SOME CHARACTERIZATIONS OF THE HEREDITARY PRETORSION CLASS OF SEMIGROUP AUTOMATA

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Abstract. Let S be a semigroup. A class of S-automata is called a hereditary pretorsion class (HPC) if it is closed under quotients, subautomata, coproducts (disjoint unions) and finite products. In this paper we present two characterizations of HPC. Specifically, we show that there is a bijective correspondence between the HPCs of S-automata, the right linear topologies on S^1 and the idempotent preradicals $\mathbf{r}$ on the category of S-automata such that the set of automata $\{M \mid \mathbf{r}(M) = M\}$ is closed under subautomata and finite products.

0. Introduction. In ring theory, there is an important correspondence between torsion modules, radicals on the category of ring modules and certain topologies on the ring. The general concept of torsion comes from the study of torsion modules in the rings of fractions and modules of fractions.

In this paper, we would like to begin the first step in obtaining similar results to semigroups and automata theories. We are able to establish the bijective correspondence between hereditary pretorsion classes of automata, right linear topologies on the semigroup adjoined with identity and certain idempotent preradicals.

Many people have studied radical and torsion theories in semigroups, monoids or their automata. For example, Marki, Mlitz and Strecker studied radicals and torsions on a collection of monoids in [**3**]. However, they defined the image of a radical as a congruence on the monoid while we defined the image of a preradical on an automaton as a subautomaton. On the other hand, Luedeman worked on torsion theories on monoids with zero and their automata in [**2**]. His approach was similar to ours but he had the advantage of defining the torsion theories on automata axiomatically as in the case of ring modules. He also made use of right ideals of monoids in his construction while we used right congruences as our main tool instead. The connection between right congruences and right ideals in rings is very intimate but it is not the case in semigroups. So we hope to present a different approach to those concepts here.

This paper is organized as follows: Section 1 provides some background definitons and preliminary results which can be found in [**1**] and [**4**]. Section 2 defines the right linear topologies on semigroups and on automata. Section 3 defines the hereditary pretorsion classes of automata and shows their correspondence with right linear topologies. Section 4 deals with preradicals on the category of automata and shows their connections with the hereditary pretorsion classes. The last section states the main correspondence theorem between the three concepts and also illustrates a particular example of a pretorsion class existence in the automata of fractions.

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1. Basic definitions and the isomorphism theorem. Assume S is a semigroup throughout this paper.

Definition 1.1. A *right congruence on* S is an equivalence relation ρ on S such that $(x, y) \in \rho$ and $s \in S$ imply $(xs, ys) \in \rho$.

Definition 1.2. (M, S, ϕ) is called an *S-automaton* if M is a set, S is a semigroup and $\phi : M \times S \to M$ is a function such that $\phi(\phi(m, s), t) = \phi(m, st)$. For simplicity, we write $\phi(m, s)$ as ms and (M, S, ϕ) as M.

Definition 1.3. Let M be an S-automaton and N a subset of M; then N is called an *S-subautomaton* (or subautomaton) of M if N is closed under ϕ; i.e. $n \in N$ and $s \in S$ imply $ns \in N$.

Definition 1.4. Let M be an S-automaton. ρ is a *right congruence on M* if it is an equivalence relation on M such that it is closed under the action by S; i.e. $(m, n) \in \rho$ and $s \in S$ imply $(ms, ns) \in \rho$.

Definition 1.5. Let M be an S-automaton and ρ a right congruence on M. Then M/ρ is a set of equivalence classes of ρ. We define the action of the elements of S on M/ρ as follows: if $[m] \in M/\rho$ and $s \in S$, then $[m]s = [ms]$. This will turn M/ρ into an S-automaton.

Definition 1.6. The *direct product* of a set of S-automata $\{M_\alpha\}$ is $\times M_\alpha$, where S operates on $\times M_\alpha$ as follows: $(m_\alpha)s = (m_\alpha s)$ where $(m_\alpha) \in \times M_\alpha$.

Definition 1.7. The *coproduct* (or direct sum) of a set of S-automata $\{M_\alpha\}$ is the disjoint union $\oplus M_\alpha$, where S operates on $\oplus M_\alpha$ in the obvious way.

Definition 1.8. A function $\alpha : M \to N$ between two S-automata M and N is called an *S-homomorphism* if it is closed under the action of S; i.e. $m \in M$ and $s \in S$ imply $\alpha(ms) = \alpha(m)s$.

Definition 1.9. The *kernel* $\ker(\alpha)$ of an S-homomorphism $\alpha : M \to N$ is the right congruence induced on M by α; i.e. $(m, n) \in \ker(\alpha)$ if and only if $\alpha(m) = \alpha(n)$.

Definition 1.10. Two S-automata M and N are said to be *isomorphic* if there is an S-homomorphism between M and N such that it is both injective and surjective.

Theorem 1.1 (Isomorphism Theorem). *If $\alpha : M \to N$ is a surjective S-homomorphism between two S-automata M and N, then $M/\ker(\alpha) \cong N$.*

Proof. Define $\Pi : M/\ker(\alpha) \to N$ by $\Pi([m]) = \alpha(m)$. ∎

For more basic results, please consult **[1]**.

2. Right linear (RL-) topologies on semigroups and on automata.

Definition 2.1. A *topological semigroup* is a semigroup S with a topology such that the semigroup operation $(s, t) \mapsto st$ is a continuous function from $S \times S$ to S (considering the corresponding product topology on $S \times S$).

Definition 2.2. A topological semigroup S is *right linearly topological* if there is a fundamental system of neighborhoods consisting of the equivalence classes of a set $\mathbf{T}$ of right congruences on S such that this set $\mathbf{T}$ satisfies the following conditions.

T1. If $\rho \in \mathbf{T}$, σ is a right congruence on M and $\sigma \supset \rho$, then $\sigma \in \mathbf{T}$.

