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CLASSIFICATION OF RESTRICTED LINEAR SPACES

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1. Introduction. The material in this paper is taken from the author's doctoral dissertation [**2**]. We will use the terminology and notation of [**3**]. Let us recall those terms which will be needed here.

We define a *restricted linear space* (RLS) as a finite set of p elements, called *points*, of which q subsets, called *lines*, are distinguished so that the following axioms hold:

(RLS-1) Any two distinct points u, v belong to exactly one common line uv.

(RLS-2) Every line contains at least two points.

(RLS-3) $q \geqq 2$.

(RLS-4) $(q - p)^2 \leqq p$.

If only (RLS-1) and (RLS-2) hold it is simply called a *finite linear space* (FLS). A *non-trivial* FLS is an FLS with $q \geqq 2$. The *square order* of an FLS is that number n defined by $n^2 \leqq p < (n + 1)^2$.

We must now define a number of special FLS's. A *near-pencil* is an FLS with all its points but one collinear. *Lin's cross* has been defined as the unique FLS with 6 points having one 4-line and one 3-line. By a *finite semi-affine plane of type* III (FSP3) [**1**, **6**] we will mean an FLS obtained from a finite affine plane (FAP) by adjoining to it one "infinite" point. If the FAP we started with had order at least 3 and if we delete a "finite" point from this FSP3, we obtain what we will call a *punctured* FSP3. It was first handled by de Witte in his doctoral dissertation [**7**] and has only recently appeared in print [**8**]. The next class of FLS's require much more explanation.

An FLS $\mathscr{L}$ is called an *inflated* FAP if and only if the following conditions hold:

(a) a subset of its points together with its induced set of lines forms an FAP, say $\mathscr{L}^*$;

(b) the complementary subset of its points together with its induced set of lines forms a non-empty FLS, say $\mathscr{L}'$;

(c) any line joining two points of $\mathscr{L}'$ contains only points of $\mathscr{L}'$;

(d) any line joining a point of $\mathscr{L}'$ and a point of $\mathscr{L}^*$ contains at least one more point of $\mathscr{L}^*$.

Because of (d) it is readily seen that this determines an injection from the set of points of $\mathscr{L}'$ into the set of "parallel classes" of $\mathscr{L}^*$. If $\mathscr{L}'$ has only one point, then $\mathscr{L}$ is an FSP3. If $\mathscr{L}'$ has s collinear points, where $1 \leqq s - 1 \leqq n =$ the order of $\mathscr{L}^*$, then $\mathscr{L}$ is obtainable from a finite projective plane (FPP) of

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order n by deleting $n + 1 - s$ collinear points. If $\mathscr{L}'$ is a near-pencil, then $\mathscr{L}$ is called a *simply inflated* FAP, and if $\mathscr{L}'$ is an FPP, then $\mathscr{L}$ is called a *projectively inflated* FAP.

The objective then is to establish the following:

THEOREM 1. *$\mathscr{L}$ is an* RLS *if and only if $\mathscr{L}$ is one of the following*:
(i) *a near-pencil,*
(ii) *an* FAP, *an* FSP3, *a punctured* FSP3, *or an* FPP *of order n with at most n points deleted and no lines deleted,*
(iii) *Lin's cross,*
(iv) *a simply inflated* FAP *or a projectively inflated* FAP.

I wish to thank Professor Paul de Witte for suggesting the problem and Professors de Witte and F. A. Sherk for their comments on and improvements to this work.

2. Prerequisites. We will now reintroduce the notation of [**3**] and list some basic formulas (P1–P4) and results that may be found in [**3**, **4**, **5**]. The *degree* of a line x (resp. point u) is the number of points lying on it (resp. lines passing through it), and is denoted by $a(x)$ (resp. $b(u)$). A *k-line* (resp. *k-point*) is a line (resp. point) of degree k. We will assume throughout that the points and lines have been given a *monotone labelling*, that is, the lines will be denoted by x_σ, $1 \leqq \sigma \leqq q$, such that $\sigma \leqq \tau$ implies $a_\sigma = a(x_\sigma) \geqq a(x_\tau) = a_\tau$, and the points will be denoted by u_α, $1 \leqq \alpha \leqq p$, such that $\alpha \leqq \beta$ implies $b_\alpha = b(u_\alpha) \geqq b(u_\beta) = b_\beta$. If it is possible to have a monotone labelling in which x_1 misses x_2, we say that the FLS is *loose*; otherwise *tight*. We will assume that the monotone labelling for any loose FLS under consideration has been given so that x_1 misses x_2. The *incidence number* $r_{\sigma\alpha} = r(x_\sigma, u_\alpha)$ of a line x_σ and a point u_α is 1 if u_α lies on x_σ and 0 otherwise. The number of lines that miss a line x_σ will be denoted by $s(x_\sigma) = s_\sigma$.

