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ON SMOOTHNESS OF MINIMAL MODELS OF QUOTIENT SINGULARITIES BY FINITE SUBGROUPS OF $SL_n(\mathbb{C})$

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Abstract. We prove that a quotient singularity $\mathbb{C}^n/G$ by a finite subgroup $G \subset SL_n(\mathbb{C})$ has a crepant resolution only if G is generated by junior elements. This is a generalization of the result of Verbitsky (*Asian J. Math.* **4**(3) (2000), 553–563). We also give a procedure to compute the Cox ring of a minimal model of a given $\mathbb{C}^n/G$ explicitly from information of G. As an application, we investigate the smoothness of minimal models of some quotient singularities. Together with work of Bellamy and Schedler, this completes the classification of symplectically imprimitive quotient singularities that admit projective symplectic resolutions.

1. Introduction. Crepant resolutions of singularities play key roles in various branches of algebraic geometry, and have been studied intensively. When one treats a quotient singularity by a finite group, crepant resolutions have particularly important meanings in the context of the McKay correspondence, which relates the geometry of a crepant resolution to the representation theory of the group. The aim of this paper is to tackle the existence problem of crepant resolutions of quotient singularities.

Let V be a finite dimensional $\mathbb{C}$-vector space and let $G \subset SL(V)$ be a finite subgroup. How can we determine the existence of a crepant resolution of the given quotient singularity V/G? When $\dim V = 2$, V/G is a well-known Kleinian singularity and therefore it has a unique crepant resolution. The existence of crepant resolutions for three-dimensional cases is also proven (cf. [**9, 27**]). However, for higher dimensional cases, crepant resolutions do not exist in general. No general criterion for existence of crepant resolutions has been known, but when V is a symplectic vector space and G is a subgroup of the symplectic group $Sp(V)$, there is a useful necessary condition. Verbitsky proved that V/G for $G \subset Sp(V)$ admits a symplectic (or equivalently crepant) resolution only if G is generated by symplectic reflections [**30**, Theorem 1.1] (see Section 3 for the definition).

One of our main results in this paper is the following.

THEOREM 1.1. *If V/G for $G \subset SL(V)$ admits a crepant resolution, then G is generated by junior elements.*

We will define a junior element $g \in G$ and give a proof of the theorem in Section 3. For symplectic cases, junior elements are nothing but symplectic reflections. Thus, the theorem is a generalization of Verbitsky's result to nonsymplectic cases.

We also suggest a procedure to determine the (non)existence of projective crepant reslutions of V/G for a given finite subgroup $G \subset SL(V)$. The idea is as follows. By a general result of birational geometry established in [**4**], it is known that V/G always admits a minimal model. Since crepant resolutions are nothing but smooth minimal models, it is enough to check whether each minimal model X is smooth or not. To this end, we compute the Cox ring $\mathrm{Cox}(X)$, which was introduced by Hu and Keel [**19**], of X from G (without constructing X explicitly) and recover X from $\mathrm{Cox}(X)$. This is done by using the similar method to one by Donten-Bury and Wiśniewski in [**15**], where the authors give the Cox ring of a symplectic resolution of V/G for four-dimensional V and G of order 32. We generalize their method to minimal models for any finite subgroups in $SL(V)$. We give the algorithm to calculate generators of Cox rings and show several examples including Kleinian singularities in Section 4. This gives a different calculation of the Cox rings of the minimal resolutions of Kleinian singularities from the ones in [**16**] and [**13**]. Most of the calculations need a help of computer software such as 'SINGULAR' [**17**] or 'Macaulay2' [**18**]. In Appendix, in Section 7, we give efficient ways of calculations.

In Section 5, we study the property of the Cox rings from the viewpoint of geometric invariant theory (GIT) and birational geometry. The spectrum $\mathfrak{X} = \mathrm{Spec}(\mathrm{Cox}(X))$ has a natural action by an algebraic torus associated to the divisor class group $\mathrm{Cl}(X)$. The crucial fact is that every minimal model can be recovered from the Cox ring as a GIT quotient of $\mathfrak{X}$ by the torus action with an appropriate linearization. The sets of generic linearizations form a fan called the GIT fan on the vector space $\mathrm{Cl}(X) \otimes \mathbb{R}$. We also discuss the structure of the GIT fan. It contains some information such as the number of minimal models. We give one example in Section 5.

Let $\chi \in \mathrm{Cl}(X) \otimes \mathbb{R}$ be a linearization that gives the minimal model X. If the (semi)stable locus $U \subset \mathfrak{X}$ associated to χ is smooth, one can check the smoothness of X by looking at the torus action on U. However, the author does not know if U is always smooth. Moreover, when the order of the group is not small, it seems almost impossible to calculate the relations of the generators of the Cox ring and to check the smoothness of U by the Jacobian criterion even if one uses a computer. Thus from these viewpoints, our method is not enough to completely answer the question raised in the second paragraph of this section.

As an application of the description of the Cox rings, we classify all symplectically imprimitive subgroups $G \subset Sp(V)$ such that V/G admits a projective crepant resolution. This was already done by Bellamy and Schedler except six types of groups all of which are subgroups of $Sp(4, \mathbb{C})$ [**10**]. In Section 6, we complete the classification by studying the remaining six cases (Theorem 6.1). It will turn out that only one group among the exceptional groups admits a projective crepant resolution and that this is not a new example.

2. Quotient singularities, minimal models and Cox rings.

Let V be a complex vector space of dimension n, and let G be a finite subgroup of $SL(V)$. Note that G contains no pseudo-reflection, that is, $g \in G$ such that $\mathrm{codim}_V V^g = 1$ where V^g denotes the fixed subspace by g. It is well-known that the quotient singularity $V/G = \mathrm{Spec}\, \mathbb{C}[V]^G$ by G is Gorenstein [**31**, Theorem 1]. Thus, we can talk about the discrepancy of exceptional divisors of a birational morphism to V/G.

DEFINITION 2.1. A minimal model of V/G is a $\mathbb{Q}$-factorial normal variety X that has only terminal singularities together with a crepant birational morphism $X \to V/G$.

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Note that a nonsingular minimal model of V/G is nothing but a crepant resolution of V/G. Throughout this paper all minimal models and crepant resolutions are assumed to be projective over V/G unless otherwise stated. The following theorem is a consequence of a result in the celebrated paper of Birker, Cascini, Hacon and McKernan.

THEOREM 2.2 (Special case of [**4**, Theorem 1.2]). *There exists a minimal model X of V/G.*

To introduce Cox rings, we should consider the divisor class group of a minimal model. As for V/G, its divisor class group $\mathrm{Cl}(V/G)$ is known.

PROPOSITION 2.3 ([**6**, Ch. 3]). *The divisor class group $\mathrm{Cl}(V/G)$ of the quotient singularity V/G is canonically isomorphic to the group $Ab(G)^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ of characters of G where $Ab(G)$ denotes the abelianization $G/[G, G]$ of G. In particular $\mathrm{Cl}(V/G)$ is a torsion group.*

Let X be a minimal model. Then, one easily sees that $\mathrm{Cl}(X)$ is also finitely generated since every divisor of X consists of exceptional divisors of π and divisors from V/G. We now assume that $\mathrm{Cl}(X)$ is torsion-free for simplicity. Let $D_1, \ldots, D_m$ be Weil divisors whose classes form a basis of $\mathrm{Cl}(X)$. We define the Cox ring of X as

$$\mathrm{Cox}(X) = \bigoplus_{(a_1,\ldots,a_m)\in\mathbb{Z}^m} H^0(X, \mathcal{O}_X(a_1D_1 + \cdots + a_mD_m)).$$

For a Weil divisor D, the vector space $H^0(X, \mathcal{O}_X(D))$ is identified with the set

$$\{f \in \mathbb{C}(X)^* \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\},$$

and the Cox ring has the natural ring structure inherited from the multiplication in $\mathbb{C}(X)$. It is known that the isomorphism class of the Cox ring is independent of the choice of D_i's (cf. [**1**, Section 1.4]). $\mathrm{Cox}(X)$ has a $\mathrm{Cl}(X)$-grading and this grading gives a torus action on $\mathrm{Cox}(X)$ in the following way. Let $T := \mathrm{Hom}(\mathrm{Cl(X)}, \mathbb{C}^*)$ be the algebraic torus. It acts on the homogeneous part $H^0(X, \mathcal{O}_X(a_1D_1 + \cdots + a_mD_m))$ of $\mathrm{Cox}(X)$ for $D = \sum_{i=1}^m a_iD_i \in \mathrm{Cl}(X)$ by multiplying by $t(D)$ for each $t \in T$. This action naturally induces an action on the spectrum $\mathfrak{X} = \mathrm{Spec}(\mathrm{Cox(X)})$.

Next, we consider GIT quotients of $\mathfrak{X}$ by T. To this end, we should choose a T-linearization on $\mathfrak{X}$. We particularly use the trivial line bundle twisted by a character of T. When we take a divisor class $D \in \mathrm{Cl}(X)$, we can regard it as a character of T by the evaluation map $T \to \mathbb{C}^*$, $t \mapsto t(D)$. We can define the GIT quotient $\mathfrak{X}/\!/_D T$ of $\mathfrak{X}$ by T with respect to a character D of T. As we will see later in Section 5, the most important feature of $\mathfrak{X}$ is that every minimal model X' of V/G can be obtained as a GIT quotient of $\mathfrak{X}$ for some D.

