach A-module for all $\epsilon > 0$,*
(2) *M is not a C-projective Banach A-module.*

Proof. (1) Fix $n > C + 1$ and let two functions $h \in M$ and $f \in C(K \backslash O)$ be defined by $h(z, t) = 1$ on E_k, $(C < k \leq n - 1)$, but $h(z, t) = z$ on E_k, $(k \geq n)$, and $f(z, t) = 1$ on E_k, $(C < k \leq n - 1)$, but $f(z, t) = \frac{1}{z}$ on E_k, $(k \geq n)$ and on $D \backslash O$. Note that $\|h\| = 1 + \frac{1}{n}$ and, for each $m \in M$, we have $\|fm\| \leq nC/(n - C)\|m\|$, because $|f| \leq nC/(n - C)$ on E_n and the multiplication by z preserves the uniform norm in $A(D)$. Now define the morphism $\rho : M \to A \hat{\otimes} M$ by the formula $\rho(m) = mf \otimes h$. Obviously $\pi(m) = mfh = m$ and $\|\rho(m)\| = \|fm\| \|h\| \leq nC/(n - C)\|m\|\left(1 + \frac{1}{n}\right) = C[1 + (C + 1)/(n - C)]\|m\|$. Since n is arbitrarily large part (1) is proved.

(2) Repeating the argument from Proposition 1 we obtain a function $g \in C(K \backslash O)$ such that $|g| \leq C$ on each annulus E_n and $g = \frac{1}{z} + a$ on $D \backslash O$, where $a \in A(D)$. As the inner circles T_n of E_n tend to the circle T of radius $\frac{1}{C}$ from D, by continuity we obtain $|g| \leq C$ on T. Hence $|1 + za| = |gz| \leq C.\ \frac{1}{C} = 1$ on T. Using the maximum modulus principle we conclude that $a \equiv 0$. Thus $g \equiv \frac{1}{z}$ on D and so $g\left(\frac{1}{C}, 0\right) \equiv C$; also by definition of the algebra A, $g\left(\frac{1}{C}, \frac{1}{n}\right) = C$, for all n. Applying the maximum modulus principle to each annulus E_n, we conclude that $g \equiv C$ on E_n. By continuity $g \equiv C$ on T giving a contradiction.

Note that both examples represent so-called non-idempotent maximal ideals; (that is $M \neq \overline{M^2}$). We know almost nothing about the exact estimates of C-projectivity in the idempotent case. If we analyse Helemskii's original proof of the projectivity of the algebra of convergent sequences c_0 (and the algebra l_1 of summable sequences) one can see that both these algebras are 1-projective [1]. The author can generalize this result to the algebra $C(K)$, where X is a semi-discrete compact set. Let X be a compact set; denote by X' the set of its accumulation points and $X^{(n+1)} = X^{(n)'}$ ($n \in \mathbf{N}$). If $X^{(n)}$ is empty for some $n \in \mathbf{N}$, we say X is a *semidiscrete compact set*. As for the algebra $C[0; 1]$ we can only see from **[1]** that the maximal ideals in it are 2-projective, but the constant 2 seems not to be the best possible.

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