P1. $\sum_\sigma a_\sigma = \sum_\alpha b_\alpha$.
P2. $p - 1 = \sum_\sigma (a_\sigma - 1)r_{\sigma\alpha}$; hence $p(p-1) = \sum_\sigma a_\sigma(a_\sigma - 1)$.
P3. If $r_{\sigma\alpha} = 0$, the number $b_\alpha - a_\sigma$ counts the number of lines passing through u_α that miss x_σ.
P4. $q - 1 = s_\sigma + \sum_\alpha (b_\alpha - 1)r_{\sigma\alpha}$.

THEOREM A. *An* FLS *is an* FAP *if and only if it is a loose* RLS.

THEOREM B. *If $\mathscr{L}$ is an* RLS *of square order n we have a_2 equal to:*
(i) 2 *if $\mathscr{L}$ is a near-pencil,*
(ii) n *if $\mathscr{L}$ is an* FAP,
(iii) $n + 1$ *otherwise.*

THEOREM C. *If $\mathscr{L}$ is an* RLS *of square order n we have a_1 equal to:*
(i) $p - 1$ *if $\mathscr{L}$ is a near-pencil,*
(ii) $n + 2 = 4$ *if $\mathscr{L}$ is Lin's cross,*

(iii) *n if $\mathscr{L}$ is an* FAP,
(iv) $n + 1$ *otherwise.*

COROLLARY. *If $\mathscr{L}$ is an* RLS *of square order n other than a near-pencil, then* $p \leqq n^2 + n + 1$.

THEOREM D. *If $\mathscr{L}$ is an* RLS *of square order n we have $b_\alpha \geqq n + 1$ for all points u_α, unless $\mathscr{L}$ is one of the following:*
(i) *a near-pencil,*
(ii) *Lin's cross,*
(iii) *an* FSP3,
(iv) *a punctured* FSP3.

COROLLARY. *If $\mathscr{L}$ is an* RLS *of square order n, then $q \geqq n^2 + n + 1$, unless $\mathscr{L}$ is one of the following:*
(i) *a near-pencil,*
(ii) *an* FAP,
(iii) *an* FSP3,
(iv) *a punctured* FSP3.

THEOREM E. *If $\mathscr{L}$ is an* RLS *of square order n other than Lin's cross, then parallelism is an equivalence relation on the set of n-lines.*

The following result is due to de Witte [**9**].

THEOREM F. *If $\mathscr{L}$ is an* FLS *of square order n other than a near-pencil, then $\mathscr{L}$ is embeddable in an* FPP *of order n if and only if $q \leqq n^2 + n + 1$.*

3. Method of proof. In order to establish Theorem 1 it is sufficient to prove:

THEOREM 2. *If $\mathscr{L}$ is an* RLS *of square order n with $q \geqq n^2 + n + 2$ other than a near-pencil or Lin's cross, then $\mathscr{L}$ is an inflated* FAP.

Proof of Theorem 1 (assuming Theorem 2). That the FLS's listed in Theorem 1 are restricted is very easy to show. So let us suppose $\mathscr{L}$ is an RLS of square order n. If $q \geqq n^2 + n + 2$, then $\mathscr{L}$ must be one of (i), (iii) or (iv) by Theorem 2 since it can be easily shown that any inflated FAP which is restricted must be either a simply or projectively inflated FAP or satisfy $q \leqq n^2 + n + 1$. On the other hand if $q \leqq n^2 + n + 1$, it follows from Theorem F that $\mathscr{L}$ is either (i) or (ii).

For the remainder of this paper we will assume that $\mathscr{L}$ is an RLS of square order n with $q \geqq n^2 + n + 2$ other than a near-pencil or Lin's cross. A line y will be called a *maximal parallel* of a line x, written $y \in M(x)$, if and only if y misses x and all lines z missing x satisfy $a(z) \leqq a(y)$. A point will be called *real* if it is an $(n + 1)$-point and *ideal* if not. A line will be called *real* if it meets every $(n + 1)$-line, *ideal* if it does not, and *hyperideal* if it misses every $(n + 1)$-line. The *weight* of a non-empty set S of points will be defined as $w(S) = \min \{b_\alpha - n - 1 | u_\alpha \in S\}$.

To prove Theorem 2 we will first observe that it is an immediate corollary of the following two theorems, whose proofs we will then undertake:

THEOREM 3. *If* $\mathscr{L}$ *is an* RLS *of square order n with* $q \geqq n^2 + n + 2$ *other than a near-pencil or Lin's cross and if no* $(n + 1)$*-line has a hyperideal maximal parallel, then* $\mathscr{L}$ *is an inflated* FAP.

THEOREM 4. *If* $\mathscr{L}$ *is an* RLS *of square order n with* $q \geqq n^2 + n + 2$ *other than a near-pencil or Lin's cross, then no* $(n + 1)$*-line has a hyperideal maximal parallel.*

Before proceeding to the proofs of these two theorems let us establish some lemmas summarizing a number of small results.