T2. If ρ and σ belong to $\mathbf{T}$, then $\rho \cap \sigma \in \mathbf{T}$.

T3. If $\rho \in \mathbf{T}$ and $u \in S$, then $(\rho : u) \in \mathbf{T}$, where

$$(\rho : u) = \{(s, t) \in S \times S \mid (us, ut) \in \rho\}.$$

DEFINITION 2.3. Let $\mathbf{T}$ be defined as above. A subset G of S is said to be *open in S* if for each $a \in G$, there exists $\rho \in \mathbf{T}$ such that $[a]_\rho$ is contained in G. This defines a topology on S and $\{[a]_\rho \mid a \in S, \rho \in \mathbf{T}\}$ is a basis. We will by abuse of language call $\mathbf{T}$ the topology for S.

LEMMA 2.1. *$\mathbf{T}$ makes the semigroup operation continuous.*

Proof. Fix a point $(s,t) \in S \times S$. We show that given $\gamma \in \mathbf{T}$ with $[st]_\gamma$, we can find $\alpha, \beta \in \mathbf{T}$ such that $a \in [s]_\alpha$ and $b \in [t]_\beta$ imply $ab \in [st]_\gamma$. Let $\alpha = \gamma$; then $a \in [s]_\alpha$ implies that $a \in [s]_\gamma$, which implies that $(a,s) \in \gamma$ and then $(ab, sb) \in \gamma$. Next, since $\gamma \in \mathbf{T}$ and $s \in S$, $(\gamma : s) \in \mathbf{T}$ by T3. Let $\beta = (\gamma : s)$. Then $b \in [t]_\beta$ implies $(b,t) \in \beta$ which again implies $(sb, st) \in \gamma$. By combining both results, we get $(ab, st) \in \gamma$. ■

Next, let S^1 be S adjoined with an identity element 1 if S does not have an identity, otherwise it is equal to S. All automata in the following will be unitary. This means that the identity element of S^1 acts as the identity operator on the given automaton, where S is the input semigroup of the automaton. Suppose S^1 is a right linearly topological semigroup with $\mathbf{T}$ as a neighborhood system of right congruences and M is an S-automaton. We may consider M as an S^1-automaton with the obvious modification.

Let $\mathbf{T}_M$ be a set of right congruences on M satisfying the following conditions.

U1. If σ is a right congruence on M, $\sigma \supset \rho$ and $\rho \in \mathbf{T}_M$, then $\sigma \in \mathbf{T}_M$.

U2. If ρ and σ belong to $\mathbf{T}_M$, then so does $\rho \cap \sigma$.

U3. If $\rho \in \mathbf{T}_M$ and $m \in M$ then $(\rho : m) \in \mathbf{T}$, where

$$(\rho : m) = \{(s,t) \in S^1 \times S^1 \mid (ms, mt) \in \rho\}.$$

This turns M into a right linear topological S-automaton with the equivalence classes of elements in $\mathbf{T}_M$ as a neighborhood system. We will again by abuse of language call $\mathbf{T}_M$ a topology. In general, if S is a right linear topological semigroup, then M is a *right linear topological S-automaton* if it is an S-automaton, equipped with a topology consisting of right congruences on M such that the map $M \times S \to M$, given by $(m,s) \mapsto ms$ is continuous. The construction here will be similar to that described as above for the right linear topological semigroup.

LEMMA 2.2. *Let S^1 be a right linear topological semigroup with topology $\mathbf{T}$. For any S-automaton M, there is a topology on M, namely, the one for which the set of open right congruences is*

$$\mathbf{T}(M) = \{\rho / \rho \text{ is a right congruence on } M \text{ and } (\rho : x) \in \mathbf{T}, \forall x \in M\},$$

where $(\rho : x) = \{(s,t) \in S^1 \times S^1 / (xs, xt) \in \rho\}$.

Proof. $\mathbf{T}(M) \neq \varnothing$ because it contains the universal congruence. Suppose $\sigma \supset \rho$ and $\rho \in \mathbf{T}(M)$. Let $x \in M$; then $(\sigma : x) \supset (\rho : x)$. Since $\rho \in \mathbf{T}(M)$, $(\rho : x) \in \mathbf{T}$. Then $(\sigma : x) \in \mathbf{T}$ also by T1. Since this is true for all $x \in M$, $\sigma \in \mathbf{T}(M)$.

Next, suppose ρ and σ belong to $\mathbf{T}(M)$; then $(\rho : x) \in \mathbf{T}$ and $(\sigma : x) \in \mathbf{T}$. Then $(\rho : x) \cap (\sigma : x) \in \mathbf{T}$ by T2. However $(\rho \cap \sigma : x) \supset (\rho : x) \cap (\sigma, x)$. We conclude that $\rho \cap \sigma \in \mathbf{T}(M)$.

Finally, U3 is satisfied automatically by the definition of $\mathbf{T}(M)$. ■

LEMMA 2.3. *Same notation as above.* $\mathbf{T}(M)$ *contains the trivial congruence if and only if* $\ker(\alpha_x) \in \mathbf{T}$, $\forall x \in M$, *where* $\alpha_x : S^1 \to xS^1$ *is defined as* $\alpha_x(s) = xs$, $\forall s \in S^1$.

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Proof. Suppose $\mathbf{T}(M)$ contains the trivial congruence ι. Then, by definition, $(\iota:x)\in\mathbf{T}$, $\forall x\in M$. Now, $(s,t)\in(\iota:x)$ if and only if $(xs,xt)\in\iota$ if and only if $(s,t)\in\ker(\alpha_x)$. We thus have $\ker(\alpha_x)\in\mathbf{T}$, $\forall x\in M$. The other direction can be proved by reversing all the previous steps. ■

LEMMA 2.4. *Suppose M is an S-automaton and $x\in M$; then clearly xS^1 is an S-subautomaton of M. Moreover, $S^1/\ker(\alpha_x)\cong xS^1$, where α_x is as in Lemma* 2.3.

