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ALMOST-RIEMANNIAN STRUCTURES ON BANACH MANIFOLDS: THE MORSE LEMMA AND THE DARBOUX THEOREM

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Introduction. In this paper we introduce the notion of an almost Riemannian manifold. Briefly speaking, an almost-Riemannian structure on a Banach manifold is a generalization of the notion of a Riemannian structure on a Hilbert manifold, common examples would be manifolds of maps modelled on the Sobolev spaces L_k^p. The successful use of weak Riemannian structures in hard problems has been given by Ebin [**2**] and Ebin and Marsden [**10**]. The most important aspect of our structures is that for a certain class of real valued functions on an L_k^p manifold of maps one can define a *gradient* for these functions. This is analogous to using a Riemannian structure to obtain a gradient. In applications these gradients also arise as solutions of linear partial differential equations on sections of vector bundles but we do not pursue this point here. We prove the existence of such structures on manifolds of mappings and then apply the theory to prove versions of the Morse lemma and the Darboux theorem. For example, if

(1) M is a Banach manifold
(2) $\langle , \rangle : TM \times TM \to R$ is a smooth weak metric on M
(3) $f : M \to R$ is a C^3 smooth function
(4) $Y : M \to TM$ a C^2 weak gradient for f; i.e., $df(x)[h] = \langle Y(x), h \rangle$
(5) $df(x_0) = Y(x_0) = 0$ (x_0 a critical point of f and a zero of Y)
(6) $Y_*(x_0) : T_{x_0}M \to T_{x_0}M$, the Fréchet derivative of Y is an isomorphism (x_0 is non-degenerate)

then the Morse lemma for f holds in a neighborhood of the critical point x_0.

This work is, in part, motivated by the author's belief that heretofore much of the infinite dimensional theory in global analysis has been phrased in much too general a context (e.g. abstract Banach manifolds, smooth real valued maps satisfying condition (C), proper Fredholm maps, etc.) to obtain the right formulation of many results in the Banach manifold setting and to push the subject beyond where it is now. In this paper we restrict our attention to a special class, albeit very common class of Banach manifold in which the Darboux Theorem and the Morse lemma can be formulated in an elegant way. In § 3 we show how the Morse lemma applies to a concrete example.

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1. Almost-inner products. Let E be a real Banach space with norm $\|\ \|$ and let $\|\ \|_w$ denote a weaker norm (which we shall call the w-norm) on E; i.e. there is a constant k with $\|\ \|_w \leqq k \|\ \|$. A symmetric bilinear form $\beta : E \times E \to R$ is said to be an *almost-inner product* on E with respect to the w-norm on E if

(a) $\beta : E \times E_w \to R$ is continuous where E_w denotes E with the w-norm,

(*b*) $y \to \sup_{\|x\|_w \leqq 1} |\beta(x, y)|$

is an equivalent norm on E,

(c) $x \to \sup_{\|y\| \leqq 1} |\beta(x, y)|$ is a norm equivalent to the w-norm,

(d) (Riesz representation property) If $\Phi \in E^*$, the dual space of E, is a continuous linear map from E_w to R then there exists a $y_\phi \in E$ so that

$$\Phi(x) = \beta(x, y_\phi)$$

for all $x \in E$.

(e) $\beta(x, x) > 0$ for all $x \neq 0$.

A Banach space with an almost-inner product β is an almost-inner product space. We immediately have

PROPOSITION 1. *The y_ϕ of property* (d) *is unique.*

Proof. If $\Phi(x) = \beta(x, y_\phi) = \beta(x, y_\phi')$ for all $x \in E$ then

$$\beta(y_\phi - y_\phi', y_\phi - y_\phi') = 0$$

which implies that $y_\phi = y_\phi'$.

Let $\hat{E}_w$ denote the completion of E_w in the w-topology. Then β extends to a symmetric bilinear form of $E \times E_w$. The map $i : x \to \beta(x, \cdot)$ embeds E into $\hat{E}_w^*$.

PROPOSITION 2. *The map i above is an isomorphism of E onto $\hat{E}_w^*$.*

Proof. Suppose $i(x_1) = i(x_2)$ then $\beta(x_1 - x_2, y) = 0$ for all $y \in E$ and in particular for $y = x_1 - x_2$. Thus by property (e) $x_1 = x_2$ and so i is injective. Let $\Phi \in \hat{E}_w^*$. Then $\Phi|E_w$ and is continuous in the w-norm, whence there is a $y_\phi \in E$ with $\beta(x, y_\phi) = \Phi(x)$ for all $x \in E$. So $i(y_\phi) = \Phi$ and i is surjective. Isomorphism is an immediate consequence of (b).