LEMMA 1. *If* $\mathscr{L}$ *is an* RLS *of square order n with* $q \geqq n^2 + n + 2$ *other than a near-pencil or Lin's cross, then*

(i) $a_1 = a_2 = n + 1$ *and* $\mathscr{L}$ *is tight; hence any two* $(n + 1)$*-lines meet each other;*

(ii) $p \leqq n^2 + n + 1$ *and so* $n \geqq 2$;

(iii) $b_\alpha \geqq n + 1$ *for all points* u_α;

(iv) *parallelism is an equivalence relation on the set of n-lines;*

(v) $p \geqq n^2 + 2$ *and* $q \leqq n^2 + 2n + 1$;

(vi) *any real point lies on at least two* $(n + 1)$*-lines;*

(vii) *any ideal line contains only ideal points; hence the weight of any ideal line is at least* 1;

(viii) *any* $(n + 1)$*-line contains at least one real point;*

(ix) *there is at least one ideal line;*

(x) *there is at least one real point lying on at least two* $(n + 1)$*-lines;*

(xi) *if x is hyperideal, then* $a(x) \leqq n - 1$;

(xii) *if y is an ideal line missing the* $(n + 1)$*-line x, then* $s(x) \geqq 1 + a(y)(w(y) - 1)$.

Proof. By Theorems A, B and C we obtain (i). By Corollary to Theorem C we get (ii), and (iii) follows from Theorem D. Statement (iv) is simply Theorem E, and (v) is a consequence of (ii) and the fact that in an RLS of square order n we have $q \leqq p + n$. By using (i), (v) and P2 we have (vi). Property (vii) is immediate from P3. By using (iii), (v) and P4 we get (viii). Statement (x) follows from (i), (vi) and (viii), and (xi) is then immediate from (x) and P3. Property (xii) is established simply by counting the lines at each point of y. It only remains to prove (ix). So let x be any $(n + 1)$-line. If x misses a line, we have nothing to prove. So suppose x meets every line. Thus $s(x) = 0$. Then since $q \geqq n^2 + n + 2$, we see by P4 that at least one point of x, say v, is ideal. Let u be a real point on x and let y be an $(n + 1)$-line passing through u but different from x (guaranteed by (viii) and (vi) respectively). Then by P3 there is a line passing through v that misses y and (ix) holds.

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LEMMA 2. *In any* FLS *if* x_σ *and* x_τ *miss each other and both of them meet* x_ρ, *then* $\sum_\alpha (b_\alpha - a_\rho)r_{\sigma\alpha}(1 - r_{\rho\alpha}) \geqq (a_\tau - 1)(a_\sigma - a_\rho + 1)$.

Proof. By P3 and P4 the expression $\sum_\alpha(b_\alpha - a_\rho)r_{\sigma\alpha}(1 - r_{\rho\alpha})$ clearly counts the number of lines which meet x_σ and miss x_ρ. Now let v be the meet of x_ρ and x_τ. Then there are $a_\sigma(a_\tau - 1)$ lines joining x_σ to x_τ that do not pass through v. At most $(a_\rho - 1)(a_\tau - 1)$ of these lines also meet x_ρ since none pass through v. Hence at least $(a_\tau - 1)(a_\sigma - a_\rho + 1)$ of the lines that meet x_σ and x_τ miss x_ρ.

COROLLARY (Transfer principle). *If* x_σ *and* x_ρ *are both k-lines,* x_σ *and* x_τ *miss each other, and both meet* x_ρ, *and* $b_\beta \geqq k$ *for the point* u_β *in common to* x_σ *and* x_ρ, *then* $\sum_\alpha (b_\alpha - k)r_{\sigma\alpha} \geqq a_\tau - 1$.

Proof. Obvious.

LEMMA 3. *If* x_σ *and* x_ρ *are k-lines, and if* x_ρ *and* x_τ *meet each other and both miss* x_σ, *then* $\sum_\alpha (b_\alpha - k - 1)r_{\sigma\alpha} \geqq a_\tau - 1$.

Proof. The proof is much the same as Lemma 2 and may be found in [**5**].

4. Proof of Theorem 3. Suppose $\mathscr{L}$ is as described in the statement of Theorem 3. To prove the theorem we need only establish:

(a) the real points together with their induced set of lines form an FAP of order n, say $\mathscr{L}^*$;

(b) the ideal points and the ideal lines form an FLS, say $\mathscr{L}'$;

(c) a line joining a real point and an ideal point contains at least two real points.