Proof. As before, define $\alpha_x:S^1\to xS^1$ by $\alpha_x(s)=xs$. Both S^1 and xS^1 are S-automata. Furthermore, $\alpha_x(st)=x(st)=(xs)t=\alpha_x(s)t$, $\forall s,t\in S$. Therefore, α_x is an S-homomorphism and it is clear that α_x is surjective. Then $S^1/\ker(\alpha_x)\cong xS^1$ by the Isomorphism Theorem. ■

3. Hereditary pretorsion classes of S-automata (HPC).

DEFINITION 3.1. A nonempty set of S-automata is called a *pretorsion class* if it is closed under quotients by right congruences, coproducts and finite products. We call its elements *pretorsion S-automata.*

DEFINITION 3.2. A pretorsion class of S-automata is called *hereditary* if it is also closed under subautomata. We then call this set an HPC.

THEOREM 3.1. *Let $\mathbf{T}$ be an RL-topology on S^1. Define*

$$\mathbf{C}=\{M/M \text{ is an } S\text{-automaton and } \ker(\alpha_x)\in\mathbf{T},\forall x\in M\}.$$

Then $\mathbf{C}$ is an HPC.

Proof. First of all, it is easy to show that $S^1/\nu\in\mathbf{C}$, where ν is the universal congruence. So $\mathbf{C}\neq\varnothing$. Next, suppose $M\in\mathbf{C}$ and ρ is a right congruence on M. We show that $M/\rho\in\mathbf{C}$; i.e. $\mathbf{C}$ is closed under quotients. Let $x\in M$; then $[x]\in M/\rho$. Consider the S-homomorphism $\alpha_{[x]}:S^1\to[x]S^1$ given by $\alpha_{[x]}(s)=[x]s=[xs]$. Then $(s,t)\in\ker(\alpha_{[x]})$ if and only if $[xs]=[xt]$. Therefore, $\ker(\alpha_{[x]})\supset\ker(\alpha_x)$. Then $\ker(\alpha_{[x]})\in\mathbf{T}$ by T1. Since $x\in M$ is arbitrary, we have $M/\rho\in\mathbf{C}$.

Next, it is clear that $\mathbf{C}$ is closed under coproducts and subautomata. We show that $\mathbf{C}$ is also closed under finite products. Let $M_1,\ldots,M_n\in\mathbf{C}$, and let $x=(x_i)\in M_1\times\ldots\times M_n$. Then

$$\begin{aligned}\ker(\alpha_x)&=\{(s,t)\in S^1\times S^1 \mid \alpha_x(s)=\alpha_x(t)\}\\&=\{(s,t)\mid x_is=x_it,\forall i=1,\ldots,n\}\\&=\bigcap\ker(\alpha_{x_i}).\end{aligned}$$

Since $M_i\in\mathbf{C}$, $\ker(\alpha_{x_i})\in\mathbf{T}$, $\forall i=1,\ldots,n$, and hence $\ker(\alpha_x)\in\mathbf{T}$ by T2. ■

THEOREM 3.2. *Let $\mathbf{C}$ be an HPC. Define*

$$\mathbf{T}=\{\rho/\rho \text{ is a right congruence of } S^1 \text{ and } S^1/\rho\in\mathbf{C}\}.$$

Then $\mathbf{T}$ is an RL-topology on S^1.

Proof. Let $M\in\mathbf{C}$ and $x\in M$; then $xS^1\in\mathbf{C}$, since $\mathbf{C}$ is closed under subautomata. Since $xS^1\cong S^1/\ker(\alpha_x)$ by Lemma 2.4, $\ker(\alpha_x)\in\mathbf{T}$. Thus, $\mathbf{T}\neq\varnothing$. Next, let $\sigma\supset\rho$ and

$\rho \in \mathbf{T}$. Define $\eta: S^1/\rho \to S^1/\sigma$ by $\eta([s]_\rho) = [s]_\sigma$. Then it is easy to show that η is a well-defined surjective S-homomorphism. By the Isomorphism Theorem, $(S^1/\rho)/\ker(\eta) \cong S^1/\sigma$. Since $\mathbf{C}$ is closed under quotient, $S^1/\sigma \in \mathbf{C}$ and hence $\sigma \in \mathbf{T}$. So T1 is established.

Now, suppose that ρ and σ belong to $\mathbf{T}$. Consider a function $\theta: S^1/\rho \cap \sigma \to S^1/\rho \times S^1/\sigma$ defined by $\theta([s]_{\rho\cap\sigma}) = ([s]_\rho, [s]_\sigma)$. Then, again, it is easy to show that θ is a well-defined S-homomorphism.

$$\begin{aligned}\ker(\theta) &= \{([s]_{\rho\cap\sigma}, [t]_{\rho\cap\sigma}) \mid [s]_\rho = [t]_\rho \text{ and } [s]_\sigma = [t]_\sigma\} \\ &= \{([s]_{\rho\cap\sigma}, [t]_{\rho\cap\sigma}) \mid t \in [s]_\rho \cap [s]_\sigma = [s]_{\rho\cap\sigma}\}.\end{aligned}$$

Therefore $\ker(\theta)$ is the trivial congruence which implies that $S^1/\rho \times S^1/\sigma$ contains an isomorphic copy of $S^1/\rho \cap \sigma$. Hence, $S^1/\rho \cap \sigma \in \mathbf{C}$, $\rho \cap \sigma \in \mathbf{T}$ and T2 is established.

