# DIAMETERS OF POLYHEDRAL GRAPHS

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**1. Setting of the problem.** The *distance* between two vertices of a connected finite graph is the smallest number of edges forming a path that joins the two vertices, and the *diameter* of the graph is the largest integer which is realized as the distance between two vertices of the graph. We are concerned here with the diameters of two graphs associated with a $d$-dimensional convex polytope $P$ (called henceforth a *$d$-polytope*). The *graph* $\Gamma(P)$ *of* $P$ is the 1-complex formed by the vertices and edges of $P$, and the *polar graph* $\Pi(P)$ *of* $P$ is the 1-complex whose vertices correspond to the $(d-1)$-faces of $P$, with two vertices joined by an edge in $\Pi(P)$ if and only if the corresponding $(d-1)$-faces intersect in a $(d-2)$-face of $P$. The diameters of $\Gamma(P)$ and $\Pi(P)$ will be denoted respectively by $\delta(P)$ (called the *diameter* of $P$) and $\phi(P)$ (called the *face-diameter* of $P$).

The class of all $d$-polytopes will be denoted by $\mathbf{P}_d$, while the subclasses $\mathbf{P}_d^v$ and $\mathbf{P}_d^f$ consist respectively of the $d$-polytopes which are *simple* (each vertex incident to $d$ edges) and those which are *simplicial* (each $(d-1)$-face incident to $d$ $(d-2)$-faces). We are interested in the maximum of $\delta(P)$ or $\phi(P)$ as $P$ ranges over various subclasses of $\mathbf{P}_d$, and especially in the relationship of $\delta(P)$ and $\phi(P)$ to the numbers $d$ and $f_s(P)$, where $f_s$ denotes the number of $s$-faces. Let us define

$$\Delta_s(d, n) = \max\{\delta(P): P \in \mathbf{P}_d \text{ and } f_s(P) \leqslant n\},$$
$$\Phi_s(d, n) = \max\{\phi(P): P \in \mathbf{P}_d \text{ and } f_s(P) \leqslant n\},$$

and similarly define $\Delta_s^v(d, n)$ and $\Phi_s^v(d, n)$ (where $P$ ranges over $\mathbf{P}_d^v$) as well as $\Delta_s^f(d, n)$ and $\Phi_s^f(d, n)$ (where $P$ ranges over $\mathbf{P}_d^f$).

Now suppose $P$ is a $d$-polytope in $\mathfrak{R}^d$, with $0 \in \operatorname{int} P$, and let $P^0$ denote the polar body

$$P^0 = \{y: \langle x, y\rangle \leqslant 1 \text{ for all } x \in P\},$$

where $\langle\ ,\ \rangle$ is the inner product in $\mathfrak{R}^d$. From the standard polarity theory (Weyl, **15**) it follows that $P^0$ is a $d$-polytope,

$$f_s(P) = f_{d-1-s}(P^0),$$
$$\Pi(P) \text{ is isomorphic with } \Gamma(P^0),$$
$$\phi(P) = \delta(P^0),$$
$$P \in \mathbf{P}_d^v \text{ if and only if } P^0 \in \mathbf{P}_d^f,$$

and

$$P \in \mathbf{P}_d^f \text{ if and only if } P^0 \in \mathbf{P}_d^v.$$

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Consequently,

$$\Phi_s = \Delta_{d-1-s}, \quad \Phi_s^v = \Delta_{d-1-s}^f, \quad \text{and } \Phi_s^f = \Delta_{d-1-s}^v.$$

In view of the above consequences of the polarity theory, attention can be confined to the functions $\Delta_s$, $\Delta_s^v$, and $\Delta_s^f$, and to the *d-polyhedral graphs* $\Gamma(P)$ for $P \in \mathbf{P}_d$. However, the polar equivalents are useful in the study of polyhedral graphs. These graphs have been studied by various authors (Balinski **1**, **2**, Brown **3**, Grünbaum and Motzkin **7**, **8**, Saaty **11**, Steinitz and Rademacher **12**, Tait **13**, and Tutte **14**), but for $d \geqslant 4$ there is at present no graph-theoretic characterization of $d$-polyhedral graphs. Thus the subject involves both combinatorial and geometric complexities, and leads to many unsolved problems.