Example 1. Let Ω be a domain in $\mathbf{R}^n$ (not necessarily bounded), with say $\partial\Omega$ a smooth submanifold of $\mathbf{R}^n$, and let $L_k^p(\Omega)$, $2 \leqq p < \infty$, $k \geqq 0$ denote the Sobelev spaces of functions on Ω with $\|\ \| : E \to R$ the L_k^p norm. For a treatment of these spaces see [**1**]. Let $L_k^p(\Omega)^0 = E$ be the closure of the C^∞ function on Ω with compact support in Ω and disjoint from the boundary $\partial\Omega$ of Ω. Then $L_{-k}^q(\Omega)$, $1/p + 1/q = 1$ is the dual space of $L_k^p(\Omega)^0$ under the bilinear pairing $L_{-k}^q(\Omega) \times L_k^p(\Omega)^0 \to R$ given by $(f, g) \to \int fg$; i.e. every Φ in

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the dual of $L_k^p(\Omega)^0$ is given by a unique $f \in L_{-k}^q(\Omega)$

$$\Phi(g) = \int_\Omega fg, \quad ||\Phi|| = ||f||.$$

The following Sobelev embedding theorem will be useful to us and it can be found in Palais [**11**] and Friedman [**5**]. If $k - m/p \geqq l - m/q$, $k \geqq l$ then $L_k^p(\Omega) \subset L_l^q(\Omega)$ and if the above inequality is strict then the inclusion is completely continuous.

Let $A : L_k^p(\Omega)^0 \to L_{-k}^p(\Omega)$ be a positive formally self adjoint elliptic operator of order $2k$ which induces an isomorphism; e.g. take $A = (I - \Delta)^k$. Take E_w to be E with the L_k^q norm. Clearly $|| \; ||_w \leqq C|| \; ||$, C a constant.

Define $\beta : E \times E \to R$ by

$$\beta(v, u) = \int_\Omega (Au)(v)$$

One readily checks that $\beta : E \times E_w \to R$ satisfied the axioms for an almost-inner product.

Example 2. This is similar to Example 1 but more general. Let Ω, $k > 0$, $p \geqq 2$, $L_k^p(\Omega)^0$, with $|| \; ||$ the L_k^p norm, all be as in Example 1. Let $0 \leqq r \leqq k$ and set $|| \; ||_w$ to be the L_{2r-k}^q norm on $L_k^p(\Omega)^0$. (By the Sobelev embedding theorem it follows immediately that $L_k^p \subset L_{2r-k}^q$.) Let $A : L_k^p(\Omega)^0 \stackrel{\approx}{\to} L_{k-2r}^p(\Omega)$ be an isomorphism induced by a self-adjoint elliptic operator of order $2r$. Define $\beta(u, v) = \int_\Omega (Au)v$. Then β is an almost-inner product on E with respect to the w-norm given by the L_{2r-k}^q norm on $L_k^p(\Omega)^0$.

Remark. Example 2 will be of interest to us later because it essentially comes up in showing that it is possible to prove the Morse lemma for non-degenerate critical points of the energy functional (these are of course geodesics parameterized by arc length) $E(\sigma) = 1/2 \int_0^1 ||\sigma'(t)||^2 dt$ defined in the path space $\Omega(P, Q)$ of paths joining points P and Q of a finite dimensional manifold V where the paths in $\Omega(P, Q)$ are taken to lie in a higher order Sobelev space than $L_1^2(I, V)$ which used by Palais [**12**].

If E is an almost-inner product space we can formulate the notion of symmetric linear operator and the adjoint of a class of continuous linear endomorphisms. An operator $A \in L(E)$ {the linear endomorphisms of E} is *symmetric* if $\beta(Ax, y) = \beta(x, Ay)$ for all $x, y \in E$ and if $A : E_w \to E_w$ is continuous.

Suppose $A \in L(E)$ and $A : E_w \to E_w$ is continuous in the w-norm. Then for each $x \in E$, $y \to \beta(Ay, x)$ is a functional on E continuous in the E_w norm. Consequently by axiom (d) there is an $A^*x \in E$ with $\beta(Ay, x) = \beta(y, A^*x)$. The adjoint map $x \to A^*x$ is linear and continuous in the E-norm and the w-norm. We will denote the set of β-symmetric maps on E by $L_s^\beta(E)$. The proof of the following proposition is straight forward.

PROPOSITION 3. *Let $A \in L_s^{\beta}(E)$ be one to one. Then $\overline{A(E)} = E$; that is, A has a dense range.*

2. Almost-Riemannian manifolds. In this and remaining sections we suppose the M is a paracompact C^{k+1} Finsler manifold (for a definition see [**14**]) modeled on a reflexive Banach space E which admits an almost-inner product β with respect to some norm $\| \ \|_w$. For each $p \in M$ let $\beta_p(,)$ be an almost-inner product for $E^p = T_pM$ (the tangent space to M at p) with respect to some norm w_p on E^p. We shall refer to E^p with this norm as E_w^p. Furthermore, suppose that there is a collection of charts $\mathscr{W} = \{U_\delta, \varphi_\delta\}_{\delta \in \Delta}$ covering M with the property that for all $p \in U_\delta$.

$D\varphi_\delta(p) : E_w^p \to E_w$ (the Fréchet derivative of φ) is continuous in the w-norm. Let $(V; \psi) \in \mathscr{W}$ be a chart about p. Define a map $\Psi : V \to L_s^{\beta}(E)$ as follows: for each $v \in E$, $D\psi_p^{-1}(v) \in T_pM$. Thus $u \to \beta_p(D\psi_p^{-1}(u); D\psi_p^{-1}(v))$ is a linear functional on E continuous in the E_w topology. Consequently, there is a $\Psi_p(v) \in E$ with

$$\beta_p(D\psi_p^{-1}(u), D\psi_p^{-1}(v)) = \beta(u, \Psi_p(v)).$$

For each p the symmetry of Ψ_p and its continuity follow immediately.