Let x_ρ be an ideal line of maximal degree. By (i) we may assume that x_ρ misses x_1 and meets x_2 since it is not hyperideal. Thus, it is appropriate to call x_2 a *transversal* of (x_1, x_ρ). By considering any point of x_ρ not on x_2, which is ideal by (vii), we see that $M(x_2)$ is not empty. Let x_τ be a maximal parallel of x_2. Then x_τ meets an $(n + 1)$-line, say y (y may or may not be x_1). Let u be the meet of x_2 and x_ρ, k the weight of the non-empty point set $x_\rho - \{u\}$, and $u_\beta \neq u$ a point of x_ρ having degree $n + 1 + k$. By (vii) we have $k \geqq 1$. By applying the transfer principle to x_2, y and x_τ we get

$$\sum_\alpha (b_\alpha - n - 1)r_{2\alpha} \geqq a_\tau - 1.$$

Through each of the points on x_ρ different from u there are at least k lines that miss x_2 and hence $s_2 \geqq k(a_\rho - 1)$. Then by P4 we have

$$q - 1 \geqq s_2 + \sum_\alpha (b_\alpha - n - 1)r_{2\alpha} + na_2 \geqq k(a_\rho - 1) + a_\tau - 1 + n^2 + n.$$

On the other hand there are exactly k lines passing through u_β that miss x_2 and by P2 we get

$$p - 1 \leqq n^2 + (a_\rho - 1) + k(a_\tau - 1).$$

Since $q \leqq p + n$ we obtain $0 \geqq (k - 1)(a_\rho - a_\tau)$. Both expressions on the

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right-hand side are non-negative. Thus we get equalities for all the above inequalities:

(1) $p = q - n = n^2 + a_\rho + k(a_\tau - 1) = n^2 + a_\tau + k(a_\rho - 1)$.

(2) $s_2 = k(a_\rho - 1)$.

(3) every point on x_ρ different from u has degree $n + 1 + k$.

(4) every ideal line missing x_2 must meet x_ρ.

(5) $\sum_\alpha (b_\alpha - n - 1)r_{2\alpha} = a_\tau - 1$.

(6) through u_β there pass one a_ρ-line, n $(n + 1)$-lines meeting x_2 and k a_τ-lines missing x_2. In fact we can say this about any point on x_ρ different from u, by considering (3).

Since x_2 could have been chosen as any transversal of (x_1, x_ρ) having degree $n + 1$, in particular one not passing through u, we see that (3) and (6) also apply to the point u when there are at least three points on x_ρ. But if $a_\rho = 2$, then $a_\tau = 2$ and we have that there are n $(n + 1)$-lines and $k' + 1$ 2-lines passing through u for some $k' \geqq 1$, and by applying P2 to the two points of x_ρ, we get $k = k'$. Thus we may improve (3) and (6) to read:

(7) each point of x_ρ has precise degree $n + 1 + k$.

(8) through every point of x_ρ, there pass one a_ρ-line, n $(n + 1)$-lines and k a_τ-lines.

By using (1) and P2 we get

(9) any point lying on an a_τ-line must be ideal.

From (8) and (9) we also obtain

(10) every $(n + 1)$-line contains at least one ideal point.

Let $u_\gamma \neq u$ be any point of x_2. Then by (i) and (8) we see that u_γ is joined to x_ρ only by $(n + 1)$-lines. Since the role of x_2 could have been played by any transversal of (x_1, x_ρ) of degree $n + 1$, in particular one not passing through u_γ, we may conclude by (4), (i) and P3 that $b_\gamma = n + 1$. Thus by (5) and (7) we get

(11) $a_\tau - 1 = \sum_\alpha (b_\alpha - n - 1)r_{2\alpha} = b(u) - n - 1 = k$.

Since any transversal of (x_1, x_ρ) of degree $n + 1$ could have played the role of x_2, it follows that any $(n + 1)$-line meeting x_ρ has exactly one ideal point, namely the one lying on x_ρ. Any $(n + 1)$-line missing x_ρ is joined to u by n $(n + 1)$-lines and one a_τ-line by (8). Thus from (9) we see that

(12) any $(n + 1)$-line has n real points and one ideal point. Let v be the ideal point of x_1. By (4) we see that every line passing through v meets either x_2 or x_ρ. Thus $b(v) \leqq n + a_\rho$, with inequality only if there is a line passing through v that meets both x_2 and x_ρ at distinct points. By (12) and (8) such a line must have degree a_τ and thus have only ideal points by (9), contradicting (12) for the line x_2. Therefore we have

(13) $b(v) = n + a_\rho$.

If y is any line joining v to x_2 but not passing through u, we obtain by P2

$$p - 1 \leqq (n - 1)n + a(y) - 1 + a_\rho(a_\tau - 1).$$

By (1) and (11) we see that $a(y) = n + 1$. Therefore

(14) through v there pass n $(n+1)$-lines (all missing x_ρ) and a_ρ a_τ-lines (all meeting x_ρ).