Finally, let $\rho \in \mathbf{T}$ and $s \in S^1$. Define $\beta: S^1 \to S^1/\rho$ by $\beta(x) = [sx]_\rho$. Then $\beta(xt) = [s(xt)]_\rho = [(sx)t]_\rho = [sx]_\rho t = \beta(x)t$ for $t \in S$. Therefore, β is a well-defined S-homomorphism. Thus, S^1/ρ contains an isomorphic copy of $S^1/\ker(\beta)$. Now

$$\begin{aligned}\ker(\beta) &= \{(x, y) \in S^1 \times S^1 \mid [sx]_\rho = [sy]_\rho\} \\ &= \{(x, y) \mid (sx, sy) \in \rho\} \\ &= \{(x, y) \mid (x, y) \in (\rho : s)\}.\end{aligned}$$

So $\ker(\beta) = (\rho : s)$. Thus, $S^1/(\rho : s) \in \mathbf{C}$ and hence $(\rho : s) \in \mathbf{T}$. ∎

THEOREM 3.3. *The set of hereditary pretorsion classes of S-automata can be parametrized by the set of right linear topologies on* S^1.

Proof. To each RL-topology $\mathbf{T}$ we have associated an HPC, namely, $\mathbf{C} = \{M \mid M$ is an S-automaton and $\ker(\alpha_x) \in \mathbf{T}, \forall x \in M\}$ by Theorem 3.1. Conversely, if $\mathbf{C}$ is an HPC, then $\mathbf{T} = \{\rho \mid \rho$ is a right congruence on S^1 and $S^1/\rho \in \mathbf{C}\}$ defines an RL-topology on S^1 by Theorem 3.2. It remains to verify that we have obtained a bijective correspondence between the set of HPC and the set of RL-topologies on S^1.

Let $\mathbf{T}$ be an RL-topology on S^1. Then we define $\mathbf{C}$ as above. Moreover, we define

$$\mathbf{T}_1 = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C}\}$$

and

$$\mathbf{T}_2 = \{\rho \mid \rho \text{ is right congruence on } S^1 \text{ and } (\rho : a) \in \mathbf{T}, \forall a \in S^1\}.$$

Then we show $\mathbf{T}_1 = \mathbf{T}$ by proving that $\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{T}$. First,

$$\begin{aligned}(\rho : s) &= \{(s, t) \in S^1 \times S^1 \mid (as, at) \in \rho\} \\ &= \{(s, t) \mid [a]s = [a]t \in S^1/\rho\} \\ &= \ker(\alpha_{[a]}).\end{aligned}$$

It is, then, clear that $\mathbf{T}_1 = \mathbf{T}_2$.

Next, let $\rho \in \mathbf{T}$ and $a \in S^1$; then $(\rho : a) \in \mathbf{T}$ by T3. Since this is true for all $a \in S^1$, $\rho \in \mathbf{T}_2$. On the other hand, if $\rho \in \mathbf{T}_2$, then $\rho = (\rho : 1) \in \mathbf{T}$. Therefore, we have established that $\mathbf{T}_2 = \mathbf{T}$. Hence $\mathbf{T}_1 = \mathbf{T}$.

Conversely, suppose $\mathbf{C}$ is an HPC. We define

$$\mathbf{T} = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C}\}.$$

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Then define

$$\mathbf{C}_1 = \{M \mid M \text{ is an } S\text{-automaton and } \ker(\alpha_x) \in \mathbf{T}, \forall x \in M\}$$

and

$$\mathbf{C}_2 = \{M \mid M \text{ is an } S\text{-automaton and each cyclic subautomaton (i.e. } xS^1 \text{ where } x \in M) \text{ is in } \mathbf{C}\}.$$

We show that $\mathbf{C}_1 = \mathbf{C}$ by showing that $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C}$.

Suppose $M \in \mathbf{C}_1$; then $\ker(\alpha_x) \in \mathbf{T}$, $\forall x \in M$. Let $x \in M$; then xS^1 is a cyclic S-subautomaton of M. Since $xS^1 \cong S^1/\ker(\alpha_x)$ and $\ker(\alpha_x) \in \mathbf{T}$, $xS^1 \in \mathbf{C}$, and hence $M \in \mathbf{C}_2$. Reversing the steps above, we establish $\mathbf{C}_1 \supset \mathbf{C}_2$.

Next, if $M \in \mathbf{C}$, then obviously each cyclic subautomaton is also in $\mathbf{C}$, since $\mathbf{C}$ is closed under subautomata. Thus, $M \in \mathbf{C}_2$. Conversely, suppose $M \in \mathbf{C}_2$; then $xS^1 \in \mathbf{C}$, $\forall x \in M$. Then $\oplus xS^1 \in \mathbf{C}$. However then M is the homomorphic image of $\oplus xS^1$ by the S-homomorphism which "forgets" the extra separating indices between the same elements in different cyclic subautomata. Hence $M \in \mathbf{C}$. ■

4. Preradicals on the category Aut(S) of S-automata. First note that the morphisms in Aut(S) are S-homomorphisms.

DEFINITION 4.1. A *preradical* $\mathbf{r}$ is a functor on Aut(S) which assigns to each S-automaton M a subautomaton $\mathbf{r}(M)$ in such a way that every S-homomorphism $\varphi: M \to N$ induces the map $\mathbf{r}(\varphi): \mathbf{r}(M) \to \mathbf{r}(N)$ by restriction.

LEMMA 4.1. *Suppose that S^1 is a right linearly topological semigroup with topology* $\mathbf{T}$. *For each S-automaton M, define*

$$\mathbf{t}(M) = \{x \in M / \ker(\alpha_x) \in \mathbf{T}\}.$$

Then $\mathbf{t}$ *is a preradical on* Aut(S).