Our attention here is confined to the functions $\Delta_0$, $\Delta_{d-1}$, $\Delta_0^v$, $\Delta_{d-1}^v$, $\Delta_0^f$, $\Delta_{d-1}^f$ and their polar equivalents, defined for $n > d > 1$. We are able to show that

$$(1) \qquad \Delta_0 = \Delta_0^f \quad \text{and} \quad \Delta_{d-1} = \Delta_{d-1}^v.$$

Grünbaum and Motzkin **(7)** observed that (for $n > d > 1$)

$$(2) \qquad \Delta_0(d, n) = \left[\frac{n-2}{d}\right] + 1,$$

where $[h]$ is the largest integer $\leqslant h$, and they suggested that $\Delta_0^v$ should be significantly less than $\Delta_0$, conjecturing in particular that among the simple 3-polytopes with a given number $2k$ of vertices, maximum diameter is achieved by the $k$-sided prisms. We show that the specific conjecture is incorrect, for indeed

$$(3) \qquad \Delta_0^v(3, 2k) = \Delta_0(3, 2k).$$

However, their suggestion may be correct when $d \geqslant 4$, for then our construction shows only that

$$(4) \qquad \Delta_0^v(d, n) \geqslant (d-1)\left[\frac{n-2}{2^d-2}\right] + 1 \qquad \text{for } n \geqslant 2^d.$$

The same construction (discussed in §2) shows that

$$(5) \qquad \Delta_{d-1}^f(d, n) \geqslant \left[\frac{n-2d}{2^d-2}\right] + 2$$

and

$$(6) \qquad \Delta_{d-1}(d, n) \geqslant (d-1)\left[\frac{n}{d}\right] - d + 2.$$

The lower bound in (6) may be very good, for we show also that

$$(7) \qquad \Delta_{d-1}(d-n) = \left[\frac{d-1}{d}n\right] - d + 2 \qquad \text{if } d \leqslant 3 \text{ or } n \leqslant d + 4.$$

This suggests that (7) *may* hold for all $n > d > 1$, but we have been unable to establish a general upper bound that is anywhere near this conjecture. A weak upper bound may be obtained from (2) in conjunction with results of Gale **(6)** and Klee **(9)** that limit the number of vertices of a $d$-polytope in terms of the number of its $(d-1)$-faces.

Sharp upper bounds for $\Delta_{d-1}(d, n)$ would be of special interest in connection with linear programming, where there is a long-standing conjecture to the effect that $\Delta_{d-1}(d, 2d) = d$. This conjecture is discussed (but not proved, except for $d \leqslant 4$) in §3 below, where reasoning suggested by Ernst Straus seems to show that if the conjecture fails, then its failure may be connected with the existence of neighbourly polytopes. The face-diameters of certain neighbourly polytopes (the cyclic polytopes) are calculated in §4.

**2. A stack of simplices.** Several of the results stated above are based on a simple construction which we now describe. For $d \geqslant 2$ and $j \geqslant 1$, let $P(d, j)$ be a $d$-polytope which is generated by $j + 1$ $(d-1)$-simplices in $\mathfrak{R}^d$, arranged in parallel hyperplanes so that adjacent simplices are antihomothetic and so that the relative boundary of each of these simplices lies in the boundary of $P(d, j)$. For the case in which $d = 3$ and $j = 2$, the Schlegel diagram of such a polytope is given in Figure 1.

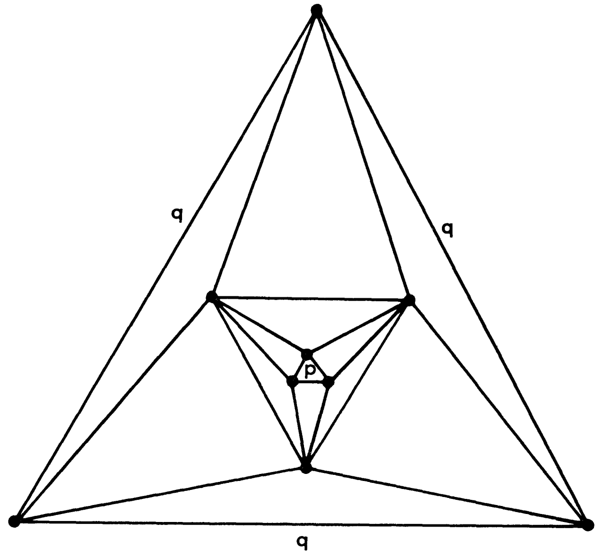


FIGURE 1

Let $S_0, S_1, \ldots, S_j$ be the $j + 1$ $(d-1)$-simplices which generate $P(d, j)$, and let $V_i$ denote the set of all vertices of $S_i$. For $1 \leqslant i \leqslant j$, let $\tau_i$ be the anti-homothety (a reflection in the origin, followed by a dilation or contraction and then by a translation) which carries $S_{i-1}$ onto $S_i$ and $V_{i-1}$ onto $V_i$. Then the edges of $P(d, j)$ are exactly the edges of the various $S_i$'s together with

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segments of the form $[v, \tau_i(w)]$, where $[v, w]$ is an edge of $S_{i-1}$. The $(d-1)$-faces of $P(d,j)$ are all simplices, namely, the sets $S_0$ and $S_j$ together with all the sets of the form $\text{con}(U \cup \tau_i(V_{i-1} \sim U))$, where $1 \leqslant i \leqslant j$ and $U$ is a *proper* subset of $V_{i-1}$. The following can then be verified.