Now if x were any transversal of (x_1, x_ρ) of degree $n+1$ not passing through u, then by (6) we have

(15) every a_τ-line meeting x_ρ must miss either x or x_2 and is thus ideal. If y is a real line with $a(y) \leqq n$, then by (14) and (15) y does not pass through v and thus meets all n $(n+1)$-lines passing through v at distinct real points by (12). Thus

(16) any real line has exactly n real points and at most one ideal point. By (16) we see that two ideal points must be joined by an ideal line and (vii) then implies that the ideal points and the ideal lines form an FLS, $\mathscr{L}'$, and thus (b) is proved. Again by (16) and (vii) a line joining a real point and an ideal point is real and has n real points. Since $n \geqq 2$ by (ii) we have proved (c). Let $\mathscr{L}^*$ be the FLS consisting of the real points and their induced lines. These lines are real lines of $\mathscr{L}$ stripped of their ideal points, if in fact they had any. By (16) the degree of every line of $\mathscr{L}^*$ is n, and since the degree of every point is $n+1$, we have that $\mathscr{L}^*$ is an FAP of order n and (a) is proved.

5. Proof of Theorem 4. Let $\mathscr{L}$ be as described in the statement of Theorem 4 and suppose that there is an $(n+1)$-line z_0 with an hyperideal maximal parallel y_0. We must derive a contradiction. Set $k = w(y_0)$ and $a = a(y_0)$. By (xi) and (vii) we have $a \leqq n-1$ and $k \geqq 1$. The proof will be carried out in several stages.

Stage 1: $p \leqq n^2 + k(a-1)$.

Simply apply P2 to an $(n+1+k)$-point on y_0.

Stage 2: *For any $(n+1)$-line x_σ we have* $\sum_\alpha (b_\alpha - n - 1)r_{\sigma\alpha} \leqq a - k - 2$.

By P4 and (xii) we obtain

$$q \geqq n^2 + n + 2 + a(k-1) + \sum_\alpha (b_\alpha - n - 1)r_{\sigma\alpha}.$$

The result follows from Stage 1 and $q \leqq p + n$.

Note 2.1: $1 \leqq k \leqq a - 2 \leqq n - 3$, *and hence* $3 \leqq a \leqq n - 1$.

Stage 3: *If x is ideal but not hyperideal, then* $a(x) \leqq a - k - 1$.

If x misses x_1 and meets x_2, then by the transfer principle we have $\sum_\alpha (b_\alpha - n - 1)r_{1\alpha} \geqq a(x) - 1$ and the result follows from Stage 2.

Stage 4: *x is ideal if and only if x is hyperideal.*

Suppose there are ideal lines which are not hyperideal, since the proof in the other direction is trivial. Let H be the set of all such lines. Let x_τ be a line of H with maximal degree, and say it meets x_1 and misses x_2. Then let u be the meet of x_τ and x_1 and let $l = w(x_\tau - \{u\})$. Let $u_\beta \neq u$ be any $(n+1+l)$-point on x_τ. Let g_β and h_β be the number of real lines and the number of lines of H respectively, which pass through u_β. Since $x_\tau \in H$, we have $h_\beta \geqq 1$. Let j

be the number of lines passing through u_β which meet x_1 and miss x_2. Then $j \geqq 1$ and there are $n + 1 - j$ lines passing through u_β meeting both x_1 and x_2. Then $g_\beta \leqq n + 1 - j \leqq n$. Since j is also the number of lines passing through u_β which meet x_2 and miss x_1, we may let x_ρ be such a line. The transfer principle then states $\sum_\alpha (b_\alpha - n - 1)r_{1\alpha} \geqq a_\rho - 1$ and by P4 we get

$$q - 1 \geqq s_1 + n^2 + n + a_\rho - 1 \geqq l(a_\tau - 1) + n^2 + n + a_\rho - 1.$$

Since $g_\beta \leqq n$ we see that every $(n + 1)$-line in $\mathscr{L}$ has at least one of its points lying on a line of H passing through u_β. Let u_γ be any real point of $\mathscr{L}$ (guaranteed by (x)). Then by (vii) u_γ does not lie on any line of H and thus the number, c_γ, of $(n + 1)$-lines passing through u_γ cannot exceed the number of points lying on the lines of H passing through u_β. Thus $c_\gamma \leqq 1 + h_\beta(a_\tau - 1)$ and by P2 we get $p - 1 \leqq n^2 + h_\beta(a_\tau - 1)$. Hence $h_\beta(a_\tau - 1) \geqq l(a_\tau - 1) + a_\rho - 1$ and therefore $h_\beta \geqq l + 1$. Suppose now that u_β lies on at least one $(n + 1)$-line. Then $b_\beta = g_\beta + h_\beta$. Now through u_β pass l lines missing x_2, whose degrees cannot exceed a_τ since they are lines of H. Of the remaining lines passing through u_β, x_ρ is one and thus P2 implies

$$p - 1 \leqq a_\rho - 1 + l(a_\tau - 1) + c_\beta n + (n - c_\beta)(n - 1),$$

where c_β is the number of $(n + 1)$-lines passing through u_β. Since $c_\beta \leqq n$, we must have $c_\beta = n$. Now all n $(n + 1)$-lines passing through u_β must miss y_0 and hence $n + 1 + l = b_\beta \geqq a + n$, from which we obtain $l \geqq a - 1$. By Stage 2 we get, for any $(n + 1)$-line x_σ passing through u_β,