If $(U, \gamma) \in \mathscr{W}$ is another chart about p let $Z = U \cap V$ and set $g = \psi \circ \gamma^{-1} : \gamma(Z) \to \psi(Z)$. Then

$$\begin{aligned} \beta(u, \Gamma_p(v)) &= \beta_p(D\gamma_p^{-1}(u), D\gamma_p^{-1}(v)) \\ &= \beta_p(D\psi_p^{-1} \circ Dg_{\gamma(p)}(u), D\psi_p^{-1}Dg_{\gamma(p)}(v)) \\ &= \beta(Dg_{\gamma(p)}(u), \Psi_p Dg_{\gamma(p)}(v)) \\ &= \beta(u, Dg_{\gamma(p)} \circ \Psi_p \circ Dg_{\gamma(p)}(v)). \end{aligned}$$

Thus $\Gamma_p = Dg_{\gamma^*(p)}\Psi_p Dg_{\gamma(p)}$ for all $p \in Z$. Since g is C^{k+1} it is possible to require $p \to \Psi_p$ to be C^s, $s \leqq k$ for each chart $(V, \psi) \in \mathscr{W}$. If this is the case we say that $p \to \beta_p$ is a C^s almost-Riemannian inner product structure for TM compatible with the given Finsler structure. The coordinate cover $\mathscr{W}$ above is said to be C^s weak smooth if for each coordinate charts (U, φ) and (V, ψ), $U \cap V \neq \emptyset$, the map $g : \psi(U \cap V) \to L(E_w)$ defined by $g(x) = D(\varphi \circ \psi^{-1})(x)$ is a C^s map. Notice that this is more than just requiring that the map $x \to D(\varphi \circ \psi^{-1})(x)$ as a map of $\psi(U \cap V) \to L(E)$ be C^s.

Definition. A C^s weak smooth coordinate cover W for M together with a C^s almost-inner product structure for TM is said to be a C^s almost-Riemannian structure for M. Manifolds with a given almost-Riemannian structure are said to be almost-Riemannian manifolds.

We would now like to state an existence theorem for almost-Riemannian structures on manifolds of maps. Let X and Y be finite dimensional Riemannian manifolds together with Riemannian connections (an RMC structure) with X compact and $\partial X = \emptyset$. Let $p \in R$, $p \geqq 2$ and $k \in Z$ be such that

$\dim X < kp$. Then one can define (see e.g. [**11**] and [**4**]) a manifold of maps $M = L_k^p(X, Y)$. The tangent space to M at a point f, T_fM can be identified with the linear space of $\{g : X \to TY | g(x) \in T_{f(x)}Y\}$ such that g has finite L_k^p norm, or in other terms the sections of f^*TY with finite L_k^p norm. Call this space $L_k^p(f^*TY) = T_fM$. An RMC structure on TX and TY induces an RMC structure on the bundles $L^j(TX, TY)$ and $L^j(TX, f^*TY)$ for all $j \geqq 1$. The jth covariant derivative of a section s of f^*TY, $\nabla^j s$ is a section of $L^j(TX, f^*TY)$. Define $\beta_f : T_fM \times T_fM \to R$ by

$$\beta_f(\xi, \eta) = \sum_{j=0}^{k} \int_X \langle \nabla^j \xi, \nabla^j \eta \rangle_j * 1$$

where $\langle , \rangle_j$ is the induced Riemannian structure on $L^j(TX, f^*TY)$ and $*1$ is the volume element of X. The L_k^p norm of $g \in T_fM$ is defined by

$$||g|| = ||g||_{L_k^p} = \left(\sum_{j=0}^{k} \int_X ||\nabla^j g||^p \right)^{1/p}$$

Let q be the conjugate index to p. For $g \in T_fM$ define a weak norm $|| \ ||_w$ by $||g||_w = (\sum_{j=0}^k \int ||\nabla^j g||^q)^{1/q}$ and call T_fM with this norm T_fM_w. The bilinear pairing β_f above extends to a continuous map $\beta_f : T_fM \times T_fM_w \to R$. For each $f \in L_k^p(X, Y)$ we get a β_f. The proof of the following is essentially contained in [**4**, Theorem 10].

THEOREM 1 (Existence). *Let X and Y be as above. Then the manifold of maps $L_k^p(X, Y)$ has the structure of an almost-Riemannian manifold with almost-inner product β.*

Remark 1. The above theorem goes through if $\partial X \neq \emptyset$. If $h \in L_k^p(X, Y)$ {this is still defined} we define the manifold $M_{\partial h}$ to be the closure in L_k^p of those maps in L_k^p which agree with h on a neighborhood of ∂X. If $f \in M_{\partial h}$, $T_fM_{\partial h}$ can be identified with the L_k^p closure of the C^∞ sections of f^*TY which have support disjoint form ∂X. Then $M_{\partial h}$ with the bilinear pairing β and weak norm given above has the structure of an almost-Riemannian manifold.