2.1. *The simplicial d-polytope $P(d,j)$ has*

| | |
|---|---|
| $d(j+1)$ | *vertices* |
| $(2^d-2)j+2$ | *$(d-1)$-faces* |
| *diameter $j$* | *(but diameter 2 when $j = 1$)* |
| *face-diameter* | $(d-1)j+1$. |

Now we can prove the following statement.

2.2. *For $n > d > 1$,*

$$\Delta_0^f(d,n) = \Delta_0(d,n) = \left[\frac{n-2}{d}\right] + 1.$$

*Proof.* As was noted by Grünbaum and Motzkin **(7)**, the fact that $\Delta_0(d,n) \leqslant [(n-2)/d] + 1$ follows at once from Balinski's observation **(2)** that a $d$-polyhedral graph must be $d$-connected, in conjunction with a theorem of Menger **(10)** and Whitney **(16)** asserting that in a $d$-connected graph, any two vertices can be joined by $d$ distinct paths which have only their end points in common.

Obviously, $\Delta_0^f(d,n) \leqslant \Delta_0(d,n)$. Thus, to complete the proof of 2.2, it suffices to exhibit, for $n > d > 1$, a simplicial $d$-polytope which has at most $n$ vertices and has diameter $[(n-2)/d] + 1$. Let $P''(d,j)$ be the simplicial $d$-polytope which is obtained from $P(d,j)$ by adding pyramidal caps on $S_0$ and $S_j$. Then $P''(d,j)$ has $d(j+1)+2$ vertices and its diameter is $j+2$; this is true also for the dipyramid $P''(d,0)$. Now suppose that $n = ld + m$ with $l \geqslant 1$ and $0 \leqslant m < d$. Let $P = P''(d, l-1)$ when $m \geqslant 2$, $P = P''(d, l-2)$ when $l \geqslant 2$ and $m = 0$ or 1, and let $P$ be a $d$-simplex when $n = d+1$. In each case, $P$ is of diameter $[(n-2)/d] + 1$ and $P$ has at most $n$ vertices.

Grünbaum and Motzkin observed **(7**, p. 158**)** that for arbitrary $n > d > 1$ there is a $d$-polyhedral graph which has $n$ vertices, has diameter $\Delta_0(d,n)$, and is of valence $\leqslant d+1$ (where the *valence* of a graph is the maximum number of edges incident at a vertex). Such a graph is obtained from a $d$-polytope generated by a stack of mutually parallel and mutually homothetic $(d-1)$-simplices in $\mathfrak{R}^d$, with suitable caps added (when necessary) on the terminal simplices in the stack. They asked how the maximum possible diameter is affected when attention is restricted to $d$-polyhedral graphs of valence $d$. In other words, what is the value of $\Delta_0^v(d,n)$?

2.3. *For $n \geqslant 2^d > 2$,*

$$(d-1)\left[\frac{n-2}{2^d-2}\right] + 1 \leqslant \Delta_0^v(d,n) \leqslant \left[\frac{n-2}{d}\right] + 1.$$

*Proof.* Clearly $\Delta_0^v \leqslant \Delta_0$, so the right-hand inequality holds for all $n > d > 1$. Now suppose that $n \geqslant 2^d > 2$ and let $j = [(n-2)/(2^d-2)] \geqslant 1$. We see from 2.1 that the simplicial $d$-polytope $P(d, j)$ has at most $n$ $(d-1)$-faces and that its face-diameter is equal to $(d-1)j + 1$. The left-hand inequality of 2.3 follows from consideration of a polytope polar to $P(d, j)$.

Obviously $\Delta_0^v(2, \cdot) = \Delta_0(2, \cdot)$.

2.4. *For $n > 3$, the numbers $\Delta_0^v(3, n)$ and $\Delta_0(3, n)$ never differ by more than* 1, *and they are equal unless $n \equiv 5 \pmod 6$.*

*Proof.* From 2.3. we see that for $n \geqslant 8$

$$2\left[\frac{n-2}{6}\right] + 1 \leqslant \Delta_0^v(3, n) \leqslant \Delta_0(3, n) = \left[\frac{n-2}{3}\right] + 1,$$

whence the numbers $\Delta_0^v(3, n)$ and $\Delta_0(3, n)$ cannot differ by more than one. For $3 < n < 8$, the same conclusion comes from considering a tetrahedron and a triangular prism. Note that a simple 3-polytope must have an even number $v$ of vertices, for $3v = 2e$, where $e$ is the number of edges. To complete the proof of 2.4, it suffices to show that for each even integer $n > 3$ there exists a simplicial 3-polytope having $n$ vertices and diameter $[(n-2)/3] + 1$. For then $\Delta_0^v(3, n)$ and $\Delta_0(3, n)$ cannot be different unless $n$ is odd and $[(n-2)/3] > [(n-3)/3]$. The desired polytopes may be obtained by polarity from the polytopes $P(3, j)$ or slight modifications of the latter. But, instead, we exhibit directly the relevant 3-polyhedral graphs of valence 3. For the case $n = 20$, such a graph is depicted in Figure 2.