3. Discrete valuations on function fields. Let V and G be as in the previous section. For an element $g \in G$, we define a discrete valuation $v_g : \mathbb{C}(V) \to \mathbb{Z} \cup \{\infty\}$ on the rational function field $\mathbb{C}(V)$ of V as follows. Let $x_1, \ldots, x_n \in V^*$ be linearly independent eigenvectors of g. Then, there are unique integers a_i for $i = 1, \ldots, n$, such that $0 \leq a_i < r$ and $g \cdot x_i = \zeta^{a_i} x_i$, where r is the order of g and $\zeta = \exp(2\pi i/r) \in \mathbb{C}$ is

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the primitive rth root of unity. For any nonzero polynomial

$$f = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}} c_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n},\ c_\alpha \in \mathbb{C},$$

we set

$$\nu_g(f) = \min_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ c_\alpha\neq 0}} \left\{\sum_i \alpha_i a_i\right\}.$$

This extends uniquely to a discrete valuation on the whole of $\mathbb{C}(V)$.

If $\pi : X \to V/G$ is a minimal model, the function field $\mathbb{C}(X)$ of X is identified with the G-invariant subfield $\mathbb{C}(V)^G$ of $\mathbb{C}(V)$. Therefore, we get a discrete valuation on $\mathbb{C}(X)$ by restricting ν_g. On the other hand, we also have another valuation on $\mathbb{C}(X)$. Let E be an irreducible exceptional divisor of π. Then, it gives the divisorial valuation ν_E on $\mathbb{C}(X)$ defined by $\nu_E(f) = \mathrm{ord}_E(\mathrm{div}(f))$ for $f \in \mathbb{C}(X)^*$.

Next, we introduce the notion of the *age* of an element $g \in G$. Let $a_1, \dots, a_n$ and r be as above. Then, we set $\mathrm{age}(g) = \frac{1}{r}\sum_{i=1}^n a_i$. Note that $\mathrm{age}(g)$ is always an integer since g is in $SL(V)$. One can easily check that age is invariant under conjugation by $GL(V)$.

REMARK. In [**20**, 2.1], age is not defined as a function on G but on $\Gamma := \mathrm{Hom}(\mu_R, G)$, where $R \in \mathbb{N}$ is a common multiple of the orders of all elements in G and μ_R is the group of the Rth roots of unity. However, the definition in [**20**] coincides with ours via the isomorphism $\Gamma \to G; f \mapsto f(\exp(2\pi i/R))$.

We call an element $g \in G$ *junior* if $\mathrm{age}(g) = 1$. The following theorem claims that the information about exceptional divisors of a minimal model can be read off from the information about G. Ito and Reid proved the following.

THEOREM 3.1 ([**20**, Theorem 1.4]). *Let $\pi : X \to V/G$ be a proper birational morphism from a $\mathbb{Q}$-factorial normal variety X. Then, π is a (not necessarily projective) minimal model if and only if there is a bijection of the sets*

$$\{\textit{an irreducible exceptional divisor of } \pi\} \cong \{\textit{a conjugacy class of junior elements in } G\}$$

such that if E is an irreducible exceptional divisor of π that corresponds to $g \in G$ via this bijection, the equality

$$\nu_E = \frac{1}{r}\nu_g|_{\mathbb{C}(X)}$$

holds.

We also give a result in relation to the existence of a smooth minimal model (i.e. a crepant resolution) of V/G.

DEFINITION 3.2. Let V be a finite dimensional symplectic $\mathbb{C}$-vector space and let G be a finite subgroup of $Sp(V)$. An element $g \in G$ is called a symplectic reflection if the codimension of the fixed subspace V^g in V is two.

Verbitsky proved that if V/G admits a (not necessarily projective) symplectic (or equivalently crepant, cf. [**22**, Proposition 3.2]) resolution, then G is generated by

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symplectic reflections [**30**, Theorem 1.1]. The aim of this section is to generalize this result.

THEOREM 3.3. *Let G be a finite subgroup of $SL(V)$ for a finite-dimensional $\mathbb{C}$-vector space V and let $\pi : X \to V/G$ be a minimal model. Then, the algebraic fundamental group $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ of the regular part of X is trivial if and only if G is generated by junior elements.*

For the definition of an algebraic (or étale) fundamental group, see e.g. [**25**, Section 1.3].

Proof. Let $H \subset G$ be the normal subgroup generated by all junior elements in G and let X' (resp. X'') be the main component (which dominates V/G) of the normalization of the fibre product $X \times_{V/G} V/H$ (resp. $X \times_{V/G} V$). Then, we have the commutative diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{\widetilde{p}'} & X' & \xrightarrow{\widetilde{p}} & X \\ {\scriptstyle \pi''}\downarrow & & {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{p'} & V/H & \xrightarrow{p} & V/G. \end{array}$$

We need the following lemma.

LEMMA 3.4. *Let E be a π-exceptional divisor. Then, the finite surjective morphism $\widetilde{p}$ is unramified at any generic point of $\widetilde{p}^{-1}(E)$.*

Proof. Let $F_1, \ldots, F_k$ be the irreducible components of $\widetilde{p}^{-1}(E)$ and let F'_i be an irreducible component of $\widetilde{p}'^{-1}(F_i)$. By construction, one has the equality of valuations on $\mathbb{C}(X)$

$$\nu_{F'_i}|_{\mathbb{C}(X)} = r_1 \nu_{F_i}|_{\mathbb{C}(X)} = r_2 \nu_E,$$

for any i, where r_1 and r_2 are the ramification indices along F'_i of $\widetilde{p}'$ and $\widetilde{p} \circ \widetilde{p}'$, respectively. Let $g \in G$ be an element in the conjugacy class corresponding to E via the bijection in Theorem 3.1. By [**20**, 2.6 and 2.8], one has $r_1 = r_2 = \sharp \langle g \rangle$. Therefore, the claim holds. □

REMARK. From this lemma one can check that $\pi' : X' \to V/H$ is crepant and thus π' is a minimal model (provided that X' is $\mathbb{Q}$-factorial). Let $C \subset G$ be the G-conjugacy class containing g that corresponds to E. Then, the decomposition of $\widetilde{p}^{-1}(E)$ into the irreducible components corresponds to the division of C into the H-conjugacy classes via the bijection in Theorem 3.1 for π'.

Now, we return to the proof of the theorem. First, we assume that $G \neq H$. By this assumption and Lemma 3.4, the map $\widetilde{p}$ is étale in codimension one of $\deg(\widetilde{p}) > 1$. By the purity of branch locus, this implies that $\widetilde{p}$ is étale over X_{reg}. Therefore, $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}})$ is nontrivial.

Conversely, we assume that $\pi_1^{\mathrm{alg}}(X_{\mathrm{reg}}) \neq 1$. Then, there is a nontrivial finite étale covering $Y_0 \to X_{\mathrm{reg}}$. By taking the normalization of X in $\mathbb{C}(Y_0)$, one can extend the covering map to a finite surjective morphism $q : Y \to X$. Let $Y \to Z \to V/G$ be the Stein factorization of $\pi \circ q$. As $Z \to V/G$ is finite étale over $(V/G)_{\mathrm{reg}}$, we can

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write $Z = V/K$ for a suitable normal subgroup K of G. Since q is étale in codimension one, the birational morphism $Y \to V/K$ is also a minimal model. By Theorem 3.1, we see that K contains all junior elements. (Any junior element $g \in G$ defines an exceptional divisor of $Y \to V/K$, and thus g is G-conjugate to an element in K.) Therefore, $H \subset K \subsetneq G$. □

If V is symplectic and $G \subset Sp(V)$, then symplectic reflections are nothing but junior elements (see e.g. [**21**, Lemma 1.1]). Therefore, the following is a generalization of Verbitsky's result.

COROLLARY 3.5 =Theorem 1.1. *If V/G admits a (not necessarily projective) crepant resolution, then G is generated by junior elements.*

Proof. If $X \to V/G$ is a crepant resolution, the topological fundamental group $\pi_1(X)$ is trivial (see [**23**, Theorem 7.8] or [**30**, Theorem 4.1]). This implies that $\pi_1^{\mathrm{alg}}(X)$ is also trivial [**25**, p28]. As $X = X_{\mathrm{reg}}$, we conclude by the theorem that G is generated by junior elements. Note that we did not use the projectivity of $X \to V/G$. □

4. Embedding of the Cox ring and description of the generators. The goal of this section is to give an explicit procedure for calculating the Cox ring of a minimal model of a given V/G. This is done by considering the Cox ring as a subring of some bigger and simpler ring. This construction is almost due to Donten-Bury and Wiśniewski. In [**15**], the authors calculated the Cox ring for a group of order 32 acting on a four-dimensional symplectic vector space (see Example 2 below in this section).

