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ON RECURRENCE RELATIONS FOR BERNOULLI AND EULER NUMBERS

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We obtain a class of recurrence relations for the Bernoulli numbers that includes a recurrence formula proved recently by M. Kaneko. Analogous formulas are also derived for the Euler and Genocchi numbers.

1. INTRODUCTION

The Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ $(n = 0, 1, 2, \dots)$ may be computed successively by means of the formulas

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n, \quad E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2x^n. \tag{1}$$

Thus the corresponding Bernoulli and Euler numbers, defined respectively by $B_n = B_n(0)$ and $E_n = 2^n E_n(1/2)$ $(n = 0, 1, 2, \dots)$, satisfy $B_0 = E_0 = 1$ and the recurrence relations

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \quad E_n + 2^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} \frac{E_k}{2^k} = 1 \quad (n \geqslant 1). \tag{2}$$

Two important properties of Bernoulli and Euler polynomials we shall make use of below are

$$B'_{n+1}(x) = (n+1)B_n(x), \qquad E'_{n+1}(x) = (n+1)E_n(x); \tag{3}$$

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad E_n(x+1) + E_n(x) = 2x^n. \tag{4}$$

We refer to [**1**] for a good account of the properties of $B_n(x)$, $E_n(x)$ and the corresponding Bernoulli and Euler numbers.

Recently a new recurrence formula for Bernoulli numbers was obtained in Kaneko [**6**], for which two proofs were given (see also Satoh [**8**]). In this note we offer a proof of Kaneko's formula which is simpler than those given in [**6, 8**] and, significantly, leads to a general class of recurrence relations for Bernoulli numbers. Analogous formulas for Euler and Genocchi numbers are also derived. Other interesting recurrence relations for Bernoulli numbers may be found in [**3, 5**] and [**7**, p.122].

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2. Two Lemmas

We first give two simple properties involving Bernoulli and Euler polynomials on which our results are based.

Lemma 1. *For an integer $n \geqslant 0$, the polynomials $P_n(x)$ and $Q_n(x)$ of degree $2n$ defined by*

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n+j}(x) \quad \text{and} \quad Q_n(x) = \sum_{j=0}^{n} \binom{n}{j} E_{n+j}(x)$$

are even functions.

Proof: It follows from (4) that

$$\begin{aligned} P_n(x+1) - P_n(x) &= \sum_{j=0}^{n} \binom{n}{j} \{B_{n+j}(x+1) - B_{n+j}(x)\} \\ &= \sum_{j=0}^{n} \binom{n}{j} (n+j)x^{n+j-1} = nx^{n-1}(x+1)^{n-1}(2x+1) \end{aligned} \tag{5}$$

and

$$\begin{aligned} Q_n(x+1) + Q_n(x) &= \sum_{j=0}^{n} \binom{n}{j} \{E_{n+j}(x+1) + E_{n+j}(x)\} \\ &= \sum_{j=0}^{n} \binom{n}{j} 2x^{n+j} = 2x^n(x+1)^n. \end{aligned} \tag{6}$$

For an integer $k \geqslant 0$ substituting $x = k$ and $x = -k-1$ into (5) and (6) we have

$$P_n(k+1) - P_n(k) = P_n(-k-1) - P_n(-k),$$
$$Q_n(k+1) + Q_n(k) = Q_n(-k-1) + Q_n(-k),$$

and so by induction for all integers $k \geqslant 1$

$$P_n(k) = P_n(-k), \quad Q_n(k) = Q_n(-k).$$

The lemma now follows as both $P_n(x)$ and $Q_n(x)$ are polynomials. ∎

Lemma 2. *For an integer $n \geqslant 0$*

$$P_n(x) - P_n(1-x) = \frac{d}{dx}[x^n(x-1)^n]; \tag{7}$$

$$Q_n(x) + Q_n(1-x) = 2x^n(x-1)^n. \tag{8}$$

PROOF: Replacing x by $-x$ in (5) it follows from Lemma 1 that

$$P_n(x) = P_n(-x) = P_n(1-x) + \frac{d}{dx}\left[x^n(x-1)^n\right].$$

Using (6) instead we have similarly

$$Q_n(x) = Q_n(-x) = -Q_n(1-x) + 2x^n(x-1)^n.$$

∎

We also need the following formula to evaluate certain derivatives. For integers $0 \leqslant m \leqslant n$

$$\frac{1}{(2m)!}\frac{d^{2m}}{dx^{2m}}\left[x^n(x-1)^n\right]_{x=1/2} = \left(-\frac{1}{4}\right)^{n-m}\binom{n}{m}. \tag{9}$$