For $n = 6j$, remove the inner triod and the outer spikes, placing $p$ at $x$

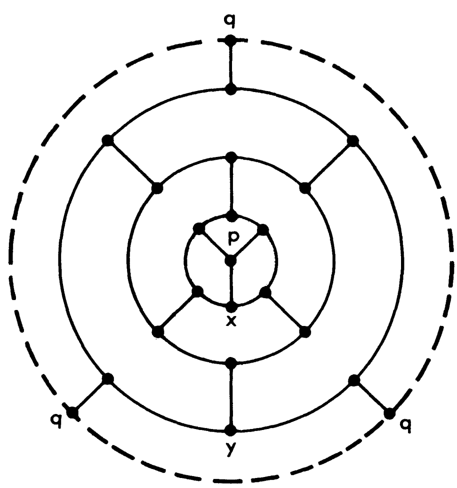


FIGURE 2

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and $q$ at $y$. Then the distance between $p$ and $q$ is $2j$. For $n = 6j + 4$, add the broken arcs and distinguish the three points labelled $q$. The distance between $p$ and $q$ is $2j + 1$. Each of the graphs described is a 3-connected planar graph and hence must be 3-polyhedral (Grünbaum and Motzkin, **8**).

We turn now to the functions $\Delta_{d-1}(d, \cdot)$. These appear to be much less tractable than the functions $\Delta_0(d, \cdot)$, principally because a restriction on the number of $(d-1)$-faces associated with a $d$-polyhedral graph does not show up in the graph-theoretic structure nearly as plainly as does a restriction on the number of vertices.

2.5. *For* $n > d > 1$,

$$\Delta_{d-1}^{v}(d, n) = \Delta_{d-1}(d, n) \geqslant (d-1)[n/d] - d + 2.$$

*Proof.* To see that $\Delta_{d-1}^{v}(d, n) \geqslant (d-1)[n/d] - d + 2$, let $j = [n/d] - 1$. The inequality is trivial when $j = 0$. When $j \geqslant 1$ we see from 2.1 that a $d$-polytope polar to $P(d, j)$ is simple; it is of diameter

$$(d-1)j + 1 = (d-1)[n/d] - d + 2,$$

and the number of its $(d-1)$-faces is

$$d(j+1) = d[n/d] \leqslant n.$$

To complete the proof of 2.5 we want to show that $\Delta_{d-1}^{v}(d, n) \geqslant \Delta_{d-1}(d, n)$, or equivalently that $\Phi_0^f(d, n) \geqslant \Phi_0(d, n)$. For the latter it suffices to show that for each $d$-polytope $P$ having $n$ vertices, there is a simplicial $d$-polytope $Q$ with $n$ vertices such that $\phi(Q) \geqslant \phi(P)$. This can be proved with the aid of the *pushing* process described in (**9**, 2.3).

Let $X$ denote the set of all vertices of $P$, let $q$ be one of these vertices, and suppose that $X'$ is obtained from $X$ by pushing $q$ to a new position $q'$. This means that $X' = (X \sim \{q\}) \cup \{q'\}$, where $q'$ is a point of con $X$ such that the segment $[q, q']$ does not intersect any $(d-1)$-flat determined by points of $X$. Now suppose that $V$ is the set of all vertices of a $(d-1)$-face of the $d$-polytope con $X'$. From the reasoning in (**9**, 2.3) it follows that either $q' \notin V$ and $V$ is contained in the set of all vertices of some $(d-1)$-face of $P$, or $q' \in V$ and the set $(V \sim \{q'\}) \cup \{q\}$ is contained in the set of all vertices of some $(d-1)$-face of $P$. From this and the definition of the face-diameter $\phi$, it is evident that $\phi(\text{con } X') \geqslant \phi(\text{con } X)$. Finally, let $Q = \text{con } X_0$, where $X_0$ is obtained from $X$ in $n$ steps by successive pushing at all of the $n$ vertices of $X$. Then $Q$ has $n$ vertices, $Q$ is simplicial, and $\phi(Q) \geqslant \phi(P)$.