As in Section 2, we can also define the Cox ring of V/G. It is defined as

$$\mathrm{Cox}(V/G) = \bigoplus_{D \in \mathrm{Cl}(V/G)} H^0(V/G, \mathcal{O}_{V/G}(D)),$$

as an $H^0(V/G, \mathcal{O}_{V/G})$-module where $\mathcal{O}_{V/G}(D)$ is the rank-1 reflexive sheaf associated to a Weil divisor class D. Note that $H^0(V/G, \mathcal{O}_{V/G}(D))$ is identified with $\{f \in \mathbb{C}(V/G)^* \mid \mathrm{div}(f) + D' \geq 0\} \cup \{0\}$, where D' is any Weil divisor on V/G that represents D. Then, $\mathrm{Cox}(V/G)$ has a $\mathrm{Cl}(V/G)$-graded ring structure that is defined similarly to the case in Section 2. However, this construction is not exactly the same because of torsions in $\mathrm{Cl}(V/G)$ (cf. Proposition 2.3). See [**1**, Section 1.4] for details.

The degree zero part of $\mathrm{Cox}(V/G)$ is $\mathbb{C}[V]^G$, and thus $\mathrm{Cox}(V/G)$ is a $\mathbb{C}[V]^G$-algebra. We have the following result.

PROPOSITION 4.1 ([**2**, Theorem 3.1]). *There is an isomorphism as $\mathbb{C}[V]^G$-algebras between* $\mathrm{Cox}(V/G)$ *and* $\mathbb{C}[V]^{[G,G]}$ *that preserves the natural gradings by $Ab(G)^\vee$.*

Let $g_1, \ldots, g_m$ be a complete system of representatives of the conjugacy classes of junior elements in G and set $\nu_i := \nu_{g_i}$. Then, for each i there is a unique irreducible exceptional divisor E_i of $\pi : X \to V/G$, such that $\nu_{E_i} = \frac{1}{r_i}\nu_i|_{\mathbb{C}(X)}$ by Theorem 3.1 where r_i is the order of g_i. Let $\mathrm{Cl}(X)^{\mathrm{free}}$ be the free abelian group $\mathrm{Cl}(X)/\mathrm{Cl}(X)^{\mathrm{tor}}$, where $\mathrm{Cl}(X)^{\mathrm{tor}}$ is the torsion part of $\mathrm{Cl}(X)$. Then, the rank of $\mathrm{Cl}(X)^{\mathrm{free}}$ is m. This follows from the short exact sequence

$$0 \to \bigoplus_{i=1}^{m} \mathbb{Z}E_i \to \mathrm{Cl}(X) \xrightarrow{\pi_*} \mathrm{Cl}(V/G) \to 0, \tag{4.1}$$

noticing that $\mathrm{Cl}(V/G)$ is a torsion group (Proposition 2.3).

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Let $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] = \bigoplus_{\bar{D}\in \mathrm{Cl}(X)^{\mathrm{free}}} \mathbb{C}t^{\bar{D}}$ be the group algebra, where $t^{\bar{D}}$'s denote the basis. Now, we construct an embedding of $\mathrm{Cox}(X)$ into $\mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ as follows. For any Weil divisor D on X and any homogeneous element $\widetilde{f} \in H^0(X, \mathcal{O}_X(D)) \subset \mathrm{Cox}(X)$, we can regard $\widetilde{f}$ as an element of $H^0(V/G, \mathcal{O}_{V/G}(\pi_* D))$ via the identification $\mathbb{C}(X) = \mathbb{C}(V/G)$ between the function fields. Let $f \in \mathbb{C}[V]^{[G,G]}$ be the corresponding element to $\widetilde{f}$ via the isomorphism which appeared in Proposition 4.1. Then, we obtain a ring homomorphism $\Theta : \mathrm{Cox}(X) \to \mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ setting $\Theta(\widetilde{f}) = f \otimes t^{\bar{D}}$, where $\bar{D}$ is the class of D in $\mathrm{Cl}(X)^{\mathrm{free}}$. The following is a generalization of [**15**, Proposition 3.8].

LEMMA 4.2. $\Theta : \mathrm{Cox}(X) \to \mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ *is injective.*

Proof. Let $\widetilde{f}$ be any element in the kernel of Θ. As Θ is compatible with the quotient map $\mathrm{Cl}(X) \to \mathrm{Cl}(X)^{\mathrm{free}}$, we may assume that all the divisor classes D_i's to which the homogeneous components of $\widetilde{f}$ belong are in the same class of $\mathrm{Cl}(X)^{\mathrm{free}}$. On the other hand, the natural map $\mathrm{Cox}(X) \to \mathrm{Cox}(V/G)$, which is obtained by the composition of Θ and the evaluation $t = 1$, is also compatible with the surjection $\mathrm{Cl}(X) \to \mathrm{Cl}(V/G)$. Therefore, we may also assume that D_i's are in the same class of $\mathrm{Cl}(X)/\bigoplus_{i=1}^m \mathbb{Z}E_i$ by (4.1). However, since the subgroup $\bigoplus_{i=1}^m \mathbb{Z}E_i$ is torsion-free, the element $\widetilde{f} \in \mathrm{Cox}(X)$ must be homogeneous. In this case, the claim is clear by definition. □

Therefore, $\mathrm{Cox}(X)$ can be realized as a subring of $\mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$. Our next task is to know which elements in $\mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ are in the image of Θ.

Let $p : V/[G,G] \to V/G$ be the quotient map. For an element f of $\mathbb{C}[V]^{[G,G]}$ that is homogeneous with respect to the $Ab(G)^\vee$-grading, consider the Weil divisor $D_f = p_*(\mathrm{div}_{V/[G,G]}(f))$ on V/G. Let $\bar{D}_f$ be the class in $\mathrm{Cl}(X)^{\mathrm{free}}$ of the strict transform of D_f by $\pi^{-1} : V/G \dashrightarrow X$.

LEMMA 4.3. *Let f and $\bar{D}_f$ be as above. Then, the equality*

$$\bar{D}_f = -\sum_{i=1}^{m} \frac{1}{r_i} \nu_i(f) \bar{E}_i$$

in $\mathrm{Cl}(X)^{\mathrm{free}}$ *holds where $\bar{E}_i$ is the class of E_i in* $\mathrm{Cl}(X)^{\mathrm{free}}$. *Moreover, $f \otimes t^{\bar{D}_f}$ is in* $\mathrm{Im}\,\Theta$.

Proof. Let $\widetilde{D}_f$ be the strict transform of D_f by $\pi^{-1} : V/G \dashrightarrow X$ and let $\widetilde{f}$ be the element of $\mathrm{Cox}(V/G)$ that corresponds to f via the isomorphism in Proposition 4.1. As f is homogeneous, some power f^r ($r \in \mathbb{N}$) is in $\mathbb{C}[V]^G$ and equals $\widetilde{f}^r \in \mathrm{Cox}(V/G)_0$. Since the pullback of rD_f by π can be written as

$$\pi^*(\mathrm{div}_{V/G}(\widetilde{f}^r)) = r\widetilde{D}_f + \nu_{E_1}(\widetilde{f}^r)E_1 + \cdots + \nu_{E_m}(\widetilde{f}^r)E_m,$$

we have $r\widetilde{D}_f = -r\sum_{i=1}^m \frac{1}{r_i}\nu_i(f)E_i$ in $\mathrm{Cl}(X)$ by Theorem 3.1. By dividing both sides by r, we obtain the desired equation.

For the second claim, one can easily check by definition that $\Theta(\widetilde{f}) = f \otimes t^{\bar{D}_f}$. □

By this lemma, we can describe generators of $\mathrm{Im}\,\Theta$ from those of $\mathbb{C}[V]^{[G,G]}$. Let $S = \{\phi_1, \ldots, \phi_k\}$ be a generating system of $\mathbb{C}[V]^{[G,G]}$ such that each ϕ_j is homogeneous with respect to the $Ab(G)$-action. We consider the following condition $(*)$:

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'For every nonzero $Ab(G)^\vee$-homogeneous $f \in \mathbb{C}[V]^{[G,G]}$, there are monomials $f_1, \ldots, f_l$ of $\phi_1, \ldots, \phi_k$ such that

(i) $f = f_1 + \cdots + f_l$, and

(ii) $\nu_i(f) \leq \nu_i(f_j)$ for all i and j.'

REMARK. Condition $(*)$ is a reformulation of condition (3.6) in the paper of Donten-Bury and Grab [**14**] where they consider $G \subset Sp(V)$ whose commutator group $[G, G]$ contains no symplectic reflections. Proposition 4.4 below is a counterpart of Theorem 3.8 in [**14**]. In the proof of Proposition 4.4 given below, we use the essentially same idea as in [**14**]. Nevertheless, we give the proof since we can avoid complexity caused by using the space of curves introduced therein.

PROPOSITION 4.4. *If the homogeneous generators $\phi_1, \ldots, \phi_k$ of $\mathbb{C}[V]^{[G,G]}$ satisfy $(*)$, then the subset*

$$\{\phi_j \otimes t^{\bar{D}_{\phi_j}}\}_{j=1,\ldots,k} \cup \{t^{\bar{E}_1}, \ldots, t^{\bar{E}_m}\}$$

of $\mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ is a generating system of $\mathrm{Im}\,\Theta$.