This follows from the Leibniz rule and the equality

$$\sum_{k=0}^{2m}(-1)^k\binom{n}{k}\binom{n}{2m-k} = (-1)^m\binom{n}{m}$$

(see [**7**, p.14]).

3. Recurrence Relations

The main result of this note is the following. We denote for integers $m, n \geqslant 0$, $[n-m]_+ = \max\{n-m, 0\}$.

THEOREM 1. *Let $n \geqslant 1$ be an integer. Then for any integer $m \geqslant 0$*

(a) $$\sum_{k=[n-m]_+}^{2n}\binom{m+n+1}{m-n+k}\binom{2m+k+1}{k}B_k = 0;$$

(b) $$\sum_{k=[n-m]_+}^{2n}\binom{m+n+1}{m-n+k}\binom{2m+k+1}{k}\frac{B_k}{2^k} = (-1)^n\frac{m+1}{2^{2n+1}}\binom{m+n+1}{n}.$$

PROOF: Let $m \geqslant 0$ be given. Applying (3) repeatedly we obtain the $(2m+1)^{\text{th}}$ derivative of $P_{m+n+1}(x)$, which by Lemma 1 vanishes at $x = 0$. Dividing the resulting summation by $(2m+1)!$ we have

$$\sum_{j=[m-n]_+}^{m+n}\binom{m+n+1}{j}\binom{m+n+j+1}{2m+1}B_{n-m+j} = 0$$

as $B_j = 0$ for odd $j \geqslant 3$. We have (a) by substituting $j = m-n+k$.

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In a similar way we calculate the $(2m+1)^{\text{th}}$ derivative of the expression $P_{m+n+1}(x) - P_{m+n+1}(1-x)$ and evaluate at $x = 1/2$. Dividing the resulting summation by $(2m+1)!$ we obtain (b) by (7), (9) and the formula in (a) as $B_j(1/2) = (1/2^{j-1})B_j - B_j$ for $j \geqslant 0$. ∎

A feature in the formulas obtained in Theorem 1 as distinct from some known results is the appearance in the coefficients of an arbitrarily chosen integer $m \geqslant 0$, by which the number of terms in the recurrence may be adjusted. The same remark applies also to those obtained in Theorems 2 and 3 below. Particularly interesting are the special cases when $m = 0$ and $m = n$. We state them separately in the following result.

Corollary 1. *For an integer $n \geqslant 1$*

(a)
$$\sum_{k=n}^{2n} \binom{n+1}{k-n}(k+1)B_k = 0;$$

(b)
$$\sum_{k=n}^{2n} \binom{n+1}{k-n}(k+1)\frac{B_k}{2^k} = (-1)^n \frac{n+1}{2^{2n+1}};$$

(c)
$$\sum_{k=0}^{2n} \binom{2n+k+1}{2k}\binom{2k}{k}B_k = 0;$$

(d)
$$\sum_{k=0}^{2n} \binom{2n+k+1}{2k}\binom{2k}{k}\frac{B_k}{2^k} = (-1)^n \frac{n+1}{2^{2n+1}}\binom{2n+1}{n}.$$

In deriving (c) and (d), we use the equality

(10)
$$\binom{2n+1}{k}\binom{2n+k+1}{k} = \binom{2n+k+1}{2k}\binom{2k}{k}.$$

Kaneko's formula is now recovered in Corollary 1(a).