2.6. *For* $d \in \{2, 3\}$ *and* $n > d$,

$$\Delta_{d-1}(d, n) = \left[\frac{d-1}{d} n\right] - d + 2.$$

*Proof.* This is evident when $d = 2$. For $d = 3$ note that if a simple 3-polytope

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has $v$ vertices, $e$ edges, and $f$ 2-faces, then $3v = 2e$ and (by Euler's theorem) $v - e + f = 2$, when $f = 2v - 4$. From 2.5, 2.4, and 2.2. we see that

$$\Delta_2(3, n) = \Delta_2^v(3, n) = \Delta_0^v(3, 2n - 4) = \Delta_0(3, 2n - 4)$$
$$= \left[\frac{(2n-4)-2}{3}\right] + 1 = \left[\frac{2n}{3}\right] - 1.$$

2.7. $$\Delta_{d-1}^f(d, n) \geqslant \left[\frac{n - 2d}{2^d - 2}\right] + 2.$$

*Proof.* This is immediate from consideration of the polytopes $P''(d, j)$ of 2.2. A slightly stronger result is established by using not only $P''(d, j)$ but also $P(d, j)$ and the polytopes $P'(d, j)$ which are obtained from $P(d, j)$ by adding a single pyramidal cap on $S_0$. The conclusion is that

$$\Delta_{d-1}^f(d, n) \geqslant \begin{Bmatrix} j \\ j+1 \\ j+2 \end{Bmatrix} \quad \text{when } n \geqslant \begin{cases} (2^d - 2)j + 2, \\ (2^d - 2)j + d + 1, \\ (2^d - 2)j + 2d. \end{cases}$$

**3. The d-step conjecture.** Since we are using $d$ for the dimension of the space, the long-standing "$m$-step conjecture" of linear programming becomes here the "$d$-step conjecture." We are able to prove it only for the previously known case $d \leqslant 4$, but our discussion may throw some new light on the problem. For $n > d > 1$, consider the (unproved) assertion

$$A(d, n)\colon \Delta_{d-1}(d, n) \leqslant \left[\frac{d-1}{d} n\right] - d + 2,$$

where the expression on the right will be denoted henceforth by $\zeta(d, n)$. For $d \leqslant 3$, $A(d, n)$ was established in 2.6. The following result was suggested by an observation of Ernst Strauss.

3.1. *For $d < n \leqslant 2d$, $A(d-1, n-1)$ implies $A(d, n)$.*

*Proof.* Let us first verify that

(*a*) $\zeta(d - 1, 2d - 1) < \zeta(d, 2d)$, and $\zeta(d - 1, n - 1) \leqslant \zeta(d, n)$ for $n > d$.

For this purpose we set

$$k = \left[\frac{d-1}{d} n\right],$$

whence $\zeta(d, n) = k - d + 2$ and

(*b*) $$(d - 1)n = kd + l \text{ with } 0 \leqslant l < d.$$

Let $j$ be defined by the condition that

(*c*) $$(d - 2)(n - 1) = (k - 1)(d - 1) + j,$$

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whence

$$j < d - 1 \text{ implies } \zeta(d-1, n-1) \leqslant (k-1) - (d-1) + 2 = \zeta(d, n)$$

and

$$j < 0 \text{ implies } \zeta(d-1, n-1) < \zeta(d, n).$$

From ($b$) and ($c$) it follows that $j = k + l + 1 - n$. When $n = 2d$, we have $k = 2(d-1)$ and $l = 0$, whence indeed $j < 0$. For $n > d$ we want to verify that $k + l < d + n - 2$. Since $l \leqslant d - 1$, the contrary assumption implies (using ($b$)) that $(d-1)n \geqslant (n-1)d + l$ or $n \leqslant d - l$, a contradiction. Thus ($a$) has been established.

Now suppose that the assertion $A(d-1, n-1)$ is valid, and consider a $d$-polytope $P$ which has $m$ $(d-1)$-faces, where $m \leqslant n$. We must show that if $x$ and $y$ are any two vertices of $P$, then the distance from $x$ to $y$ (in the graph $\Gamma(P)$) is at most $\zeta(d, n)$. Note that each $(d-1)$-face $F$ of $P$ is a $(d-1)$-polytope which has at most $m - 1$ $(d-2)$-faces, for each $(d-2)$-face of $F$ is the intersection of $F$ with another $(d-1)$-face of $P$. Thus if $x$ and $y$ lie on a common $(d-1)$-face of $P$ it follows that

$$\text{dist}(x, y) \leqslant \Delta_{d-2}(d-1, m-1) = \zeta(d-1, m-1) \leqslant \zeta(d, n),$$

where ($a$) justifies the last inequality. It remains only to consider the case in which no $(d-1)$-face of $P$ contains both $x$ and $y$. Under our hypotheses, this can happen only if $m = n = 2d$ and each of $x$ and $y$ is on exactly $d$ of the $(d-1)$-faces of $P$. Let $u$ be an arbitrary vertex of $P$ that is a neighbour of $x$, so that the segment $[x, u]$ is an edge of $P$. Then $u$ lies on a $(d-1)$-face $F$ which does not include $x$, whence $F$ does include $y$ and we see (using ($a$)) that

$$\text{dist}(x, y) \leqslant 1 + \Delta_{d-2}(d-1, 2d-1) = 1 + \zeta(d-1, 2d-1) \leqslant \zeta(d, 2d).$$

3.2. *Suppose $n > d > 1$, and $d \leqslant 3$ or $n \leqslant d + 4$; then*

$$\Delta_{d-1}(d, n) = \left[\frac{d-1}{d} n\right] - d + 2.$$

*Proof.* For $d \leqslant 3$, this was 2.6. Starting with the result for $d = 3$, we then apply 3.1 to establish the validity of $A(4, n)$ for $4 < n \leqslant 8$, of $A(5, n)$ for $5 < n \leqslant 9$, etc. It remains to show that for $4 \leqslant d < n \leqslant d + 4$ there exists a $d$-polytope having $n$ $(d-1)$-faces and diameter $(d, n)$. This will be accomplished in §4, using polytopes polar to the cyclic polytopes. See 4.4.