Remark 2. The standard infinite dimensional proof of the existence of Riemannian structures on Hilbert manifolds which admit smooth partitions of unity does not appear to go through for Banach manifolds M modelled on almost-inner product spaces, even if M has a C^s weak smooth coordinate cover. That is, it does not necessarily follow that one can patch together an almost-inner product using a partition of unity. One has to assume more about M, namely that M has a Fredholm structure (e.g. see [**19**; **3**]). In this case the Hilbert space result goes through.

Remark 3. In Theorem 1 above and in Remark 1 we can take the w-norm to be the $L_{2r-k}{}^q$ norm $0 \leqq r \leqq k$ as we add in Example 2 of Section 1 with β again the bilinear pairing of L_r^p and L_r^q, i.e. if $f \in L_k^p(X, Y)$, ζ, η sections of

f^*TY define

$$\beta_f(\zeta, \eta) = \sum_{j=0}^{r} \int_X \langle \nabla^j \zeta, \nabla^j \eta \rangle * 1.$$

We can integrate the expression on the right by parts to obtain

$$\beta_f(\zeta, \eta) = \sum_{j=0}^{r} \int_X \langle A^j \zeta, \eta \rangle * 1$$

where $A^j = (-\mathrm{Div})^j \nabla^j$. If A is the elliptic operator $A = \sum_{j=0}^{r} A^j$ of order $2r$, then

$$\beta_f(\zeta, \eta) = \int_X \langle A \zeta, \eta \rangle * 1$$

and we are almost exactly in the situation of Example 2 in Section 1.

3. The Morse lemma. Throughout this and the next section M will be a C^s ($s \geqq 3$) almost-Riemannian manifold modelled in an almost-inner product space E.

Definition. Let $f : M \to R$ be a real valued map of class C^k. f is said to be weak C^k smooth if for each point $p \in M$ there exists a coordinate neighborhood (U, φ) of M about p so that

(a) $g(x) = D(f\varphi^{-1})(x) : E \to R$

extends to a continuous linear map of E_w into R for all $x \in \varphi(U)$; i.e. $g(x) \in L(E_w, R)$.

(b) $x \to g(x)$ is a C^{k-1} map of $\varphi(U)$ into $L(E_w, R)$.

Since M is an almost-Riemannian manifold and thus has a C^s weak smooth coordinate cover this notion does not depend on the choice of a weakly smooth coordinate chart. In this section we shall only consider weak smooth mappings.

If $f : M \to R$ is C^k weak smooth then $Df(x) : T_xM \to R$ is continuous in the w-topology of T_xM and therefore there exists an element say $\nabla f(x) \in T_xM$ with the property that $Df(x)h = \beta_x(\nabla f(x), h)$ where $\beta_x : T_xM \times T_xM \to R$ is the almost-inner product. This gradient will be of class C^{k-1} if f is C^k weak smooth. We might remark at this point that $\nabla f(x)$ will in practice be the same gradient as used by the author in [**17**] and [**18**]. In the later two references $\nabla f(x)$ was obtained, not from an almost-inner product but from solving a linear partial differential equation. We now proceed first with the definition of non-degenerate critical point for $f : M \to R$ and then with the proof of the Morse lemma.

Definition. Let $f : M \to R$ be C^2 weak smooth. A point $p \in M$ for f is *critical* if the derivative $Df(p) : T_pM \to R$ is zero, or equivalently if $\nabla f(p) = 0$. If p is critical then we can view the Fréchet derivative of ∇f at p, say $\nabla f_*(p)$

as a linear endomorphism of T_pM into itself $(\nabla f_*(p) \in L(T_pM))$. We say that p is *non-degenerate* if $\nabla f_*(p)$ is an *isomorphism*.

There is an equivalent way of looking at the notion of non-degenerate critical point. If p is critical then the second derivative of f, $H(p) = D^2f(p)$ can be regarded as a real valued bilinear form on T_pM, the so-called Hessian of f at p; $H(p) : T_pM \times T_pM \to R$.

Since f is C^2 weak smooth $H(p)$ extends to a continuous map in one variable in the w-topology of T_pM. Consequently $H(p)(u, v) = \beta_p(Au, v)$ where $A : T_pM$ is a linear endomorphism. p is non-degenerate if A is an isomorphism.

Remark. This definition of a non-degeneracy is geometric; it is a point condition which does not depend on the choice of coordinate chart. This was not true of earlier definitions of non-degeneracy employed by the author for general Banach spaces, e.g. see [**15**; **16**].

We are now ready to state and prove:

THEOREM 2 (Morse lemma). *Let $f : M \to R$ be a C^3 weak smooth map with $p \in M$ a non-degenerate critical point of f. Let $f_\varphi : \mathcal{O} \to R$ be a representation of f in some coordinate chart (φ, U) about p, with $\varphi(p) = 0$ where $f_\varphi = f \circ \varphi^{-1}$, $\mathcal{O} = \varphi(U)$. Then there exists an origin preserving diffemorphism ψ, $(\psi(0) = 0)$ of a neighborhood of $0 \in E$ with*

$$f_\varphi \circ \psi(x) = (1/2)D^2f_\varphi(0)(x, x) + f_\varphi(0).$$

Proof. By Taylor's theorem we have

$$f_\varphi(\zeta) = f_\varphi(0) + \int_0^1 (1 - \lambda)D^2f_\varphi(\lambda\zeta)(\zeta, \zeta)d\lambda.$$