Proof. Suppose $M \in \text{Aut}(S)$. Then clearly $M \supset \mathbf{t}(M)$. Let $x \in \mathbf{t}(M)$ and $u \in S$. Then $\ker(\alpha_x) \in \mathbf{T}$ by definition. Now

$$\begin{aligned}(\ker(\alpha_x):u) &= \{(a,b) \in S^1 \times S^1 \mid (ua, ub) \in \ker(\alpha_x)\}\\ &= \{(a,b) \mid (xu)a = (xu)b\}\\ &= \ker(\alpha_{xu}) \in \mathbf{T} \text{ by T3.}\end{aligned}$$

Thus $xu \in \mathbf{t}(M)$ and $\mathbf{t}(M)$ is an S-subautomaton of M.

Moreover, let M and N be S-automata and $\varphi: M \to N$ be an S-homomorphism. Consider the induced map $\mathbf{t}(\varphi): \mathbf{t}(M) \to N$ where $\mathbf{t}(\varphi)$ is equal to φ restricted to $\mathbf{t}(M)$. Since $\ker(\alpha_x) \in \mathbf{T}$, $\forall x \in \mathbf{t}(M)$, $\mathbf{T}(\mathbf{t}(M))$ contains the trivial congruence by Lemma 2.3. Therefore $\ker(\mathbf{t}(\varphi)) \in \mathbf{T}(\mathbf{t}(M))$ by U1, and thus $(\ker(\mathbf{t}(\varphi)):x) \in \mathbf{T}$, $\forall x \in \mathbf{t}(M)$. However,

$$\begin{aligned}\ker(\alpha_{\mathbf{t}(\varphi)(x)}) &= \{(u,v) \in S^1 \times S^1 \mid \mathbf{t}(\varphi)(x)u = \mathbf{t}(\varphi)(x)v\}\\ &= \{(u,v) \mid (xu, xv) \in \ker(\mathbf{t}(\varphi))\}\\ &= (\ker(\mathbf{t}(\varphi)):x).\end{aligned}$$

Therefore, $\ker(\alpha_{\mathbf{t}(\varphi)(x)}) \in \mathbf{T}$ and hence $\mathbf{t}(\varphi)(x) \in \mathbf{t}(N)$. So the induced map is $\mathbf{t}(\varphi): \mathbf{t}(M) \to \mathbf{t}(N)$ and we conclude that it is a preradical on Aut(S). ■

DEFINITION 4.2. A preradical $\mathbf{r}$ on Aut(S) is *idempotent* if $\mathbf{r}(\mathbf{r}(M)) = \mathbf{r}(M)$ for all $M \in \text{Aut}(S)$.

LEMMA 4.2. *The preradical* $\mathbf{t}$ *we defined in Lemma* 4.1 *is idempotent.*

Proof. Clear. ■

Next, we would like to relate the preradical $\mathbf{t}$ with HPC.
Define $\mathbf{C_t} = \{M \mid M \text{ is an } S\text{-automaton and } \mathbf{t}(M) = M\}$.

THEOREM 4.3. *C_t is an HPC.*

Proof. $\mathbf{C_t} \neq \varnothing$ since $\mathbf{t}$ is idempotent by Lemma 4.2. Suppose $M \in \mathbf{C_t}$ and N is an S-subautomaton of M. It suffices to show that $\mathbf{t}(N) \supset N$. Let $x \in N$; then $x \in M$ and $x \in \mathbf{t}(M)$ since $M = \mathbf{t}(M)$. But then $\ker(\alpha_x) \in \mathbf{T}$ by definition of $\mathbf{t}(M)$ and hence $x \in \mathbf{t}(N)$. So $\mathbf{C_t}$ is closed under subautomata.

Secondly, suppose $M \in \mathbf{C_t}$ and ρ is a right congruence on M. We show that $M/\rho \in \mathbf{C_t}$. Define $\pi : M \to M/\rho$ by $\pi(m) = [m]_\rho$. Then π is a well-defined surjective S-homomorphism. Since $\mathbf{t}$ is a preradical, we have the induced map $\mathbf{t}(\pi) : \mathbf{t}(M) \to \mathbf{t}(M/\rho)$. Since $M \in \mathbf{C_t}$, the induced map is $\mathbf{t}(\pi) : M \to \mathbf{t}(M/\rho)$. But $\mathbf{t}(\pi)(M) = \pi(M) = M/\rho$. Therefore $\mathbf{t}(M/\rho) \supset M/\rho$ and hence $M/\rho \in \mathbf{C_t}$.

Thirdly, suppose $\{M_\alpha\}$ is a collection of S-automata in $\mathbf{C_t}$. Since $\mathbf{t}(M_\alpha) = M_\alpha, \forall \alpha$, the induced map of each inclusion is $\mathbf{t}(M_\alpha) \to \mathbf{t}(\oplus M_\alpha)$ which is then $M_\alpha \to \mathbf{t}(\oplus M_\alpha)$. It follows that $\mathbf{t}(\oplus M_\alpha) = \oplus M_\alpha$ by the universal mapping property of coproducts. Hence $\mathbf{C_t}$ is closed under coproducts.

Finally, we show that $\mathbf{C_t}$ is closed under finite products. Suppose $M_1, \ldots, M_n \in \mathbf{C_t}$; then $\mathbf{t}(M_1 \times \ldots \times M_n) = \{x = (x_i) \in M_1 \times \ldots \times M_n \mid \ker(\alpha_x) \in \mathbf{T}$. Let $x = (x_i) \in M_1 \times \ldots \times M_n$. As before, $\ker(\alpha_x) = \bigcap \ker(\alpha_{x_i})$. By hypothesis, $M_i \in \mathbf{C_t}$. So $\mathbf{t}(M_i) = M_i, \forall i = 1, \ldots, n$, and then $\ker(\alpha_{x_i}) \in \mathbf{T}$. Therefore, $\ker(\alpha_x) \in \mathbf{T}$ by T2. Hence $\mathbf{t}(M_1 \times \ldots \times M_n) = M_1 \times \ldots \times M_n$. ■