Let us consider the three conjectures:

$C_1(d) = \bigcap_{d<n\leqslant 2d} A(d, n)$,
$C_2(d) = A(d, 2d)$,
$C_3(d)$: *If $P$ is a $d$-polytope having exactly $2d$ $(d-1)$-faces and $x$ and $y$ are vertices of $P$ that are not on the same $(d-1)$-face, then $x$ can be joined to $y$ by a path consisting of $d$ or fewer edges of $P$.*

The term "$d$-step conjecture" usually refers to $C_2(d)$ or to the slightly weaker conjecture $C_3(d)$. (The latter appears as Problem 1 in Dantzig's list **(4)** of unsolved problems connected with linear programming.) The 4-step conjecture (and, in fact, $C_2(4)$) is validated by 3.2, but not the 5-step conjecture.

Now suppose the conjecture $C_2(d-1)$ is valid but $C_2(d)$ fails for some $d$-polytope $P$. Thus, $P$ has $m$ $(d-1)$-faces for some $m \leqslant 2d$; but there are two vertices $x$ and $y$ of $P$ such that the distance from $x$ to $y$ (in the graph $\Gamma(P)$) is greater than $\zeta(d, m)$. From the reasoning of 3.1 we see that necessarily $m = 2d$, $x$ and $y$ are each on exactly $d$ $(d-1)$-faces of $P$ and are not on the same $(d-1)$-face, and the assertion $A(d-1, 2d-1)$ is false. Further, every $(d-1)$-face of $P$ that includes $y$ and also includes a neighbour $u$ of $x$, as well as every one that includes both $x$ and a neighbour of $y$, must have at least $2d-1$ $(d-2)$-faces and hence must intersect every other $(d-1)$-face of $P$ in a $(d-2)$-face. Such a surprising situation cannot arise for $d = 3$, but for $d \geqslant 4$ this sort of behaviour is exhibited by polytopes polar to the neighbourly polytopes. Thus, it may be reassuring, in connection with the $d$-step conjecture, to know that the conjecture is not contradicted by the most tractable polytopes of this sort. This is established in §4.

**4. Facial structure of cyclic polytopes.** The reasoning of §3 suggests that if the $d$-step conjecture fails, then there probably exists, for some $k \leqslant d$, a $k$-polytope $P$ such that $f_{k-1}(P) = 2k$, $\delta(P) > k$, and each two $(k-1)$-faces of $P$ intersect in a $(k-2)$-face of $P$. The polar of $P$ would then be a $k$-polytope which has $2k$ vertices, has face-diameter $> k$, and is 2-neighbourly. Here a polytope is called *$m$-neighbourly* provided each $m$ of its vertices determine an $(m-1)$-face. Because of this connection of neighbourliness with the $d$-step conjecture, and because the neighbourly polytopes seem destined for an important role in other studies of the facial structure of polytopes, it seems worthwhile to determine the face-diameters of the most tractable of the neighbourly polytopes, and thus to see that they do not contradict the $d$-step conjecture.

A *cyclic $d$-polytope* is the convex hull of a set $V$ of $d+1$ or more points of the *moment curve* $M_d$ in $\Re^d$, where $M_d$ is the set of all points of the form $(r, r^2, \ldots, r^d) \in \Re^d$ (for $r \in \Re$). Every cyclic $d$-polytope is $[d/2]$-neighbourly (Gale **5**). The points of the set $V$ are linearly ordered by means of their first coordinates, and Gale **(5)** shows that a $d$-pointed subset $D$ of $V$ is the set of all the vertices of some $(d-1)$-face of con $V$ if and only if each two points of $V \sim D$ are separated by an even number (possibly zero) of points of $U$. (Gale states this only when $d$ is even, but his reasoning applies also when $d$ is odd.) Since a cyclic polytope is necessarily simplicial, the problem of determining its face-diameter is thus purely combinatorial in nature. We shall introduce some terminology that will facilitate the discussion.

From now on, $V$ will denote a finite set which is linearly ordered by means of an antireflexive relation $<$. The first and last elements of $V$ will be denoted

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by $\alpha$ and $\omega$ respectively. A *cluster* is a subset $C$ of $V$ such that $\emptyset \neq C \neq V$ and

(i) no point of $C$ is between two points of $V \sim C$
or (ii) no point of $V \sim C$ is between two points of $C$.