Proof. First note that $t^{\bar{E}_1}, \ldots, t^{\bar{E}_m}$ are in $\mathrm{Im}\,\Theta$ since E_i's are effective divisors. Take any homogeneous element $f \otimes t^{\bar{D}}$ in $\mathrm{Im}\,\Theta$ with a Weil divisor D on X. By the condition $(*)$, we can write $f = f_1 + \cdots + f_l$ satisfying the conditions and hence can also write

$$f \otimes t^{-\sum_{i=1}^m \frac{1}{r_i}\nu_i(f)\bar{E}_i} = \sum_{j=1}^{l} f_j \otimes t^{-\sum_{i=1}^m (\frac{1}{r_i}\nu_i(f_j)\bar{E}_i - \sum_i a_{i,j}\bar{E}_i)},$$

for some $a_{i,j} \in \mathbb{Q}_{\geq 0}$. Since the images of the RHS and $\sum_{j=1}^l f_j \otimes t^{-\sum_{i=1}^m \frac{1}{r_i}\nu_i(f_j)\bar{E}_i}$ by the natural map $\mathrm{Im}\,\Theta \to \mathbb{C}[V]^{[G,G]} = \mathrm{Cox}(V/G)$ are the same, the sum $\sum_i a_{i,j}\bar{E}_i$ must be in $\bigoplus_{i=1}^m \mathbb{Z}\bar{E}_i$.

Take $\widetilde{f}$ so that $\Theta(\widetilde{f}) = f \otimes t^{\bar{D}}$. Then, the inequality $\sum_{i=1}^m \frac{1}{r_i}\nu_i(f)E_i + D \geq 0$ must be satisfied since $\widetilde{f}$ is in $H^0(X, \mathcal{O}_X(D))$. Thus, we can write $f \otimes t^{\bar{D}} = f \otimes t^{-\sum_{i=1}^m \frac{1}{r_i}\nu_i(f)\bar{E}_i} \cdot t^{\sum_i b_i\bar{E}_i}$ for some $b_i \in \mathbb{Q}_{\geq 0}$. The same argument as above shows that b_i is in $\mathbb{Z}_{\geq 0}$. Since each f_j is a monomial of $\phi_1, \ldots, \phi_k$, each $f_j \otimes t^{\bar{D}_{f_j}} = f_j \otimes t^{-\sum_{i=1}^m \frac{1}{r_i}\nu_i(f_j)\bar{E}_i}$ is also a monomial of $\phi_j \otimes t^{\bar{D}_{\phi_j}}$'s. Therefore, we obtain a desired expression of $f \otimes t^{\bar{D}}$. □

Therefore, we can construct the Cox ring explicitly if we find generators of $\mathbb{C}[V]^{[G,G]}$ satisfying $(*)$. Now, we give an algorithm for finding such generators from any generators of $\mathbb{C}[V]^{[G,G]}$.

Let $\phi_1, \ldots, \phi_k$ be generators of $\mathbb{C}[V]^{[G,G]}$. We may assume that they are homogeneous with respect to the $Ab(G)$-action. For each $i \in \{1, \ldots, m\}$, let $x_{i,1}, \ldots, x_{i,n} \in V^*$ be linearly independent eigenvectors for the $\langle g_i \rangle$-action. When we write an element $\phi \in \mathbb{C}[V]^{[G,G]}$ as the sum of monomials of $x_{i,1}, \ldots, x_{i,n}$, let $\min_i(\phi)$ be the sum of the monomials whose values of ν_i are minimal among these monomials.

Consider the ring homomorphism $\alpha : \mathbb{C}[X_1, \ldots, X_k] \to \mathbb{C}[V]^{[G,G]}$, $X_j \mapsto \phi_j$ and let I be the kernel of α. We give a grading on $\mathbb{C}[X_1, \ldots, X_k]$ by setting $\deg_i(X_j) = \nu_i(\phi_j)$. For an inhomogeneous polynomial h, let $\deg_i(h)$ denote the minimal degree of h. We can

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define the minimal part $\min_i(h)$ for $h \in \mathbb{C}[X_1, \ldots, X_k]$, and let $\min_i(I) \subset \mathbb{C}[X_1, \ldots, X_k]$ be the ideal generated by the set $\{\min_i(h) \mid h \in I\}$. One sees that, for each nonzero $h \in \min_i(I)$, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$ and $\deg_i(h) < \deg_i(\widetilde{h})$.

On the other hand, consider the ring homomorphism $\beta_i : \mathbb{C}[X_1, \ldots, X_k] \to \mathbb{C}[V]$, $X_j \mapsto \min_i(\phi_j)$ and let J be the kernel of β_i. Let $\underline{\min}_i(J)$ denote the ideal generated by homogeneous elements in J with respect to the $Ab(G)$-action where the $Ab(G)$-action on $\mathbb{C}[X_1, \ldots, X_k]$ is the natural lift of the $Ab(G)$-action on $\mathbb{C}[V]^{[G,G]}$. Thus, we obtain two ideals of the polynomial ring associated to $\phi_1, \ldots, \phi_k$: the ideal $\min_i(I)$ of the minimal terms of the relations, and the ideal $\underline{\min}_i(J)$ of the $Ab(G)^\vee$-homogeneous relations of the minimal terms.

The reason for introducing $\min_i(I)$ and $\underline{\min}_i(J)$ is explained as follows. Fix i, and assume that we are given a nonzero $Ab(G)^\vee$-homogeneous element $f \in \mathbb{C}[V]^{[G,G]}$. When we take a lift $h \in \mathbb{C}[X_1, \ldots, X_k]$ of f such that $\alpha(h) = f$, the inequality $\deg_i(h) \leq \nu_i(f)$ always holds. If $\deg_i(h) = \nu_i(f)$, then the condition $(*)$ is satisfied for f and i. However, $\deg_i(h) < \nu_i(f)$ can happen in general. This case happens for the reason that the image of $\min_i(h)$ by β_i cancels. Thus, $\underline{\min}_i(J)$ is the set of such 'bad' polynomials. Nevertheless, if such h is in $\min_i(I)$, then we can take another 'better' polynomial $\widetilde{h}$ such that $\alpha(\widetilde{h}) = f$ and $\deg_i(h) < \deg_i(\widetilde{h})$. The algorithm we will give below can produce generators of $\mathbb{C}[V]^{[G,G]}$ such that $\min_i(I) = \underline{\min}_i(J)$ (Proposition 4.9).

We have the following lemma.

LEMMA 4.5. *For each i, the two ideals* $\min_i(I)$ *and* $\underline{\min}_i(J)$ *are* $\deg_i$*-homogeneous. Moreover, the inclusion* $\min_i(I) \subset \underline{\min}_i(J)$ *holds.*

Proof. The $\deg_i$-homogeneity for $\min_i(I)$ is clear by definition. To prove the claim for $\underline{\min}_i(J)$, give a grading on $\mathbb{C}[V]$ such that the degree of $x_{i,j}$ is $\nu_i(x_{i,j})$. Then, the map β_i sends elements which are in distinct $\deg_i$-homogeneous components to ones that are homogeneous and have distinct degrees. Therefore, the kernel J is $\deg_i$-homogeneous.

Let h be any $Ab(G)^\vee$-homogeneous element of J. Let $h = \sum_j h_j$ be the unique decomposition of h into the sum of monomials in $\mathbb{C}[X_1, \ldots, X_k]$. Since every monomial of $\mathbb{C}[X_1, \ldots, X_k]$ is $Ab(G)^\vee$-homogeneous, all h_j's are in the same $Ab(G)^\vee$-homogeneous component. Since every monomial is also $\deg_i$-homogeneous, the sum of the monomials h_j which take the same value, say d, of $\deg_i$ is exactly the $\deg_i$-homogeneous component of degree d of h. By the $\deg_i$-homogeneity of J, this $\deg_i$-homogeneous component of h is contained in J. Thus every $Ab(G)^\vee$-homogeneous element of J is decomposed into the sum of elements of J which are both $Ab(G)^\vee$-homogeneous and $\deg_i$-homogeneous. This implies that $\underline{\min}_i(J)$ is $\deg_i$-homogeneous.

For the claim of the inclusion, it suffices to show that each nonzero $\min_i(h)$ with $h \in I$ is in $\underline{\min}_i(J)$. We may assume that $\min_i(h)$ is $Ab(G)^\vee$-homogeneous since α is $Ab(G)^\vee$-graded homomorphism. Set $d = \deg_i(h)$. As mentioned above, we can take $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$ and $\deg_i(\widetilde{h}) > d$. We see that $\nu_i(\alpha(\widetilde{h})) > d$ by the definition of $\deg_i$. Since $\alpha(\widetilde{h}) = \alpha(h)$, we also have $\nu_i(\alpha(h)) > d$. This implies that $\beta_i(h)$ vanishes since $\beta_i(h)$ is exactly the part of $\alpha(h)$ whose value of ν_i is d. $\square$

For a subset $A \subset \{1, \ldots, m\}$, let R_A be the polynomial ring $\mathbb{C}[X_1, \ldots, X_k, \{t_i\}_{i \in A}]$. The grading $\deg_i$ $(i \in A)$ on $\mathbb{C}[X_1, \ldots, X_k]$ naturally extends to one on R_A by setting $\deg_i(t_j) = -\delta_{i,j}$ where $\delta_{i,j}$ is Kronecker delta. For nonzero $h \in \mathbb{C}[X_1, \ldots, X_k]$, let h_A be the element in R_A obtained by homogenizing h by t_i's with respect to $\deg_i$'s respectively. That is, if $h = \sum_l h_l$ where h_l's are monomials, then $h_A = \sum_l (h_l \prod_{i \in A} t_i^{\deg_i(h_l) - \deg_i(h)})$.