Let $B_\zeta : E \times E \to R$ be the bilinear form defined by

$$B_\zeta(h, k) = \int_0^1 (1 - \lambda)D^2f_\varphi(\lambda\zeta)(h, k)d\lambda.$$

Since f is C^3 weak smooth we can write $B_\zeta(h, k) = \beta(A_\zeta h, k)$ where β is the almost-inner product on E and $\zeta \to A_\zeta$ is a C^1 map of $\mathcal{O}$ into the linear operators of E to itself and from E_w to itself. Moreover A_ζ is symmetric for each ζ. From earlier remarks it is easy to see that $A_0 = (1/2) \nabla f_{\varphi *}(0)$, where ∇f_φ is the gradient of f_φ with respect to β and $\nabla f_{\varphi *}(0)$ is its Fréchet derivative at 0.

Since p is non-degenerate $A_0 : E \to E$ is an isomorphism and so for t close to 0, A_t will also be an isomorphism. Let $Q_t = A_t^{-1}A_0$. Consequently $A_tQ_t = A_0$, $A_0 = Q_t{*}A_t{*}$. Note that $Q_0 = I$, the identity and so by the Taylor expansion for the square root Q_t has a C^1 square root S_t in some neighborhood $V_0 \subset \mathcal{O}$ of the origin. Now since the equations $A_0 = A_tQ_t = Q_t^*A_t^*$ are also satisfied by S_t (in fact by any polynomial in Q_t and hence by a limit of such) we have that $S_t^*A_t^* = A_tS_t$, whence $S_t^*A_t^*S_t = A_tS_t^2 = A_tQ_t = A_0$. Thus $A_t^* = R_t^*A_0R_t$ where $R_t = S_t^{-1}$. Recall that $f(x) = f_\varphi(0) + \beta(A_xx, x) = f_\varphi(0) + \beta(x, A_x^*x) = \beta(x, R_x^*A_0R_xx) + f_\varphi(0) = \beta(R_x(x), A_0R_x(x)) + f_\varphi(0)$.

Let $\gamma(x) = R_x(x)$. Then $D\gamma(0)h = R_0(h) = h$ since $Q_0 = S_0 = R_0 = I$. Thus by the inverse function theorem γ has local inverse ψ restricted to a subneighborhood $V \subset V_0$. Then

$$\begin{aligned} f_\varphi \circ \psi(x) &= \beta(x, A_0 x) + f_\varphi(0) \\ &= (1/2)\beta(x, \nabla f_{\varphi *}(0)(x)) + f_\varphi(0) \\ &= (1/2)D^2 f(0)(x, x) + f_\varphi(0) \end{aligned}$$

which completes the proof of the Morse lemma.

Example 1. Let $D \subset \mathbf{R}^2$ be the unit disc and let $\varphi : R \to R$ be a C^∞ real valued function with $\varphi(0) = 1$ and let $L_1^p(D)^0$, $p > 2$ be the closure in the L_1^p norm of the C^∞ real valued functions on D with support in the interior of D. The Sobelev embedding theorem tells us that there is continuous inclusion of $L_1^p(D)^0$ into $C(D)$ the continuous functions on the unit disc. The Banach space $E = L_1^p(D)^0$ can be given an almost-inner product. Let the weak norm on E be the L_1^q norm where $1/p + 1/q = 1$. Then $\beta(u, v) = \int_D \langle \nabla u, \nabla v\rangle$ is an almost-inner product for E with respect to the given weak norm.

Denote by $\varphi(f) : D \to R$ the map $\varphi \circ f$. The function $f \to \varphi(f)$ will then be C^∞. Consider the C^∞ functional $\epsilon : L_1^p(D)^0 \to R$ defined by

$$\epsilon(f) = -1/2 \int_D \varphi(f)|\nabla f|^2$$

where $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ and $|\nabla f|^2 = (\partial f/\partial x)^2 + (\partial f/\partial y)^2$.

It is straightforward to check that ϵ is C^∞ weak smooth. The first derivative of ϵ is a linear map given by the formula

$$h \mapsto D\epsilon_f(h) = -1/2 \int_D \varphi'(f)|\nabla f|^2 h - \int_D \varphi(f)\langle \nabla f, \nabla h\rangle$$

where $\varphi'(f) = \varphi' \circ f$ and $\varphi' = d\varphi/dt$. But $\operatorname{div}((\varphi(f)h)\nabla f) = \varphi(f)h\triangle f + \varphi'(f)|\nabla f|^2 h + \varphi(f)\langle \nabla f, \nabla h\rangle$ where $\triangle$ = Laplacian.