$$a - k - 2 \geqq \sum_\alpha (b_\alpha - n - 1)r_{\sigma\alpha} \geqq b_\beta - n - 1 = l \geqq a - 1,$$

a contradiction. Therefore, there are no $(n + 1)$-lines passing through u_β. Then by P2 we obtain

$$\begin{aligned} p - 1 &\leqq g_\beta(n - 1) + a_\rho - 1 + (h_\beta - 1)(a_\tau - 1) + (b_\beta - g_\beta - h_\beta)(a - 1) \\ &= g_\beta(n - a) + a_\rho - 1 + (h_\beta - 1)(a_\tau - a) + (n + l)(a - 1) \\ &\leqq g_\beta(n - a) + a_\rho - 1 + l(a_\tau - a) + (n + l)(a - 1) \\ &= g_\beta(n - a) + a_\rho - 1 + l(a_\tau - 1) + n(a - 1), \end{aligned}$$

where we have used the fact that $a_\tau < a$ by Stage 3. Because $q - 1 \geqq n^2 + n + a_\rho - 1 + l(a_\tau - 1)$, we get $g_\beta(n - a) \geqq n(n + 1 - a)$, which contradicts $g_\beta \leqq n$ since we have $a \leqq n - 1$.

Note 4.1: *x is real if and only if x meets some $(n + 1)$-line.*

Stage 5: *Every $(n + 1)$-line contains only real points.*

If an $(n + 1)$-line x contains at least one ideal point, say u, then by P3 and Stage 4 all $(n + 1)$-lines must pass through u, contradicting (x).

Note 5.1: *Ideal points lie on at least one ideal line.*

Stage 6: *If x is ideal, then $a(x) \leqq a$.*

Obvious.

Note 6.1: *Any n-line is real.*

Stage 7: *There are exactly* $n + 1$ *real lines passing through each point.*

If u lies outside any $(n + 1)$-line, the result is true for u by Note 4.1. If u lies on all the $(n + 1)$-lines, then by (x) u is real and certainly all lines passing through u are real.

Stage 8: *A real line* x *meets* $1 + na(x)$ *real lines.*

Immediate from Stage 7 and P4.

Note 8.1: *Any* $(n + 1)$*-line meets* $n^2 + n + 1$ *(real) lines. Thus for any* $(n + 1)$*-line* x *we have* $s(x) = s = q - n^2 - n - 1$ *and hence*

(α) $n \geqq s \geqq 1 + (k - 1)a \geqq 1$;

(β) $s \leqq p - n^2 - 1 \leqq k(a - 1) - 1$.

(α) follows from (v) and (xii), and (β) from Stage 1.

Note 8.2: *There are* $n^2 + n + 1$ *real lines and* s *(hyper-) ideal lines.*

Note 8.3: *Any n-line meets* $n^2 + 1$ *real lines and misses* n *real lines.*

Stage 9: *There are at least* $k + 2$ $(n + 1 + k)$*-points lying on* y_0.

Let t denote the number of $(n + 1 + k)$-points lying on the line y_0. The number of lines that meet y_0 and miss x_1 is at least $ka - t + 1$. Thus we have $s = s_1 \geqq 1 + ka - t$ and by (β) we get $t \geqq k + 2$.

Stage 10: *There are at least* $n + k + 3 - a$ *n-lines passing through any* $(n + 1 + k)$*-point on* y_0.

Let u_β be any $(n + 1 + k)$-point lying on y_0 and let d_β be the number of n-lines passing through it. Then by P2 we get

$$p - 1 \leqq d_\beta(n - 1) + (n + 1 - d_\beta)(n - 2) + k(a - 1).$$

By (β) we get $s \leqq d_\beta - n - 2 + ka - k$ and by (α) we have $d_\beta \geqq n + k + 3 - a$.

Stage 11: $2(a - 1) \leqq n$.

Let x be any n-line joining y_0 to x_1 (guaranteed by Stage 10 and Note 6.1). Applying Lemma 2 to x_1, y_0 and x we get $n \geqq (a - 1)(n + 1 - n + 1)$.

Stage 12: *If two n-lines meet at a point u, then there is at most one ideal line meeting both and not passing through u.*

Let z_1 and z_2 be the n-lines and y_1, y_2 two ideal lines not passing through u and meeting both z_1 and z_2. We must derive a contradiction. Without loss of generality the lines y_1 and y_2 meet z_1 at two distinct points, say v and w respectively. By Note 8.3 it suffices to show that z_2 misses at least $n + 1$ real lines. Since y_1 joins v to z_2, at least two real lines passing through v miss z_2 by Stage 7 and similarly for w. By P3 there is at least one real line missing z_2 passing through each other point of z_1 different from u, which is a contradiction.

Stage 13: $k = 1$.