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Consider the ideal $I = \mathrm{Ker}\,\alpha$ as above and let $I_A \subset R_A$ be the homogeneous ideal generated by the set $\{h_A \mid h \in I\}$.

Now we define a collection $\{S_p\}_{p=0,1,\dots}$ of subsets each of which consists of finitely many elements in $\mathbb{C}[V]^{[G,G]}$ inductively by taking the following steps.
Step (0)
Set $S = S_0 := \{\phi_1, \dots, \phi_k\}$ and go to *Step* (0, 1).

Step $(0, i)$ $(i = 1, \dots, m)$
Compute $\min_i(I)$ and $\underline{\min}_i(J)$ for S. Write $\underline{\min}_i(J) = \min_i(I) + (h_1, \dots, h_l)$ with $\deg_i$-homogeneous $h_j \notin \min_i(I)$.

- If $\min_i(I) = \underline{\min}_i(J)$ and $i = m = 1$, then set $S_{p+1} = S_{p+2} = \cdots := S_p$.
- If $\min_i(I) = \underline{\min}_i(J)$ and $i = 1 < m$, then replace S by $S_{p+1} := S_p$ and go to *Step* $(0, 2)$.
- If $\min_i(I) = \underline{\min}_i(J)$ and $i > 1$, then replace S by $S_{p+1} := S_p$ and go to *Step* $(1, i)$.
- If $\min_i(I) \subsetneq \underline{\min}_i(J)$, then replace S by $S_{p+1} := S_p \cup \{\alpha(h_1), \dots, \alpha(h_l)\}$ and go to *Step* $(0, i)$ again.

Step $(i', i)(1 \le i' < i \le m)$
Compute the two ideals $\widetilde{I}_{i',i} := I_{\{1,\dots,i',i\}} \cap (t_{i'}, t_i)$ and $\widetilde{I}'_{i',i} := (I_{\{1,\dots,i',i\}} \cap (t_{i'})) + (I_{\{1,\dots,i',i\}} \cap (t_i))$ for S. Write

$$\widetilde{I}_{i',i} = \widetilde{I}'_{i',i} + (h_1, \dots, h_l),$$

where h_j's are elements in $R_{\{1,\dots,i',i\}} \setminus \widetilde{I}'_{i',i}$ which are homogeneous with respect to $\deg_j$ for each $j \in \{1, \dots, i', i\}$. (Note that $\widetilde{I}_{i',i}$ and $\widetilde{I}'_{i',i}$ are $\deg_j$-homogeneous ideals for each $j \in \{1, \dots, i', i\}$.)

- If $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$, $i = m$ and $i' = m - 1$, then set $S_{p+1} = S_{p+2} = \cdots := S_p$.
- If $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$, $i < m$ and $i' = i - 1$, then replace S by $S_{p+1} := S_p$ and go to *Step* $(0, i + 1)$.
- If $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$, $i < m$ and $i' < i - 1$, then replace S by $S_{p+1} := S_p$ and go to *Step* $(i' + 1, i)$.
- If $\widetilde{I}_{i',i} \supsetneq \widetilde{I}'_{i',i}$, then replace S by $S_{p+1} := S_p \cup \{\alpha(\min_i(h_1|_{t=1})), \dots, \alpha(\min_i(h_l|_{t=1}))\}$ and go to *Step* (i', i) again.

We should perform the above algorithm in the following order of the steps possibly with repetition in each step:

$$(0) \to (0, 1) \to (0, 2) \to (1, 2) \to (0, 3) \to (1, 3) \to (2, 3) \to (0, 4) \to (1, 4) \to \cdots$$

Note that each S_p is not unique since it involves several choices. A concrete procedure for performing the algorithm above (with a computer) will be given in Appendix in Section 7.

The relation between $\widetilde{I}_{i',i}$ and $\widetilde{I}'_{i',i}$ is similar to that between $\underline{\min}_i(J)$ and $\min_i(I)$. Fix i, and assume that we are given nonzero $h \in \underline{\min}_i(J)$. Then, we have $\deg_i(h) < \nu_i(f)$ where $f := \alpha(h)$. If S satisfies $\underline{\min}_i(J) = \min_i(I)$, then we can take another lift $\widetilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ of f such that $\deg_i(\widetilde{h}) = \nu_i(f)$. However, $\deg_{i'}(h) > \deg_{i'}(\widetilde{h})$ for another i' can happen in general. Such 'bad' h defines an element of $\widetilde{I}_{i',i}$. If this element is also in $\widetilde{I}'_{i',i}$, then we can replace the lift $\widetilde{h}$ by 'better' one (see the proof of Proposition 4.10). The algorithm can produce S such that $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$ (Proposition 4.12). The idea of this is as follows. Consider the situation where $\widetilde{I}_{i',i} \supsetneq \widetilde{I}'_{i',i}$ in *Step* (i', i) for $S = S_p$. Then each

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$\min_i(h_j|_{t=1})$ is 'bad' in the sense above. In S_{p+1}, however, it can be replaced by the newly introduced variable which is 'better'. We will show that we finally get the equality of the ideals after finitely many steps. The situation for *Step* $(0, i)$ is similar.

We will show that the algorithm terminates in finite time and gives generators of the Cox ring (cf. Corollary 4.13). To this end, we define conditions on S as follows.

DEFINITION 4.6. Fix $S = S_p$.

- Let $A \subset \{1, \dots, m\}$ be any subset. We say that S satisfies $(*A)$ if for every nonzero $Ab(G)^\vee$-homogeneous $f \in \mathbb{C}[V]^{[G,G]}$, there is $h \in \mathbb{C}[X_1, \dots, X_k]$ such that $f = \alpha(h)$ and $\nu_i(f) = \deg_i(h)$ for every $i \in A$.
- Let $A \subset \{1, \dots, m\} \setminus \{i\}$ be any subset and let $h \in \mathbb{C}[X_1, \dots, X_k]$ be any element. We say that h satisfies $(*A, i, p)$ if there is $\widetilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(h) < \deg_i(\widetilde{h})$, and $\deg_j(h) \le \deg_j(\widetilde{h})$ for every $j \in A$.

Note that if $f = \alpha(h)$, the inequality $\nu_i(f) \ge \deg_i(h)$ always holds. However, the equality does not hold in general.

LEMMA 4.7. *Fix $S = S_p$ and let $A \subset \{1, \dots, m\} \setminus \{i\}$ be any subset. Assume that S_p satisfies $(*A)$. Then, S_p satisfies $(*A \cup \{i\})$ if and only if h satisfies $(*A, i, p)$ for any $\deg_i$-homogeneous element $h \in \underline{\min}_i(J)$.*

Proof. We first assume that S_p satisfies $(*A \cup \{i\})$. Take any $\deg_i$-homogeneous element h from $\underline{\min}_i(J)$. Then one sees that $\nu_i(\alpha(h)) > \deg_i(h)$. On the other hand, there is $\widetilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ such that $\alpha(\widetilde{h}) = \alpha(h)$ and $\nu_j(\alpha(\widetilde{h})) = \deg_j(\widetilde{h})$ for every $j \in A \cup \{i\}$ since S_p satisfies $(*A \cup \{i\})$. Therefore, h satisfies $(*A, i, p)$.

To prove the converse, we show that S_p satisfies $(*A \cup \{i\})$ assuming that h satisfies $(*A, i, p)$ for any $\deg_i$-homogeneous $h \in \underline{\min}_i(J)$. For each nonzero $Ab(G)^\vee$-homogeneous $f \in \mathbb{C}[V]^{[G,G]}$, by assumption there is $h' \in \mathbb{C}[X_1, \dots, X_k]$ such that $f = \alpha(h')$ and $\nu_j(f) = \deg_j(h')$ for every $j \in A$. If $\nu_i(f) = \deg_i(h')$, we are done. So we assume otherwise. Then, the $\deg_i$-minimal part $h = \min_i(h')$ is in $\underline{\min}_i(J)$. Since h satisfies $(*A, i, p)$, there is $\widetilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(h) < \deg_i(\widetilde{h})$, and $\deg_j(h) \le \deg_j(\widetilde{h})$ for every $j \in A$. Therefore replacing h in h' by $\widetilde{h}$ increases the value of $\deg_i$ without decreasing the values of $\deg_j$ for $j \in A$. Repeating this process gives h' such that $\nu_i(f) = \deg_i(h')$ by induction on $\nu_i(f) - \deg_i(h')$. □

One can show that the Cox ring of a minimal model X is finitely generated using [**4**, Corollary 1.1.9]. Therefore, we can take finitely many homogeneous generators $\widetilde{f}_1, \dots, \widetilde{f}_s$ of $\operatorname{Im}\Theta$. Set $f_j := \widetilde{f}_j|_{t=1} \in \mathbb{C}[V]^{[G,G]}$. To check whether or not the condition $(*A)$ is satisfied, we only have to check it for these f_i's by the following lemma.