Since by Green's theorem $\int_D \operatorname{div}((\varphi(f)h)\nabla f) = 0$ it follows that

$$D\epsilon_f(h) = \int_D (\varphi(f)\triangle f + (1/2)\varphi'(f)|\nabla f|^2)h.$$

Clearly $f \equiv 0$ is a critical point of ϵ. Let us compute the Hessian of ϵ at $\mathbf{0}$. First

$$D^2\epsilon_f(h, k) = \int_D (\varphi'(f)k\triangle f + \varphi(f)\triangle k)h + \int_D \varphi'(f)h\langle \nabla f, \nabla k\rangle + 1/2 \int_D \varphi''(f)kh|\nabla f|^2$$

so for $f = 0$

$$D^2\epsilon_0(h, k) = \int_D (\triangle k)h = -\int_D \langle \nabla h, \nabla k \rangle.$$

So $D^2\epsilon_0(h, k) = -\beta(h, k)$ which immediately implies that 0 is non-degenerate (here we are using the alternate interpretation of non-degeneracy which was given following the definition of non-degenerate critical point). Applying the Morse lemma we have the existence of a local coordinate chart ψ about 0 with

$$\epsilon \circ \psi(f) = -1/2 \int_D \langle \nabla f, \nabla f \rangle.$$

Thus 0 is a strict local maximum for ϵ.

Remark. We could not hope for a Morse lemma in the case $p = 2$, where $E = L_1{}^2(D)^0$, a Hilbert space, for which the Palais definition [**13**] of non-degeneracy would work. In fact ϵ is not in general a C^1 functional on this space since there is no continuous inclusion of $L_1{}^2(D)^0$ into $C(D)$.

Example 2. We take this example from Palais' exposition [**12**] of the infinite dimensional version of the existence and geometry of geodesics on finite dimensional Riemannian manifolds. Notationally we shall follow this paper. Let V be a C^∞ complete Riemannian manifold.

Let $H_1(I, V) = L_1{}^2(I, V)$ where I is the unit interval, and let $\Omega(P, Q)$ be the C^∞ Riemannian submanifold of $H_1(I, V)$ consisting of those $\sigma : I \to V$, $\sigma \in H_1(I, V)$ with $\sigma(0) = P$, $\sigma(1) = Q$. Let $\Gamma(P, Q) \subset L_2{}^p(I, V)$, $p \geqq 2$ be the submanifold of $L_2{}^p(I, V)$ consisting of those $\sigma \in L_2{}^p$ with $\sigma(0) = P$ and $\sigma(1) = Q$. Clearly $\Gamma(P, Q) \subset \Omega(P, Q)$. Define the C^∞ energy functional $J : \Omega(P, Q) \to R$ by $J(\sigma) = 1/2\int_0{}^1 ||\sigma'(t)||^2 dt$ where $||\sigma'(t)||^2 = \langle \sigma'(t), \sigma'(t)\rangle_{\sigma(t)}$, $\langle , \rangle : TV \times TV \to R$ the Riemannian structure on V.

For a study of the functional J including a proof of its differentiability properties see [**12**]. The critical points of γ are the geodesics of V joining P and Q parameterized by arc length. J natural restricts to a C^∞ functional $\tilde{J}$ on $\Gamma(P, Q)$.

Now the Riemannian structure on $\Omega(P, Q)$ is given as follows. Suppose $\sigma \in \Omega(P, Q)$ and h and k are in the tangent space to $\Omega(P, Q)$ at σ. Then

$$\langle h, k \rangle_\sigma = \int_0^1 \left\langle \frac{Dh}{\partial t}, \frac{Dk}{\partial t} \right\rangle dt$$

where $D/\partial t$ denotes the covariant derivative along σ induced by the Riemannian structure on V. It is shown in [**17**] as well as in [**18**] that the gradient $\nabla\gamma(\sigma)$ of γ on $\Omega(P, Q)$ is for each σ a vector field γ over σ, (here $\gamma : I \to TV$, $\gamma(t) \in T_{\sigma(t)}V$) which satisfies the second order equation $D^2\gamma/\partial t^2 = D\sigma'/\partial t$. Now $\Gamma(P, Q)$ is not an infinite dimensional Riemannian manifold but can be given the structure of an almost-Riemannian manifold.

Let $\sigma \in \Gamma(P, Q) = \Gamma$ and let $h \in T_\sigma\Gamma$. Define

$$||h||_w = \left(\int_0^1 ||h(t)||^q\right)^{1/q}$$

which is the L^q norm. Recall that $T_\sigma\Gamma$ can be identified with the L_2^p sections of the bundle $\sigma^*TY = L_2^p(\sigma^*TY)$, which vanish at 0 and 1. So the completion of $T_\sigma\Gamma$ in the w-norm is just the L^q sections of σ^*TY which vanish at 0 and 1, i.e. $L^q(\sigma^*TY)^0$. Define the almost-inner product $\beta : T\Gamma \times T\Gamma \to R$ by

$$\beta_\sigma(h, k) = \int_0^1 \left\langle \frac{Dh}{\partial t}, \frac{Dk}{\partial t}\right\rangle dt = -\int_0^1 \left\langle \frac{D^2h}{\partial t^2}\ k\right\rangle$$

Note that $D^2h/\partial t^2 \in L^p(\sigma^*TY)$, and that β is an almost-inner product similar to that of Example 2 of Section 1. By definition

$$\tilde{J}(\sigma) = 1/2 \int_0^1 ||\sigma'(t)||^2\,dt.$$

Consequently

$$D\tilde{J}_\sigma(h) = \int_0^1 \left\langle \sigma', \frac{Dh}{\partial t}\right\rangle dt = -\int_0^1 \left\langle \frac{D\sigma'}{\partial t}, h\right\rangle dt$$

where $D\sigma'/\partial t \in L^p(\sigma^*TY)$.