Let u and v be two distinct $(n + 1 + k)$-points lying on the line y_0 (guaranteed by Stage 9) and let $d(u)$ and $d(v)$ denote the respective number of n-lines passing through them. By Stage 10 and (iv) we see that every n-line passing through u meets an n-line passing through v and vice versa. By Stage 12 we can conclude that any ideal line different from y_0 passing through u misses all n-lines passing through v and vice versa. Without loss of generality, suppose now that $d(u) \leqq d(v)$. Then, other than y_0, the degree of any ideal line passing through u cannot exceed $b(v) - d(v)$. By applying P2 to the point u we obtain

$$\begin{aligned} p - 1 &\leqq d(u)(n - 1) + (n + 1 - d(u))(n - 2) + a - 1 \\ &\quad + (k - 1)(n + k - d(v)) \\ &\leqq n^2 - n - 3 + a + (k - 1)(n + k) - d(v)(k - 2). \end{aligned}$$

By (α) and (β) we have $1 + (k - 1)a \leqq p - n^2 - 1$ and thus

$$(n + k + 1 - d(v))(k - 2) \geqq 2 + a(k - 2).$$

By Stage 10 we have $n + k + 1 - d(v) \leqq a - 2$. Therefore, $k \geqq 2$ is impossible.

Stage 14: y_0 *is the only ideal line* (*i.e.* $s = 1$ *and so* $q = n^2 + n + 2$).

Suppose instead that x_π is an ideal line different from y_0 having maximal degree. Let u be the meet of y_0 and x_π if they have a point in common, or any point of x_π if y_0 misses x_π. Let $l = w(x_\pi - \{u\})$ and let t be the number of $(n + 1 + l)$-points of x_π different from u. Then

$$s = s_1 \geqq 2 + t(l - 1) + (a_\pi - t - 1)l = (a_\pi - 1)l - t + 2.$$

By applying P2 at an $(n + 1 + l)$-point $v \neq u$ of x_π we see that

$$p - 1 \leqq (n + 1)(n - 1) + l(a_\pi - 1) = n^2 - 1 + l(a_\pi - 1),$$

and by (β) we get $t \geqq 3$. We also have $s \geqq 2 + (l - 1)(a_\pi - 1)$ and thus by (β) again we get $a \geqq (l - 1)(a_\pi - 1) + 4$. Now let us suppose that there are at most two n-lines passing through v which meet y_0. Therefore, by applying P2 to v we obtain

$$\begin{aligned} p - 1 &\leqq 2(n - 1) + (a - 2)(n - 2) + (n + 1 - a)(n - 1) + l(a_\pi - 1) \\ &= n^2 + 1 - a + l(a_\pi - 1), \end{aligned}$$

and by (β) we have $s \leqq 1 + l(a_\pi - 1) - a$. But $s \geqq 2 + (l - 1)(a_\pi - 1)$, from which we see that $a_\pi \geqq a + 2$, a contradiction. Hence, there are at least three n-lines passing through any $(n + 1 + l)$-point of x_π different from u and meeting y_0. Now let y_1, y_2, y_3 be three n-lines passing through v and meeting y_0. Since $t \geqq 3$ we may let w be yet another $(n + 1 + l)$-point of x_π different from both u and v. Suppose z is an n-line passing through w and meeting y_0. Then by (iv) we see that z must meet at least two of y_1, y_2, y_3, say z meets y_1 and y_2. But

z, y_1, y_2 are certainly not concurrent and thus z must meet at least one of them, say y_1, at a point not lying on y_0. This contradicts stage 12 for the n-lines y_1 and z and the ideal lines y_0 and x_π. Therefore we must have $s = 1$.

Stage 15: *All ideal points lie on* y_0 *and have degree* $n + 2$.

By Note 5.1 and Stage 14 we see that every ideal point lies on y_0 and Stage 14 proves the rest.

Stage 16: *No* 2-*line meets* y_0.

Suppose that a point v lies on y_0 and a 2-line. Then from P2 we see that

$$p - 1 \leqq n(n - 1) + 1 + (a - 1) = n^2 - n + a,$$

which together with (v) gives a contradiction.

Note 16.1: *A real line has at least* 2 *real points.*

Stage 17: *The real points of* $\mathscr{L}$ *together with all real lines stripped of their ideal points, if they had any, form an* FLS, *say* $\mathscr{L}^*$.

This is immediate from (vii) and note 16.1.

Stage 18: *Any* n-*line* x *in* $\mathscr{L}$ *determines a partition* Π_x *of the points of* $\mathscr{L}^*$ *into* $n + 1$ *lines of* $\mathscr{L}^*$, *such that no line of* Π_x *different from* x *passes through the ideal point of* x *in* $\mathscr{L}$, *if there is in fact an ideal point lying on* x.

This follows directly from Note 8.3 and Stage 15.