LEMMA 4.8. *Fix $S = S_p$ and let $A \subset \{1, \dots, m\}$ be any subset. Then S satisfies $(*A)$ if and only if for each $f \in \{f_1, \dots, f_s\}$, there is $h \in \mathbb{C}[X_1, \dots, X_k]$ such that $f = \alpha(h)$ and $\nu_i(f) = \deg_i(h)$ for every $i \in A$.*

Proof. The 'only if' part is trivial. We show the converse. By the argument in the proof of Proposition 4.4, we see that each $\widetilde{f}_j \in \operatorname{Im}\Theta$ is a product of $f_j \otimes t^{\bar{D}_{f_j}}$ and $t^{\bar{E}_i}$'s. Therefore, for any nonzero $f \in \mathbb{C}[V]^{[G,G]}$, the element $f \otimes t^{\bar{D}_f} \in \operatorname{Im}\Theta$ is expressed as a polynomial of $f_j \otimes t^{\bar{D}_{f_j}}$'s and $t^{\bar{E}_i}$'s. By evaluating $t = 1$, we obtain an expression of f as a polynomial of f_j's such that the values of ν_j's $(j \in A)$ of each monomial are greater than or equal to those of f. Replacing f_j's by the expressions as polynomials of elements in S satisfying $(*A)$, we obtain an expression of f satisfying $(*A)$. □

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The following proposition shows the termination of *Step* $(0, i)$.

Proposition 4.9. *Fix* $S = S_p \subset \mathbb{C}[V]^{[G,G]}$ *and* $i \in \{1, \ldots, m\}$. *Then the equality* $\min_i(I) = \underline{\min}_i(J)$ *holds if and only if* S *satisfies* $(*\{i\})$. *Moreover, Step* $(0, i)$ *ends in finite time (i.e. the equality* $\min_i(I) = \underline{\min}_i(J)$ *holds for* S_q *with* $q \gg p$*).*

Proof. First assume that S satisfies $(*\{i\})$. Let h be a $\deg_i$-homogeneous element of $\underline{\min}_i(J)$. Then, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $\alpha(\widetilde{h}) = \alpha(h)$ and $\deg_i(\widetilde{h}) > \deg_i(h)$ by Lemma 4.7. Since h is $\deg_i$-homogeneous, h is the minimal part of $h - \widetilde{h} \in I$ and therefore $h \in \min_i(I)$. By using the fact $\underline{\min}_i(J)$ is a $\deg_i$-homogeneous ideal (Lemma 4.5), we conclude that $\min_i(I) = \underline{\min}_i(J)$.

Conversely, we assume that $\min_i(I) = \underline{\min}_i(J)$ holds. Then, for any $h \in \underline{\min}_i(J)$, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$ and $\deg_i(h) < \deg_i(\widetilde{h})$. Thus, h satisfies $(*\emptyset, i, p)$, and S satisfies $(*\{i\})$ by Lemma 4.7.

To prove the second claim, it is enough to show that S_q satisfies $(*\{i\})$ with $q \gg p$. By Lemma 4.8, we only have to check the condition $(*\{i\})$ for each $f \in \{f_1, \ldots, f_s\}$. Similarly to the proof of Lemma 4.7, take $h' \in \mathbb{C}[X_1, \ldots, X_k]$ and $h = \min_i(h') \in \underline{\min}_i(J)$ such that $\alpha(h') = f$. Then, one can write $h = h'' + \sum_{j=1}^{l} a_j h_j$, where $h'' \in \min_i(I)$, $a_j \in \mathbb{C}[X_1, \ldots, X_k]$, and h_j's are ones in *Step* $(0, i)$. We may assume that h, h'' and each $a_j h_j$ have the same value of $\deg_i$ since $\min_i(I)$ and h_j's are $\deg_i$-homogeneous. Since $h'' \in \min_i(I)$, there is $\widetilde{h''} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h'' - \widetilde{h''} \in I$ and $\deg_i(h'') < \deg_i(\widetilde{h''})$. Let $X_{k_1}, \ldots, X_{k_l} \in \mathbb{C}[X_1, \ldots, X_{|S_{p+1}|}]$ be the new variables associated to S_{p+1} corresponding to $h_1, \ldots, h_l$, respectively. Then replacing h'' and h_j's in h by $\widetilde{h''}$ and X_{k_j} respectively increases the value of $\deg_i$. Repeating this process gives $h' \in \mathbb{C}[X_1, \ldots, X_{|S_q|}]$ for $q \gg p$ such that $\nu_i(f) = \deg_i(h')$ by induction on $\nu_i(f) - \deg_i(h')$. □

The condition $(*\{1, \ldots, i', i\})$ is in fact characterized by the equality of the ideals $\widetilde{I}_{i',i}$ and $\widetilde{I}'_{i',i}$.

Proposition 4.10. *Fix* $S = S_p \subset \mathbb{C}[V]^{[G,G]}$ *and* $1 \le i' < i \le m$. *Assume that* S *satisfies* $(*\{1, \ldots, i' - 1, i\})$. *Then, the equality* $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$ *holds if and only if* S *satisfies* $(*\{1, \ldots, i', i\})$.

Proof. First assume that S satisfies $(*\{1, \ldots, i', i\})$. Let h' be any element in $\widetilde{I}_{i',i}$ which is homogeneous with respect to $\deg_j$ for each $j = 1, \ldots, i', i$. If $t_{i'} | h'$ or $t_i | h'$, then clearly h' is in $\widetilde{I}'_{i',i}$. So we assume otherwise. Set $h = \min_i(h'|_{t=1}) \in \min_i(I)$. Since S satisfies $(*\{1, \ldots, i', i\})$, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(h) < \deg_i(\widetilde{h})$, and $\deg_j(h) \le \deg_j(\widetilde{h})$ for $j = 1, \ldots, i'$ by Lemma 4.7. Set $h'' = (h - \widetilde{h})_{\{1,\ldots,i',i\}} \in I_{\{1,\ldots,i',i\}}$. Note that both $h'|_{t_i=0}$ (which is nonzero since $t_i \nmid h'$) and $h''|_{t_i=0}$ become h when one substitutes $t = 1$. Therefore, by the homogeneity of h' and h'', one sees that $h'|_{t_i=0} = h''|_{t_i=0} \prod_{j=1}^{i'} t_j^{l_j}$ in $R_{1,\ldots,i',i}$ for some $l_j \ge 0$ $(j = 1, \ldots, i')$. Note also that $h'|_{t_i=0}$ is divisible by $t_{i'}$ since h' is in $\widetilde{I}'_{i',i}$ while $h''|_{t_i=0}$ is not divisible by $t_{i'}$ by the choice of $\widetilde{h}$. Thus, $l_{i'} > 0$ and

$$h' = h'' \prod_{j=1}^{i'} t_j^{l_j} + (h' - h'' \prod_{j=1}^{i'} t_j^{l_j}) \in I_{\{1,\ldots,i',i\}} \cap (t_{i'}) + I_{\{1,\ldots,i',i\}} \cap (t_i) = \widetilde{I}'_{i',i}.$$

Conversely, we assume that $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$ holds. To prove that S satisfies $(*\{1, \ldots, i', i\})$, it is enough to show that, for any $\deg_i$-homogeneous $h \in \underline{\min}_i(J)$, there exists

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$\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(\widetilde{h}) > \deg_i(h)$, and $\deg_j(\widetilde{h}) \geq \deg_j(h)$ for $j = 1, \ldots, i'$ by Lemma 4.7.

Since S satisfies $(*\{1, \ldots, i'-1, i\})$, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(\widetilde{h}) > \deg_i(h)$, and $\deg_j(\widetilde{h}) \geq \deg_j(h)$ for $j = 1, \ldots, i'-1$. Let $h' = (h - \widetilde{h})_{\{1,\ldots,i',i\}} \in I_{\{1,\ldots,i',i\}}$. If $h'|_{t_{i'}=t_i=0} \neq 0$ in $\mathbb{C}[X_1, \ldots, X_k]$, then $\deg_{i'}(\widetilde{h}) \geq \deg_{i'}(h)$ and thus we are done. So we assume otherwise. Then, h' is in $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$, and we can write $h' = h'_1 t_{i'}^l + h'_2 t_i$ for some $h'_j \in I_{\{1,\ldots,i',i\}}$ and $l \in \mathbb{Z}_{>0}$, such that $t_{i'} \nmid h'_1$. Therefore, one sees that $(h'_1|_{t_i=0})|_{t=1} = (h'|_{t_i=0})|_{t=1} = h$ and $h'_1|_{t_{i'}=t_i=0} \neq 0$. Then, the new $\widetilde{h} := (h'_1 - h'_1|_{t_i=0})|_{t=1} \in \mathbb{C}[X_1, \ldots, X_k]$ satisfies the desired condition. □

To show that Step $(\boldsymbol{i'}, \boldsymbol{i})$ $(i' \neq 0)$ terminates, we introduce a subset of the polynomial ring as follows. Consider the situation where one has obtained S_{p+j} from S_{p+j-1} for $j = 1, \ldots, c$ by performing Step $(\boldsymbol{i'}, \boldsymbol{i})$ $(i' \neq 0)$ and assume that $S_p \subsetneq S_{p+1} \subsetneq \cdots \subsetneq S_{p+c}$. For $j = 1, \ldots, c$, define $B_{p+j,i',i} \subset \mathbb{C}[X_1, \ldots, X_{|S_{p+j-1}|}]$ as the set

$$\left\{h \in \min_i(I) \mid h \text{ is } \deg_i\text{-homogeneous and does not satisfy } (*A, i, p+j)\right\},$$

where I is defined with respect to S_{p+j-1} and $A = \{1, \ldots, i'\}$. Note that any element of $B_{p+j,i',i}$ does not satisfy $(*A, i, p+j-1)$, either.