The cluster $C$ is called *central* provided it includes neither $\alpha$ nor $\omega$; otherwise $C$ is called *terminal.* Each terminal cluster $C$ has a *left half* $L(C)$ [a *right half* $R(C)$], i.e. a subset maximal with respect to satisfying (i) or (ii) and including $\alpha$ but not $\omega$ [$\omega$ but not $\alpha$]. Either $L(C)$ or $R(C)$ may be empty, but they cannot both be empty, for the terminal cluster $C$ is a proper subset of $V$. Note that every central cluster satisfies (ii) but not (i). If a terminal cluster includes both $\alpha$ and $\omega$, it satisfies (i) but not (ii). If a terminal cluster includes only one of $\alpha$ and $\omega$, it satisfies both (i) and (ii).

Now let us define $\alpha^- = \omega; \omega^+ = \alpha$. For each $p \in V \sim \{\alpha\}$, $p^-$ is the immediate predecessor of $p$; for each $p \in V \sim \{\omega\}$, $p^+$ is the immediate successor of $p$. For each cluster $C$ in $V$, the *left end point* $l_C$ and the *right end point* $r_C$ are points of $C$ which are defined by the conditions that $l_C^- \notin C$ and $r_C^+ \notin C$. Note that if $C$ is a central cluster, then $l_C^- < r_C^+$. If $C$ is a terminal cluster, then $r_C^+ \leqslant l_C^-$, with equality if and only if $C$ omits but a single point of $V$. If the terminal cluster $C$ includes both $\alpha$ and $\omega$, then $r_C \in L(C)$ and $l_C \in R(C)$.

When $D$ is a proper subset of $V$, a *$D$-cluster* is a maximal cluster in $D$. The set $D$ will be called *admissible* provided every central $D$-cluster consists of an even number of points. Equivalently, $D$ is admissible provided that for each two points $p$ and $q$ of $V$, there is an even number (possibly zero) of points of $D$ between $p$ and $q$. (Compare this with Gale's description **(5)** of the $(d-1)$-faces of a cyclic $d$-polytope.) Finally, an *admissible pair* is an ordered pair $(X, Y)$ of admissible sets whose symmetric difference consists of exactly one point from each set; that is,

$$Y = (X \sim \{x\}) \cup \{y\} \text{ for some } x \in X \sim Y \text{ and } y \in Y \sim X.$$

When the sets $X$ and $Y$ are both of cardinality $d$, the following result describes the pairs of $(d-1)$-faces of a cyclic $d$-polytope such that the intersection of the two $(d-1)$-faces is a $(d-2)$-face. The proof consists of a routine vertification based on the remarks and definitions in the preceding paragraphs.

4.1. *Suppose $X$ and $Y$ are admissible subsets of $V$ whose symmetric difference consists of exactly one point from each set; say $Y = (X \sim \{x\}) \cup \{y\}$ with $x \in X \sim Y$ and $y \in Y \sim X$. Let $C$ be the $X$-cluster which includes $x$. Then at most one of the following four statements is true, and the pair $(X, Y)$ is admissible if and only if exactly one is true:*

*$C$ is a central cluster, $x$ has an even number of predecessors in $C$, and $y = r_C^+$;*
*$C$ is a central cluster, $x$ has an even number of successors in $C$, and $y = l_C^-$;*
*$C$ is a terminal cluster; $x \in L(C)$ with an odd number of successors in $L(C)$ or $x \in R(C)$ with an even number of predecessors in $R(C)$; $y = r_C^+$;*

*$C$ is a terminal cluster; $x \in L(C)$ with an even number of successors in $L(C)$ or $x \in R(C)$ with an odd number of predecessors in $R(C)$; $y = l_C^-$.*

Now we shall define a *d-admissible chain* in $V$ to be a finite sequence of sets $(D_0, \ldots, D_k)$, each of cardinality $d$, such that each pair $(D_{i-1}, D_i)$ is admissible. If $V$ is of cardinality $n > d$, then the face-diameter of a cyclic $d$-polytope having $n$ vertices is equal to the smallest number $k$ such that whenever $X$ and $Y$ are admissible sets of cardinality $d$ in $V$, then there is a $d$-admissible chain $(D_0, \ldots, D_k)$ with $D_0 = X$ and $D_k = Y$. Here we are concerned primarily with the case in which $d < n \leqslant 2d$.