Theorem 2. *Let $n \geqslant 1$ be an integer. Then for any integer $m \geqslant 0$*

$$\sum_{k=[n-m]_+}^{2n} \binom{m+n}{m-n+k}\binom{2m+k}{k}\frac{E_k}{2^k} = \left(-\frac{1}{4}\right)^n \binom{m+n}{n}.$$

Proof: Let $m \geqslant 0$ be given. We calculate the $(2m)^{\text{th}}$ derivative of the expression $Q_{m+n}(x) + Q_{m+n}(1-x)$ using (3) as in the proof of Theorem 1 and evaluate at $x = 1/2$. Dividing the resulting summation by $(2m)!$ we have by (8) and (9)

$$\sum_{j=[m-n]_+}^{m+n} \binom{m+n}{j}\binom{m+n+j}{2m}\frac{E_{n-m+j}}{2^{n-m+j}} = \left(-\frac{1}{4}\right)^n \binom{m+n}{m}.$$

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The theorem follows by substituting $j = m - n + k$. ∎

Again we have the following two interesting special cases.

COROLLARY 2. *For an integer* $n \geqslant 1$

(a) $$\sum_{k=n}^{2n} \binom{n}{k-n} \frac{E_k}{2^k} = \left(-\frac{1}{4}\right)^n;$$

(b) $$\sum_{k=0}^{2n} \binom{2n+k}{2k} \binom{2k}{k} \frac{E_k}{2^k} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}.$$

In deriving (b), we use the equality

(11) $$\binom{2n}{k}\binom{2n+k}{k} = \binom{2n+k}{2k}\binom{2k}{k}.$$

We may compare the recurrence relations for Bernoulli and Euler numbers in (2) with those given in Corollaries 1 and 2.

Finally we recall that the Genocchi numbers may be defined by $G_0 = 0$ and $G_n = nE_{n-1}(0)$ $(n = 1, 2, \ldots)$. We refer to [**4**] for an interesting exposition on G_n and related polynomials and to [**2**, p.49] for a table of the first few Genocchi numbers. It follows from the second formula in (1) that the Genocchi numbers satisfy the recurrence relation

(12) $$2G_n + \sum_{k=0}^{n-1} \binom{n}{k} G_k = 0 \quad (n \geqslant 2).$$

THEOREM 3. *Let* $n \geqslant 1$ *be an integer. Then for any integer* $m \geqslant 0$

(a) $$\sum_{k=[n-m]_+}^{2n} \binom{m+n}{m-n+k}\binom{2m+k}{k} G_k = 0;$$

(b) $$\sum_{k=[n-m]_+}^{2n} \binom{m+n+1}{m-n+k}\binom{2m+k+1}{k}\frac{G_k}{2^k} = (-1)^n \frac{m+1}{2^{2n}}\binom{m+n+1}{n}.$$

PROOF: Let $m \geqslant 0$ be given. By Lemma 1 the $(2m+1)^{\text{th}}$ derivative of $Q_{m+n}(x)$ vanishes at $x = 0$. So (a) follows by a calculation similar to that in the proof of Theorem 2 and using the equality

$$\binom{2m+k}{2m+1} = \frac{k}{2m+1}\binom{2m+k}{k}.$$

On the other hand, (b) follows directly from the two formulas obtained in Theorem 1, as $G_n = 2(1-2^n)B_n$ for $n \geqslant 0$. ∎

In particular we have the following consequences.

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Corollary 3. *For an integer $n \geqslant 1$*

(a)
$$\sum_{k=n}^{2n} \binom{n}{k-n} G_k = 0;$$

(b)
$$\sum_{k=n}^{2n} \binom{n+1}{k-n} (k+1) \frac{G_k}{2^k} = (-1)^n \frac{n+1}{2^{2n}};$$

(c)
$$\sum_{k=1}^{2n} \binom{2n+k}{2k} \binom{2k}{k} G_k = 0;$$

(d)
$$\sum_{k=0}^{2n} \binom{2n+k+1}{2k} \binom{2k}{k} \frac{G_k}{2^k} = (-1)^n \frac{n+1}{2^{2n}} \binom{2n+1}{n}$$

In deriving (c) and (d), we use again (10) and (11). We may compare (12) with those given in Corollary 3.

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