Stage 19: *If* x *and* y *are two distinct* n-*lines of* $\mathscr{L}$ *meeting in an ideal point, then* Π_x *and* Π_y *have exactly one line of* $\mathscr{L}$ *in common.*

Let u be the ideal point in common to x and y. Then by Stage 18 passing through each point of y there is exactly one line of Π_x, the one through u being x itself. Since Π_x contains $n + 1$ lines, there must be exactly one of them in Π_y.

Stage 20: *Let* u *be any fixed ideal point of* $\mathscr{L}$. *For every* n-*line* x *of* $\mathscr{L}$ *passing through* u *let us adjoin to* $\mathscr{L}^*$ *a new point* $[x]$ *lying on all the lines of* Π_x (*and only those lines*). *Then the resulting structure, denoted* $\mathscr{L}'$, *is an* FLS *with* $p' = p^* + d(u)$, $q' = q^* = n^2 + n + 1$ *and* $b_\alpha' = n + 1$ *for all* α.

By Stage 19 we see that $\mathscr{L}'$ is an FLS.The values of p' and q' are obvious and $b_\alpha' = n + 1$ follows from Stages 15 and 18.

Stage 21: $\mathscr{L}'$ *can be embedded in an* FPP *of order* n, *say* $\mathscr{L}''$.

Since $p^* = p - a$ we see that

$$\begin{aligned} p' = p - a + d(u) &\geqq p + n + 4 - 2a \\ &\geqq p + 2(a - 1) + 4 - 2a = p + 2 \end{aligned}$$

by Stages 10, 11, 13 and 20. Therefore, $\mathscr{L}'$ is an RLS of square order n with $q' = n^2 + n + 1$ and by Theorem F we have $\mathscr{L}'$ is embeddable in an FPP of order n.

Stage 22: *If two* n-*lines of* $\mathscr{L}$ *both meet* y_0, *they must meet each other.*

Suppose the parallel n-lines x_ρ and x_σ both meet y_0, say at u and v respectively. Then passing through u there is another line, say x_τ, that misses x_σ by P3. By Lemma 3 and Stage 16 we get $\sum_\alpha (b_\alpha - n - 1)r_{\sigma\alpha} \geqq a_\tau - 1 \geqq 2$, which contradicts Stage 15.

Stage 23: *Final contradiction.*

Let $\mathscr{L}^*$, $\mathscr{L}'$ and $\mathscr{L}''$ be created as above for the fixed ideal point u of $\mathscr{L}$. Any n-line of $\mathscr{L}$ containing an ideal point different from u is mapped onto an $(n-1)$-line of $\mathscr{L}'$ by Stage 22. Now for each ideal point of $\mathscr{L}$ different from u we choose one n-line. These correspond to $a - 1$ distinct $(n-1)$-lines of $\mathscr{L}'$ which meet pairwise by Stage 22. In embedding $\mathscr{L}'$ into $\mathscr{L}''$ we must add to each of these $a - 1$ lines two more points which must all be distinct. Thus as in Stage 21 we have

$$n^2 + n + 1 = p'' \geqq p' + 2(a-1) \geqq p + n + 4 - 2a + 2(a-1),$$

which contradicts the definition of n.

De Witte has found a different method of completing the proof of Theorem 4 from Stage 22 on:

Stage 22′: *In the above embedding the* $n+1$ *lines of* $\mathscr{L}^*$ *originating from the* $n+1$ *real lines passing through any fixed ideal point* v *of* $\mathscr{L}$ *are mapped onto* $n+1$ *concurrent lines of* L''.

Let v be any ideal point of $\mathscr{L}$ whatsoever. Then by Stage 10 there are at least two n-lines passing through v, say x and y. In creating $\mathscr{L}^*$ the n-lines x and y are mapped onto parallel $(n-1)$-lines. The $n+1$ real lines passing through v are mapped onto pairwise parallel lines of $\mathscr{L}^*$. They are in turn mapped onto $n+1$ lines of $\mathscr{L}''$, which meet pairwise. Let X be this set of lines of $\mathscr{L}''$. Let v'' be the meet in $\mathscr{L}''$ of x'' and y'', which correspond to x and y in $\mathscr{L}$, and suppose a line z'' of X does not pass through v''. Then z'' meets x'' and y'' at points of $\mathscr{L}'' - \mathscr{L}^*$. Thus there is exactly one such line z'' and every other line of X passes through v''. Since there are n lines of X passing through v'' and z'' meets each in a distinct point, z'' must contain at least n points of $\mathscr{L}'' - \mathscr{L}^*$. But by Note 16.1 this gives z'' at least $n+2$ points, which is impossible.

Stage 23′: *Final contradiction.*

By applying the argument of Stage 22′ to two different ideal points of $\mathscr{L}$, say v and w, we see that the $n+1$ real lines passing through each are mapped onto $n+1$ concurrent lines of $\mathscr{L}''$, say concurrent at v'' and w''. This means that the line $v''w''$ of $\mathscr{L}''$ corresponds to both a real line passing through v and one passing through w. This is impossible since the mapping $\mathscr{L}^* \to \mathscr{L}''$ is an embedding and thus injective.

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