LEMMA 4.11. *Assume that S_p satisfies $(*\{1, \ldots, i'-1, i\})$. Then, there are integers $m_1 < m_2 < \cdots < m_c$ such that the inequality $\deg_{i'}(h) > m_j$ holds for any $h \in B_{p+j,i',i}$ and $j = 1, \ldots, c$.*

Proof. Take any element $h \in B_{p+1,i',i}$. Since S_p satisfies $(*\{1, \ldots, i'-1, i\})$, there is $\widetilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \widetilde{h} \in I$, $\deg_i(h) < \deg_i(\widetilde{h})$ and $\deg_j(h) \leq \deg_j(\widetilde{h})$ for $j \in \{1, \ldots, i'-1\}$. Then $h' := (h - \widetilde{h})_{\{1,\ldots,i',i\}}$ is in $\widetilde{I}_{i',i}$ since h does not satisfy $(*\{1, \ldots, i', i\}, i, p)$. Therefore, we can write $h' = h'' + \sum_{j=1}^{l} a_j h_j$ with $h'' \in \widetilde{I}'_{i',i}$ and $a_j \in R_{\{1,\ldots,i',i\}}$ where h_j's are ones in Step $(\boldsymbol{i'}, \boldsymbol{i})$. We may assume that h'' and each a_j are $\deg_{j'}$-homogeneous for all $j' \in \{1, \ldots, i', i\}$. As we saw in the 'only if' part of the proof of Proposition 4.10, we can take $\widetilde{h''} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $\min_i(h''|_{t=1}) - \widetilde{h''} \in I$, $\deg_i(\min_i(h''|_{t=1})) < \deg_i(\widetilde{h''})$ and $\deg_j(\min_i(h''|_{t=1})) \leq \deg_j(\widetilde{h''})$ for $j \in \{1, \ldots, i'\}$.

On the other hand, let $X_{k_1}, \ldots, X_{k_l} \in \mathbb{C}[X_1, \ldots, X_{|S_{p+1}|}]$ be the new variables associated to S_{p+1} corresponding to $h_1, \ldots, h_l$, respectively. Then, $\min_i(h_j|_{t=1}) - X_{k_j}$ is in I, $\deg_i(\min_i(h_j|_{t=1})) < \deg_i(X_{k_j})$, and $\deg_{j'}(\min_i(h_j|_{t=1})) \leq \deg_{j'}(X_{k_j})$ for $j' \in \{1, \ldots, i'\}$. Moreover, we can show that the inequality

$$\deg_{j'}(h) \leq \min_{j=1,\ldots,l}\{\deg_{j'}(\min_i((a_j h_j)|_{t=1}))\},$$

for $j' \in \{1, \ldots, i'-1\}$ holds. Indeed, otherwise h' would be divisible by $t_{j'}$ for some $j' \in \{1, \ldots, i'-1\}$ by the $\deg_{j'}$-homogeneity of h_j's. However, this is contrary to the construction of h'.

Similarly, by the $\deg_i$-homogeneity of h'' and $a_j h_j$'s, the following inequality

$$\deg_i(h) = \deg_i(\min_i((h'' + \sum_{j=1}^{l} a_j h_j)|_{t=1})) < \deg_i(\widetilde{h''} + \sum_{j=1}^{l} a_j|_{t=1} X_{k_j})$$

holds. (Otherwise it would be contrary to the construction of h'.)

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If $\deg_{i'}(h) \leq m_1 := \min_{j=1,\dots,l}\{\deg_{i'}(\min_i(h_j|_{t=1}))\}$ holds, then

$$\deg_{i'}(h) = \deg_{i'}(\min_i((h'' + \sum_{j=1}^{l} a_j h_j)|_{t=1})) \leq \deg_{i'}(\widetilde{h}'' + \sum_{j=1}^{l} a_j|_{t=1} X_{k_j}).$$

This is contrary to the fact that h does not satisfy $(*\{1, \dots, i'\}, i, p+1)$ and hence $\deg_{i'}(h) > m_1$.

Since $S_{p+1} \subsetneq S_{p+2}$, there are $h_{l+1}, \dots, h_{l'} \in \mathbb{C}[X_1, \dots, X_{|S_{p+1}|}]$ such that $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i} + (h_{l+1}, \dots, h_{l'})$ as in *Step* (i', i) for $S = S_{p+1}$. By construction $h_{l+1}, \dots, h_{l'}$ come from elements in $\widetilde{I}_{i',i} \setminus \widetilde{I}'_{i',i}$. Therefore, the elements $\min_i(h_j|_{t=1})$ with $j = l+1, \dots, l'$ do not satisfy $(*\{1, \dots, i'\}, i, p+1)$. The same argument as above shows that $m_2 := \min_{j=l+1,\dots,l'}\{\deg_{i'}(\min_i(h_j|_{t=1}))\} > m_1$. The integers $m_3, \dots, m_c$ are defined similarly and the claim about the inequality for $j = 2, \dots, c$ follows from the same argument as the case $j = 1$. □

The following proposition shows the termination of *Step* (i', i).

Proposition 4.12. *Assume that S_p satisfies $(*\{1, \dots, i'-1, i\})$. Then, Step (i', i) ends in finite time (i.e. the equality $\widetilde{I}_{i',i} = \widetilde{I}'_{i',i}$ holds for S_q with $q \gg p$).*

Proof. It is enough to show that S_q satisfies $(*\{1, \dots, i', i\})$ for $q \gg p$ by Proposition 4.10. By Lemma 4.8, we only have to check the condition $(*\{1, \dots, i', i\})$ for each $f \in \{f_1, \dots, f_s\}$. Similarly to the 'if' part of the proof of Lemma 4.7, take $h' \in \mathbb{C}[X_1, \dots, X_k]$ and $h = \min_i(h') \in \underline{\min}_i(J)$, such that $\alpha(h') = f$. By Lemma 4.11, h is not in $B_{q,i',i}$ for $q \gg p$. Therefore, h satisfies $(*\{1, \dots, i'\}, i, q)$ for $q \gg p$. □

Corollary 4.13. *The algorithm ends in finitely many steps, that is, $S_\infty := \bigcup_p S_p$ is a finite set. Moreover, the subset $\{\phi \otimes t^{\bar{D}_\phi}\}_{\phi \in S_\infty} \cup \{t^{\bar{E}_1}, \dots, t^{\bar{E}_m}\}$ of $\mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ is a generating system of* $\mathrm{Im}\,\Theta$.

Proof. The first claim follows from Propositions 4.9 and 4.12. As we see that S_∞ satisfies $(*\{1, \dots, m\})(=(*))$ by Proposition 4.10, the second claim follows from Proposition 4.4. □

This theorem makes it possible for us to calculate Cox rings of minimal models of any quotient singularities at least theoretically. Before we try concrete examples, we state one application of the above construction of Cox rings.

In the previous section, we showed that the simply-connectedness of the regular part of a minimal model is determined by whether G is generated by junior elements. By using the embedding of the Cox ring above, we can show that the torsion-freeness of the divisor class group of the minimal model can also be read from the property of G.

Proposition 4.14. *The divisor class group* $\mathrm{Cl}(X)$ *of the minimal model X of V/G is torsion-free if and only if G is generated by* $[G, G]$ *and junior elements.*

Proof. Let H be the subgroup of G generated by $[G, G]$ and junior elements. First, assume $H \neq G$. Then, there is an element $f \in \mathbb{C}[V]^H \setminus \mathbb{C}[V]^G \subset \mathbb{C}[V]^{[G,G]}$ that is homogeneous with respect to $Ab(G)$-action. Let $\bar{D}_f \in \mathrm{Cl}(X)^{\mathrm{free}}$ be the divisor class associated to f. Then, by Lemma 4.3, one has $\bar{D}_f = -\sum_{i=1}^{m} \frac{1}{r_i}\nu_i(f)\bar{E}_i \in \bigoplus_{i=1}^{m} \mathbb{Z}\bar{E}_i$. This

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implies that the integral effective Weil divisor $D = D_f + \sum_{i=1}^{m} \frac{1}{r_i}\nu_i(f)E_i$ is torsion in $\mathrm{Cl}(X)$. We can show that D is nonzero in $\mathrm{Cl}(X)$ as follows. Assume D were linearly trivial. Then, D would be defined by a regular function $f' \in H^0(\mathcal{O}_X) = \mathbb{C}[V]^G$, and we could write $D = D_{f'} + \sum_{i=1}^{m} \frac{1}{r_i}\nu_i(f')E_i$ (cf. the proof of Lemma 4.3). Therefore, $\pi_* D_f$ and $\pi_* D_{f'}$ (both of which are the same as $\pi_* D$) would give the same classes in $\mathrm{Cl}(V/G)$. This is contrary to the choices of f and f'.