THEOREM 4.4. *The set of hereditary pretorsion classes of S-automata can also be parametrized by the set of idempotent preradicals* $\mathbf{r}$ *such that* $\mathbf{C_r} = \{M \mid M$ *is an S-automaton and* $\mathbf{r}(M) = M\}$ *is closed under subautomata and finite products.*

Proof. Let $\mathbf{C}$ be an HPC. Define $\mathbf{T}$ by

$$\mathbf{T} = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C}\}.$$

Then define $\mathbf{t}$ on Aut(S) as $\mathbf{t}(M) = \{x \in M \mid \ker(\alpha_x) \in \mathbf{T}\}$ and $\mathbf{C_t} = \{M \mid \mathbf{t}(M) = M\}$. Suppose $M \in \mathbf{C_t}$, then $\mathbf{t}(M) = M$ which implies that $\ker(\alpha_x) \in \mathbf{T}, \forall x \in M$. Then $xS^1 \cong S^1/\ker(\alpha_x) \in \mathbf{C}$ for all $x \in M$ by definition of $\mathbf{T}$. So $\oplus xS^1 \in \mathbf{C}$ and hence M, being the homomorphic image of $\oplus xS^1$, belongs to $\mathbf{C}$. On the other hand, if $M \in \mathbf{C}$, then $\ker(\alpha_x) \in \mathbf{T}, \forall x \in M$ by the proof of Theorem 3.3. Then $\mathbf{t}(M) = M$ and $M \in \mathbf{C_t}$. Hence $\mathbf{C} = \mathbf{C_t}$.

Conversely, let $\mathbf{t}$ be an idempotent preradical on Aut(S) such that $\mathbf{C_t}$ is closed under subautomata and finite products. Then $\mathbf{C_t}$ is an HPC because the proof in Theorem 4.3 on closure under quotients and coproducts does not depend on the particular definition of $\mathbf{t}$. By Theorem 3.3, we get an RL-topology

$$\mathbf{T} = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C_t}\}.$$

We define $\mathbf{t}_1$ on Aut(S) by $\mathbf{t}_1(M)=\{x \in M \mid \ker(\alpha_x) \in \mathbf{T}\}$. We would like to show that $\mathbf{t}_1=\mathbf{t}$. Suppose $M \in \text{Aut}(S)$. Let $x \in \mathbf{t}_1(M)$; then $\ker(\alpha_x) \in \mathbf{T}$ which implies that $S^1/\ker(\alpha_x) \in \mathbf{C}_\mathbf{t}$. Then $xS^1 \in \mathbf{C}_\mathbf{t}$. Now $\oplus xS^1 \in \mathbf{C}_\mathbf{t}$ and so is $\mathbf{t}_1(M)$. Therefore $\mathbf{t}(\mathbf{t}_1(M))=\mathbf{t}_1(M)$. But $M \supset \mathbf{t}_1(M)$; so $\mathbf{t}(M) \supset \mathbf{t}_1(M)$. On the other hand, we show that $\mathbf{t}_1(M)$ is the largest subautomaton of M belonging to $\mathbf{C}_\mathbf{t}$. Suppose $M \supset N \supset \mathbf{t}_1(M)$ and $N \in \mathbf{C}_\mathbf{t}$. Let $x \in N \backslash \mathbf{t}_1(M)$; then $\ker(\alpha_x) \notin \mathbf{T}$ and thus $xS^1 \cong S^1/\ker(\alpha_x) \notin \mathbf{C}_\mathbf{t}$. Since $N \in \mathbf{C}_\mathbf{t}$ and xS^1 is a subautomaton of N, $xS^1 \in \mathbf{C}_\mathbf{t}$. So we have reached a contradiction. Furthermore, $\mathbf{t}(M) \in \mathbf{C}_\mathbf{t}$ since $\mathbf{t}$ is idempotent. We then have $\mathbf{t}_1(M) \supset \mathbf{t}(M)$. Hence $\mathbf{t}_1=\mathbf{t}$. ∎

5. Semigroups of fractions and automata of fractions. We summarize here the previous results from Sections 3 and 4 as follows.

THEOREM 5.1. *The set of hereditary pretorsion classes of S-automata can be parameterized by both the set of right linear topologies on S^1 and the set of idempotent preradicals $\mathbf{t}$ on Aut(S) such that $\mathbf{C}_\mathbf{t}=\{M/\mathbf{t}(M)=M\}$ is closed under subautomata and finite products.*

Next, we turn our attention to the concepts of semigroups of fractions and automata of fractions. Then we construct a specific example for an HPC and use the Correspondence Theorem 5.1 to develop some interesting results. We first present some preliminary definitions and results while details and proofs may be found in [**1**, **4**, **5**].

DEFINITION 5.1. Let S be a semigroup as always and A be a subsemigroup. A *right semigroup of fractions* is a monoid $S[A^{-1}]$ together with a semigroup homomorphism $\phi: S \to S[A^{-1}]$ satisfying the following conditions.

S1. $\phi(a)$ is invertible for all $a \in A$ with respect to the identity.

S2. Every element in $S[A^{-1}]$ has the form $\phi(s)\phi(a)^{-1}$ with $s \in S$ and $a \in A$.

S3. $\phi(s)=\phi(t)$ if and only if $sa=ta$ for some $a \in A$.