It follows from 4.1 that for each admissible pair $(X, Y)$ there is a unique cluster $C \subset X$ such that

$$Y = (X \sim \{l_C\}) \cup \{r_C^+\} \quad \text{or} \quad Y = (X \sim \{r_C\}) \cup \{l_C^-\}.$$

The cluster $C$ is contained in some $X$-cluster but $C$ need not be an $X$-cluster. Let $X^\tau$ denote the (unique) terminal $X$-cluster if there is one, and otherwise $X^\tau = \emptyset$. The pair $(X, Y)$ will be called *central* provided one of the following three conditions is satisfied:

(i) $C$ is central,
(ii) $\alpha \in C = L(X^\tau)$ with $Y = (X \sim \{\alpha\}) \cup \{r_C^+\}$,
(iii) $\omega \in C = R(X^\tau)$ with $Y = (X \sim \{\omega\}) \cup \{l_C^-\}$.

In short, the admissible pair $(X, Y)$ is central if and only if its admissibility is independent of the fact that for some purposes the points $\alpha$ and $\omega$ are adjacent to each other. It can be verified that $(X, Y)$ is central if and only if $(Y, X)$ is central.

4.2. *Suppose that $X$ and $Y$ are admissible subsets of $V$, with*

$$\text{card } V = n > d = \text{card } X = \text{card } Y > 1.$$

*Then for some $k \leqslant n - d$ there exists a d-admissible chain $(D_0, \ldots, D_k)$ with $D_0 = X$ and $D_k = Y$. If* card $L(X^\tau)$ *and* card $L(Y^\tau)$ *are of the same parity, then the chain can be chosen so that every pair $(D_{i-1}, D_i)$ is central $(1 \leqslant i \leqslant k)$. If* card $L(X^\tau) \geqslant$ card $L(Y^\tau)$ *and these two numbers are of different parity, then the chain can be chosen so that $(D_{i-1}, D_i)$ is central for $2 \leqslant i \leqslant k$ and $D_1 = (D_0 \sim \{x\}) \cup \{l_{D_0\tau}^-\}$ for some $x \in L(D_0)$.*

*Proof.* Let the statement of 4.2 be denoted by $S(d, n)$. The reader can verify that $S(2, n)$ is valid for all $n > 2$ and that $S(d, d + 1)$ is valid for all $d > 1$. Now suppose, for a given $n > 3$, that $S(c, n - 1)$ is known whenever $n - 1 > c > 1$. Consider the case of $S(d, n)$ with $n - 1 > d > 1$ (since $S(d, d + 1)$ is also known). If $\alpha \in X \cap Y$, we obtain the desired $d$-admissible chain by applying $S(d - 1, n - 1)$ to the set $V \sim \{\alpha\}$ (in the induced ordering) and its admissible subsets $X \sim \{\alpha\}$ and $Y \sim \{\alpha\}$. If $\alpha \notin X \cup Y$, the desired chain is obtained by applying $S(d, n - 1)$ to the set $V \sim \{\alpha\}$ and its admissible subsets $X$ and $Y$; in this case the chain $(D_0, \ldots, D_k)$ has $k \leqslant (n-1) - d$ and every pair $(D_{i-1}, D_i)$ is central $(1 \leqslant i \leqslant k)$. In the remaining case, $\alpha$ is in exactly one of the sets $X$ and $Y$ and it suffices to consider the case in which $\alpha \in X \sim Y$. Let $D_0 = X$ and define

$$D_1 = (X \sim \{\alpha\}) \cup \{r_{X^\tau}{}^+\} \text{ when card } L(X^\tau) \text{ is even,}$$
$$D_1 = (X \sim \{\alpha\}) \cup \{l_{X^\tau}{}^-\} \text{ when card } L(X^\tau) \text{ is odd.}$$

Then $\alpha \notin D_1 \cup Y$ and the chain can be completed in the desired fashion by applying $S(d, n-1)$ to the set $V \sim \{\alpha\}$ and its admissible subsets $D_1$ and $Y$.

4.3. *Suppose the cyclic d-polytope $G(d, n)$ is the convex hull of $n$ points from the moment curve $M_d$, where $n > d > 1$. Then the face-diameter of $G(d, n)$ is $\leqslant n - d$, and*

$$\phi(G(d, n)) = n - d \qquad \textit{for } d < n \leqslant 2d.$$

*Proof.* The inequality is immediate from 4.2. For equality when $n = 2d$, let card $V = n$, let $X$ consist of the first $d$ points of $V$, and let $Y$ consist of the last $d$ points of $V$. Then $X$ and $Y$ are both admissible. The desired conclusion follows from the fact that card$(Y \sim X) = n - d$, in conjunction with the definition of a $d$-admissible chain.

We have not determined the exact face-diameter of the cyclic polytope $G(d, n)$ for $n > 2d$, though it appears that $\phi(G(d, n)) = [n/2]$ when $n > 2d$.

4.4. *For* $2d \geqslant n > d > 1$,

$$\Delta_{d-1}(d, n) \geqslant \left[\frac{d-1}{d} n\right] - d + 2.$$

*Proof.* Use 4.3, polarity, and the fact that $\zeta(d, n) = n - d$ for $d$ and $n$ as described.

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