From this formula it is fairly evident that $\tilde{J}$ is C^∞ weak smooth. Consequently γ restricted to $\Gamma(P, Q)$ has a gradient $\nabla\gamma(\sigma) = v_\sigma$ with respect to β. This gradient also satisfied the second order equation $D^2v_\sigma/\partial t^2 = D\sigma'/\partial t$.

It follows from a standard regularity theorem (e.g. again see [**12**]) that all critical points of J are in fact C^∞. Thus if $\sigma \in \Omega(P, Q)$ is critical for J, $\sigma \in \Gamma(P, Q)$. It is shown in [**17**] and [**7**] that both $\nabla\tilde{J}_*(\sigma) : T_\sigma\Omega \to T_\sigma\Omega$ and $\nabla\tilde{J}_*(\sigma) : T_\sigma\Gamma \to T_\sigma\Gamma$ (the Fréchet derivatives of J and $\tilde{J}$ at σ) are of the form identity plus a completely continuous linear map with $\nabla\tilde{J}_*$ the restriction of ∇J_* to $T_\sigma\Gamma \subset T_\sigma\Omega$. From this it follows that *σ is non-degenerate for J (in the sense of Palais) if and only if $\sigma \in \Gamma$ is non-degenerate in our sense for $\tilde{J}$*. It is a classical theorem of Marston Morse that for almost all P and Q, $J : \Omega(P, Q) \to R$ has only non-degenerate critical points. Thus, *for almost all P, Q, $J : \Gamma(P, Q) \to R$ has non-degenerate critical points in the sense used in this section.*

Thus the Morse lemma holds in the space $\Gamma(P, Q)$ for non-degenerate σ as well as in the space $\Omega(P, Q)$.

Remark 1. The choice of L_2^p maps above was entirely arbitrary. We could, for this example, have used any L_k^p space of maps of I to V, $p \geqq 2$, $k \geqq 1$.

Remark 2. $\tilde{J}$ above does not satisfy the condition (C) of Palais and Smale. It was shown in [**18**] that on manifolds M modelled on Banach spaces E where E is not isomorphic to E^*, the existence of a Morse lemma negates the possibility of the functional satisfying condition (C).

4. The Darboux theorem. Suppose M is a Banach manifold modelled on a Banach space E. A symplectic structure on M is a closed 2-form Ω such that the associated mapping $\overline{\Omega} : TM \rightarrow T^*M$ defined by $\overline{\Omega}(X) = i_X\Omega$, $(i_X\Omega)(u) = \Omega(X, u)$ is an isomorphism. i_X is the interior product of X and Ω. Unfortunately, this definition of symplectic structure implies that the model space E is isomorphic to its dual space E^*. In [**9**] Marsden introduced the notion of weak simplectic form and showed that the Darboux theorem, which states that there is a local change of variables with respect to which the simplectic structure is constant, fails for such forms. We would like to present an alternate definition of symplectic structure on an almost-Riemannian manifold and then prove the Darboux theorem in this context.

If M is finite dimensional Darboux's theorem is phrased as follows: Every point in M has a coordinate neighborhood U with coordinate functions $(x_1, \ldots x_\eta, y_1, \ldots y_\eta)$ such that

$$\Omega = \sum_{i=1}^{\eta} dx_i \wedge dy_i \quad \text{on } U.$$

Definition. Suppose M is an almost-Riemannian manifold. A two form $\Omega : TM \times TM \rightarrow R$ is weak continuous if for each $x \in M$ it is continuous as a fibre map from $T_xM \times T_xM_w$ to R; i.e., Ω is fibrewise continuous in the w-topology in one variable. Let (U, φ) be a coordinate system on the weak smooth structure of M. The pull back form $\Omega^\varphi = (\varphi^{-1})^*\Omega$ is a two form on $V = \varphi(U) \subset E$. Let $L(E, E_w;R)$ be the normed space of continuous bilinear maps from $E \times E_w$ to R. Ω^φ induces a map $\hat{\Omega}^\varphi : V \rightarrow L(E, E_w;R)$. We say that Ω is C^s *weak smooth* if $\hat{\Omega}^\varphi$ is C^s smooth for every such coordinate chart (of course just one would suffice).

If Ω is weak smooth on M then $\Omega(x)(u, v) = \beta_x(\Omega^\#(x)u, v)$ where $\Omega^\# : TM \rightarrow TM$ is a weak smooth bundle homomorphism (in fact $\Omega^\#$ is skew-symmetric with respect to β). We say that Ω is *non-degenerate* if the induced bundle homomorphism $\Omega^\#$ is a bundle isomorphism.

Definition. A C^s symplectic structure on an almost-Riemannian manifold M is a C^s weak smooth non-degenerate closed two form Ω.

For the proof of the Darboux theorem we shall follow Weinstein [**20**] whose proof is an adoption of an idea of J. Moser. Since the theorem concerns only local behavior it suffices to consider a symplectic structure Ω on a neighborhood of 0 in an almost-inner product space E.