LEMMA 5.1. *$S[A^{-1}]$ exists if and only if the subsemigroup A satisfies*:

A1. *If $a \in A$ and $s \in S$, then there exists $b \in A$ and $t \in S$ such that $at=sb$.*

A2. *If $a \in A$ and $as=at$ with $s,t \in S$, then there exists $b \in A$ such that $sb=tb$.*

LEMMA 5.2. *When $S[A^{-1}]$ exists, it has the following universal property*: *for every semigroup homomorphism $\lambda: S \to T$ such that $\lambda(a)$ is invertible in T, for all $a \in A$, there exists a unique homomorphism $\sigma: s[A^{-1}] \to T$ such that $\sigma \circ \phi=\lambda$. We then conclude that $S[A^{-1}]$ is unique up to isomorphism.*

When $S[A^{-1}]$ exists, it has the form $S[A^{-1}]=S \times A/\sim$ where $\sim$ is the equivalence relation defined on $S \times A$ as $(s,a) \sim (t,b)$ if there exist $u,v \in S$ such that $su=tv$ and $au=bv \in A$. If we denote elements of $S[A^{-1}]$ by $[s,a]$, then $\phi: S \to S[A^{-1}]$ is defined as $\phi(s)=[sa,a]$ for any $a \in A$. Moreover, it is easy to show that the identity in $S[A^{-1}]$ is $[a,a]$ for any $a \in A$.

DEFINITION 5.2. A subsemigroup A of S is called a *right denominator set* when it satisfies A1 and A2.

DEFINITION 5.3. Suppose $S[A^{-1}]$ exists and M is an S-automaton. An *automaton of fractions with respect to A* is an $S[A^{-1}]$-automaton $M[A^{-1}]$.

It has the form $M[A^{-1}] = M \times A/\sim$, where $\sim$ is the equivalence relation defined on $M \times A$ as $(m, a) \sim (n, b)$ if there exist $s, t \in S$ such that $ms = nt$ and $as = bt \in A$. We denote elements of $M[A^{-1}]$ by $[m, a]$. Moreover, $S[A^{-1}]$ operates on $M[A^{-1}]$ by $[m, a][s, b] = [mt, bc]$, where $[s, b] \in S[A^{-1}]$, $t \in S$, $c \in A$ and $at = sc$.

LEMMA 5.3. *Suppose A is a denominator set. If N is an $S[A^{-1}]$-automaton, then N is an S-automaton with the S-operation defined by $n\,.\,s = n\phi(s)$, where ϕ is as defined in Definition* 5.1.

LEMMA 5.4. *Suppose that $M[A^{-1}]$ is an $S[A^{-1}]$-automaton. Then the function $\mu_M : M \to [A^{-1}]$, defined by $\mu_M(m) = [ma, a]$ for any $a \in A$ is a well-defined S-homomorphism. Moreover μ_M satisfies the following universal property: for each $S[A^{-1}]$-automaton N and S-homomorphism $\alpha : M \to N$ with the S-action as defined in Lemma* 5.3, *there exists a unique $S[A^{-1}]$-homomorphism $\sigma : M[A^{-1}] \to N$ such that $\sigma\mu_M = \alpha$. Moreover, each element $[m, a] \in M[A^{-1}]$ can be expressed as $\mu_M(m)\phi(a)^{-1}$.*

Let $\mu_M : M \to M[A^{-1}]$ be defined as above.

DEFINITION 5.4. $\ker(\mu_M)$ is called the *A-torsion* of M.

LEMMA 5.5. $\ker(\mu_M) = \{(m, n) \in M \times M / \exists c \in A\ (mc = nc)\}$.

DEFINITION 5.5. M is an *A-torsion S-automaton* if M is an S-automaton and $\ker(\mu_M)$ is the universal congruence on M. M is *A-torsion-free* if $\ker(\mu_M)$ is the trivial congruence.

THEOREM 5.2. *Let* **C** *be the set of A-torsion S-automata and their coproducts. Then* **C** *is an HPC.*

Proof. We show that **C** is closed under subautomata, quotients, coproducts and finite products. It is clear that subautomata of A-torsion S-automata are A-torsion by using the definition of $\ker(\mu_M)$. Suppose $\{M_\alpha\}$ is a set of A-torsion S-automata and N is a subautomaton of $\oplus M_\alpha$. Then since $\oplus M_\alpha$ is a disjoint union $N = \oplus N_\alpha$, where $N_\alpha = N \cap M_\alpha, \forall \alpha$. It is clear that each N_α is an S-subautomaton of M_α and thus is A-torsion. Hence N is a coproduct of A-torsion S-automata.

Next, suppose M is A-torsion and ρ is a right congruence of M. Let $\hat{M} = M/\rho$. Consider the map $\mu_{\hat{M}} : \hat{M} \to \hat{M}[A^{-1}]$ defined as before. By Lemma 5.5, $\ker(\mu_{\hat{M}}) = \{([m], [n]) \in \hat{M} \times \hat{M} \mid \exists c \in A([m]c = [n]c)\}$. Since M is A-torsion, if $m, n \in M$, then there exists $c \in A$ such that $mc = nc$. Then $[m]c = [n]c$. Thus $([m], [n]) \in \ker(\mu_{\hat{M}})$. We then conclude that M/ρ is also A-torsion. Now suppose $\{M_\alpha\}$ is a set of A-torsion S-automata and ρ is a right congruence on $\oplus M_\alpha$. Let $\mathbf{D} = \{\mathbf{D}_j\}$ be the set of equivalence classes of ρ. For each α, let $E_\alpha = \{\mathbf{D}_j \mid \mathbf{D}_j \cap M_\alpha \neq \emptyset\}$. We claim that E_α is an A-torsion S-automaton for each α. First of all, $E_\alpha \neq \emptyset$. Assume not, then there is an α such that $\mathbf{D}_j \cap M_\alpha = \emptyset$ for all j. Then $\bigcup \mathbf{D}_j = (\bigcup M_\beta) \backslash M_\alpha$ which is absurd. Next, suppose $\mathbf{D}_j \in E_\alpha$ and $s \in S$; then $\mathbf{D}_j \cap M_\alpha \neq \emptyset$. So there is $x \in \mathbf{D}_j \cap M_\alpha$, and $xs \in \mathbf{D}_j s \cap M_\alpha$. This implies that $\mathbf{D}_j s \in E_\alpha$ and so E_α is an S-automaton. Suppose $\mathbf{D}_j$ and $\mathbf{D}_k$ are arbitrary elements of E_α; then $\mathbf{D}_j \cap M_\alpha \neq \emptyset$ and $\mathbf{D}_k \cap M_\alpha \neq \emptyset$. Let $d_j \in \mathbf{D}_j \cap M_\alpha$ and $d_k \in \mathbf{D}_k \cap M_\alpha$. By hypothesis, M_α is A-torsion, and so there is $c \in A$ such that $d_j c = d_k c$. Then $\mathbf{D}_j c = \mathbf{D}_k c$ and $(\mathbf{D}_j, \mathbf{D}_k) \in \ker(\mu_{E_\alpha})$. Hence, we conclude that $\ker(\mu_{E_\alpha})$ is the universal congruence. We then have $\oplus E_\alpha \in \mathbf{C}$. Since $\mathbf{D}$ is the homomorphic image of $\oplus E_\alpha$ under the S-homomorphism which forgets the extra separating indices from the same elements $\mathbf{D}_j$ but in different automata E_α, $\oplus M_\alpha/\rho = \mathbf{D} \in \mathbf{C}$. Hence, **C** is closed under quotients.

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Finally, we show that $\mathbf{C}$ is closed under finite products. We first show that the finite product of A-torsion S-automata is A-torsion. However, it suffices to show that it is true for only two A-torsion S-automata, say, M and N. Suppose (m, n) and (x, y) are arbitrary elements of $M \times N$. Since M and N are A-torsion, there exist $c, d \in A$ such that $mc = xc$ and $nd = yd$. Moreover, $c \in A$ and $d \in S$ imply that there exist $b \in A$ and $e \in S$ such that $ce = db$ by A1 in Lemma 5.2. Therefore, $mce = xce$ and $ndb = ydb$. Let $g = db \in A$; then $mg = xg$ and $ng = yg$. We then conclude that $\ker(\mu_{M\times N})$ is the universal congruence and $M \times N$ is A-torsion. Next, suppose $\oplus M_\alpha$ and $\oplus N_\beta$ are coproducts of A-torsion S-automata. Then it is easy to show that $(\oplus M_\alpha) \times (\oplus N_\beta) \cong \oplus(M_\alpha \times N_\beta)$. Since $M_\alpha \times N_\beta$ is A-torsion by the proof above, $(\oplus M_\alpha) \times (\oplus N_\beta)$ is a coproduct of A-torsion S-automata. ■

COROLLARY 5.1. *Let* $\mathbf{C}$ *be the set of* A*-torsion* S*-automata and their coproducts; then* $\mathbf{C}$ *is the set of* S*-automata each cyclic subautomaton of which is* A*-torsion.*

Proof. Since $\mathbf{C}$ is an HPC by Theorem 5.2, we can define

$$\mathbf{T} = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C}\}$$

by the Correspondence Theorem 5.1. Then

$$\mathbf{C} = \{M \mid M \text{ is an } S\text{-automaton and } \ker(\alpha_x) \in \mathbf{T}, \forall x \in M\}$$

by Theorem 5.1 again. However, $\ker(\alpha_x) \in \mathbf{T}, \forall x \in M$ if and only if $S^1/\ker(\alpha_x) \in \mathbf{C}, \forall x \in M$. This is equivalent to $xS^1 \in \mathbf{C}, \forall x \in M$, which is then equivalent to xS^1 being A-torsion for all $x \in M$. ■

DEFINITION 5.6. Elements of $\mathbf{C}$, as defined in Corollary 5.1, are called *pre-*A*-torsion* S*-automata.*

A-torsion S-automata are obviously also pre-A-torsion. Moreover, pre-A-torsion S-automata are pretorsion since $\mathbf{C}$ is an HPC.

Notice that

$$\begin{aligned}\mathbf{C} &= \{M \mid M \text{ is an } S\text{-automaton and each cyclic subautomaton is } A\text{-torsion}\}\\ &= \{M \mid xS^1 \text{ is } A\text{-torsion for all } x \in M\}\\ &= \{M \mid \ker(\mu_{xS^1}) = \text{universal congruence for all } x \in M\}.\end{aligned}$$

We define $\mathbf{t}_2$ on Aut(S) as

$$\mathbf{t}_2(M) = \{x \in M \mid \ker(\mu_{xS^1}) = \text{universal congruence}\}.$$

Then $\mathbf{C} = \{M \mid \mathbf{t}_2(M) = M\}$. Since $\mathbf{C}$ is an HPC, we can define, via the Correspondence Theorem 5.1, an RL-topology

$$\mathbf{T} = \{\rho \mid \rho \text{ is a right congruence on } S^1 \text{ and } S^1/\rho \in \mathbf{C}\}.$$

Then we get our original $\mathbf{t}$ which is defined as $\mathbf{t}(M) = \{x \in M \mid \ker(\alpha_x) \in \mathbf{T}\}$.

COROLLARY 5.2. $\mathbf{t}_2 = \mathbf{t}$.

Proof. $x \in \mathbf{t}(M)$ if and only if $\ker(\alpha_x) \in \mathbf{T}$. Then this is equivalent to $S^1/\ker(\alpha_x) \in \mathbf{C}$. Then $xS^1 \in \mathbf{C}$ since $S^1/\ker(\alpha_x) \cong xS^1$. But this is equivalent to $\ker(\mu_{xS^1}) =$ universal congruence and it is the same as $x \in \mathbf{t}_2(M)$. Hence $\mathbf{t}_2$ is also an idempotent preradical such that $\mathbf{C} = \{M \mid \mathbf{t}_2(M) = M\}$ is closed under subautomata and finite products. ■

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