THEOREM 3 (Darboux). *Let Ω be a C^s symplectic structure defined on a neighborhood on* 0 *in an almost-inner product space E, and let Ω_1 be the symplectic structure which is constant on E and equal to Ω at* 0. *Then there are neighborhoods U and V of* 0 *and a C^s differmorphism $f : U \rightarrow V$ such that $f(0) = 0$ and $f^*(\Omega_1) = \Omega$.*

Proof (following Weinstein). Let $\omega = \Omega_1 - \Omega$, $\Omega_t = \Omega + t\omega$, $t \in [0, 1]$. By

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the compactness of [0, 1] and the openness of invertibility there is an open ball W about 0 such that on W all the Ω_t are symplectic structures. By Poincare's lemma there is a 1-form ϕ on W such that $d\phi = \omega$ and $\phi(0) = 0$. An examination of the proof of the lemma [**8**] shows that $\phi : TM \to R$ is fibrewise continuous in the w-norm. Thus there is a smooth vector field $\phi^{\#}$ such that for each $x \in M$, $\phi_x(v) = \beta(\phi^{\#}(x), v)$ for all $v \in E$.

Recall that

$$\Omega_t(x)(u, v) = \beta(\Omega_t^{\#}(x)u, v)$$

where $\Omega_t^{\#} : W \times E \to W \times E$ is a smooth bundle homomorphism which is fibrewise continuous in the w-norm. Set $X_t(x) = -[\Omega_t^{\#}(x)]^{-1}\phi^{\#}(x)$. So X_t is a time dependent vector field with $-\Omega_t(x)(X_t(x), v) = \phi_x(v)$. Moreover X_t vanishes at 0. We may integrate the vector field X_t to a family $\{f_t\}$ of mappings partially defined on W. The compactness of [0, 1] and the openness of the domain of $\{f_t\}$ in $W \times [0, 1]$ imply the existence of a neighborhood U of 0 on which all the f_t are defined.

Recall that if Ω is any symplectic form and f_t is the flow of X_t with $\phi = -i_{X_t}\Omega$

$$\frac{d}{dt}(f_t^*\Omega) = f_t^*(L_{X_t}\Omega) = f_t^*(i_{X_t}d\Omega + d(i_{X_t}\Omega))$$

$$f_t^*\{d(i_{X_t}\Omega)\} = f_t^*(-d\phi).$$

Thus for the Ω_t under consideration and any arbitrary $t_0 \in [0, 1]$

$$\frac{d}{dt}(f_t^*\Omega_t)|_{t=t_0} = f_{t_0}^*\left(\frac{d\Omega_t}{dt}|_{t=t_0}\right) + \frac{d}{dt}(f_t^*\Omega_{t_0})|_{t=t_0}.$$

which by means of the above formula equals $f_{t_0}^*(\omega) + f_{t_0}^*(-d\phi) = 0$. Whence $f_1^*(\Omega_1) = f_0^*(\Omega_0) = \Omega_0 = \Omega$ since $f_0(x) = x$ for all x. Setting $f = f_1$, $V = f(U)$ gives the desired result.

5. The canonical symplectic form induced by an almost-Riemannian structure. Let E be a reflexive Banach space ($E \cong E^{**}$) and let $N = U \times E^*$ where $U \subset E$ is open. Then it is well known that there is a canonical simplectic form Ω on N defined by

$$\Omega_{(u,v)}[h_1, h_2), (k_1, k_2)] = k_2(h_1) - h_2(k_1)$$

where (h_1, h_2), $(k_1, k_2) \in E \times E^*$ (e.g. see Lang [**8**]). Suppose now that E is an almost-inner product space with almost-inner product β. Let $M = U \times E$. One readily checks that M is also an almost-Riemannian manifold. In fact $E \times E$ is an almost-inner product space with induced inner product $\bar{\beta}[(h_1, h_2), (k_1, k_2)] = \beta(h_1, k_1) + \beta(h_2, k_2)$.

Using the almost-inner product β on E we can duplicate the above construction and define a symplectic structure Ω on M as follows (Ω is symplectic in the sense of almost-Riemannian manifolds)

$$\Omega_{(u,v)}[(h_1, h_2), (k_1, k_2)] = \beta(k_2, h_1) - \beta(h_2, k_1) = \bar{\beta}[(-h_2, h_1), (k_1, k_2)].$$

Now suppose $x \to \beta_x$ is another almost-Riemannian structure on U. Then for $(e, h) \in E \times E$ there is a unique $e(x)$ with $\beta_x(e, h) = \beta(e(x), h)$. Then $(x, e) \xrightarrow{\varphi} (x, e(x))$ is a diffeomorphism of $U \times E$ onto $U \times E$ or of M onto M. We can use the diffeomorphism φ to pull back the canonical symplectic two-form Ω defined immediately above to a simplectic two form Ω^φ given by

$$\Omega^\varphi(x, e)[(h_1, h_2), (k_1, k_2)] = \{D_x\beta_x(e, h_1)\}[k_1] - \{D_x\beta_x(e, k_1)\}[h_1] + \beta_x(k_2, h_1) - \beta_x(h_2, k_1)$$

where D_x denotes the Frechet derivative of the map $x \to \beta_x$ and Ω^φ is the symplectic form on $M = U \times E$ induced by $x \to \beta_x$.

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