Next, assume that $H = G$. Let D be any Weil divisor which is a torsion in $\mathrm{Cl}(X)$. Let $f \in \mathbb{C}[V]^{[G,G]} \cong \mathrm{Cox}(V/G)$ be the defining section of $\pi_* D$ and let D' be the strict transform of $\pi_* D$ on X. Clearly, one can write $D - D' = \sum_{i=1}^{m} a_i E_i$ for some $a_i \in \mathbb{Z}$. By Lemma 4.3, we have $\bar{D}' = -\sum_{i=1}^{m} \frac{1}{r_i}\nu_i(f)\bar{E}_i$ in $\mathrm{Cl}(X)^{\mathrm{free}}$. On the other hand, we have $\bar{D}' = \bar{D} - \sum_i a_i \bar{E}_i = -\sum_i a_i \bar{E}_i$ in $\mathrm{Cl}(X)^{\mathrm{free}}$ since D is a torsion. Therefore, $-\sum_{i=1}^{m} \frac{1}{r_i}\nu_i(f)\bar{E}_i = -\sum_i a_i \bar{E}_i$. As $\{E_i\}_i$ is a $\mathbb{Q}$-basis of $\mathrm{Cl}(X)_{\mathbb{Q}}$, the condition $\nu_i(f) \equiv 0 \bmod r_i$ $(\forall i)$ must be satisfied. This is equivalent to $f \in \mathbb{C}[V]^H = \mathbb{C}[V]^G = \mathrm{Cox}(V/G)_0$. Hence, the class of D in $\mathrm{Cl}(X)$ is contained in $\bigoplus_{i=1}^{m} \mathbb{Z}E_i$. However, $\bigoplus_{i=1}^{m} \mathbb{Z}E_i$ is torsion-free and thus D must be 0. □

Now, we calculate Cox rings for several examples. For this, consider the Laurent polynomial ring $R := \mathbb{C}[V]^{[G,G]}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ over $\mathbb{C}[V]^{[G,G]}$. By Proposition 4.4, we can regard $\mathrm{Im}\,\Theta$ as a subring of R by identifying $f \otimes t^{\bar{D}_f} \in \mathrm{Im}\,\Theta$ with $f t_1^{\nu_1(f)} \cdots t_m^{\nu_m(f)}$. Most of the calculations in the examples below are done with a computer making use of the softwares 'Macaulay 2'[**18**] or 'SINGULAR'[**17**]. See Appendix 7.1 for how to perform the algorithm above and see Appendix 7.2 for how to calculate the relations of the generators of the Cox rings efficiently.

REMARK. In the following examples, a finite group G is realized as a subgroup of the matrix group $SL_n(\mathbb{C})(= SL(V))$. The letters $x, y, \ldots$ denote the dual basis to the standard basis of $V = \mathbb{C}^n$. Originally, we should let G act on V^* as the dual representation of G on V. However, for convenience, we will let G act on V^* by identifying the dual basis of V^* with the standard basis of V. This difference will not produce any effect on the result since one representation and its dual give rise to isomorphic quotient singularities.

EXAMPLE 1. Kleinian (or ADE) singularities
Case 1. A_m-singularity ($m \geq 1$)

- Cyclic group $G = \langle g_1 = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right) \rangle$, $\zeta = \exp(2\pi i/(m+1))$.
- Representatives of junior elements: $g_k := g_1^k$ $(k = 1, \ldots, m)$.
- x (resp. y) is a ζ (resp. ζ^{-1})-eigenvector of g_1.

The order r_k of g_k is given by $r_k = \frac{m+1}{\gcd(k, m+1)}$, and the valuations are given by

$$\nu_k(x) = \frac{k r_k}{m+1} \qquad \text{and} \qquad \nu_k(y) = r_k - \frac{k r_k}{m+1}.$$

Since $x t_1^{\frac{r_1}{m+1}} \cdots t_m^{\frac{m r_m}{m+1}}, y t_1^{r_1 - \frac{r_1}{m+1}} \cdots t_m^{r_m - \frac{m r_m}{m+1}}, t_1^{-r_1}, \ldots, t_m^{-r_m} \in R$ clearly have no relations, the algorithm trivially means that these are the generators of the Cox ring of the minimal model (or the crepant resolution) of V/G.

Case 2. D_m-singularity ($m \geq 4$)

- Binary dihedral group $G = \left\langle g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, g_{m-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$,
  $\zeta = \exp(2\pi i/2(m-2))$, $[G, G] = \langle g_1^2 \rangle$,
  $Ab(G) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\times 2} & \text{if } m \text{ is even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } m \text{ is odd.} \end{cases}$
- Representatives of junior elements: $g_k := g_1^k$ $(k = 1, \dots, m-2)$, $g_{m-1}, g_m := g_{m-1}g_1$.
- x (resp. y) is a ζ (resp. ζ^{-1})-eigenvector of g_k $(k = 1, \dots, m-2)$.
- $x + iy$ (resp. $x - iy$) is a i (resp. $-i$)-eigenvector of g_{m-1}.
- $x + i\zeta y$ (resp. $x - i\zeta y$) is a i (resp. $-i$)-eigenvector of g_m.

The order r_k of g_k is given by $r_k = \begin{cases} 2(m-2)/\gcd(k, 2(m-2)) & \text{if } k = 1, \dots, m-2 \\ 4 & \text{if } k = m-1, m. \end{cases}$

In this case, we can take $x^{m-2} + (iy)^{m-2}$, $x^{m-2} - (iy)^{m-2}$, and xy as homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. We can directly calculate the valuations as follows.

$$\nu_k(x^{m-2} + (iy)^{m-2}) = \begin{cases} kr_k/2 & \text{if } k = 1, \dots, m-2 \\ m-2 & \text{if } k = m-1 \\ m & \text{if } k = m \end{cases}$$

$$\nu_k(x^{m-2} - (iy)^{m-2}) = \begin{cases} kr_k/2 & \text{if } k = 1, \dots, m-2 \\ m & \text{if } k = m-1 \\ m-2 & \text{if } k = m \end{cases}$$

$$\nu_k(xy) = \begin{cases} r_k & \text{if } k = 1, \dots, m-2 \\ 2 & \text{if } k = m-1, m \end{cases}$$

Now, we apply the algorithm to $S = \{x^{m-2} + (iy)^{m-2}, x^{m-2} - (iy)^{m-2}, xy\}$. We use the notations in the algorithm above. First, the kernel of $\alpha : \mathbb{C}[X_1, X_2, X_3] \to \mathbb{C}[V]^{[G,G]}$ is a principal ideal $I = (X_1^2 - X_2^2 - 4(iX_3)^{m-2})$ and one sees that

$$\min_k(I) = \begin{cases} (X_1^2 - X_2^2) & \text{if } k = 1, \dots, m-3 \\ (X_1^2 - X_2^2 - 4(iX_3)^{m-2}) & \text{if } k = m-2 \\ (X_1^2 - 4(iX_3)^{m-2}) & \text{if } k = m-1 \\ (-X_2^2 - 4(iX_3)^{m-2}) & \text{if } k = m. \end{cases}$$

On the other hand, one sees that

$$\min_k(x^{m-2} + (iy)^{m-2}) = \begin{cases} x^{m-2} & \text{if } k = 1, \dots, m-3 \\ x^{m-2} + (iy)^{m-2} & \text{if } k = m-2 \\ \frac{1}{2^{m-1}}(x + iy)^{m-2} & \text{if } k = m-1 \\ \frac{m-2}{2^{m-1}}(x + i\zeta y)^{m-3}(x - i\zeta y) & \text{if } k = m, \end{cases}$$

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$$\min_k(x^{m-2}-(iy)^{m-2}) = \begin{cases} x^{m-2} & \text{if } k=1,\dots,m-3 \\ x^{m-2}-(iy)^{m-2} & \text{if } k=m-2 \\ \dfrac{m-2}{2^{m-1}}(x+iy)^{m-3}(x-iy) & \text{if } k=m-1 \\ \dfrac{1}{2^{m-1}}(x+i\zeta y)^{m-2} & \text{if } k=m, \end{cases}$$

and

$$\min_k(xy) = \begin{cases} xy & \text{if } k=1,\dots,m-2 \\ \frac{1}{4i}(x+iy)^2 & \text{if } k=m-1 \\ \frac{\zeta^{-1}}{4i}(x+i\zeta y)^2 & \text{if } k=m. \end{cases}$$

From these, one can check that $\min_k(I)=\min_k(J)$ for all k. One can also check that each Step (i', i) ends at one try. Therefore, by Corollary 4.13, we obtain a generating system of the Cox ring

$$(x^{m-2}+(iy)^{m-2})t_1^{m-2}\cdots t_{m-1}^{m-2}t_m^{m},\ (x^{m-2}-(iy)^{m-2})t_1^{m-2}\cdots t_{m-2}^{m-2}t_{m-1}^{m}t_m^{m-2},$$
$$(xy)t_1^{r_1}\cdots t_{m-2}^{r_{m-2}}t_{m-1}^{2}t_m^{2},\ t_1^{-r_1},\dots,t_m^{-r_m}.$$

If we rename these elements as $X_1, X_2, X_3, Y_1, \dots, Y_m$ in order, then they have a single relation

$$X_1^2Y_m - X_2^2Y_{m-1} - 4(iX_3)^{m-2}\prod_{k=1}^{m-3}Y_k^{m-2-k} = 0.$$

Case 3. E_6-singularity

- Binary tetrahedral group $G=\left\langle g_1=\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, g_2=\frac{1}{\sqrt{2}}\begin{pmatrix}\zeta & \zeta \\ \zeta^3 & \zeta^7\end{pmatrix}\right\rangle$,
  $\zeta=\exp(2\pi i/8)$, $[G,G]=\left\langle g_1, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right\rangle$, $Ab(G)\cong\mathbb{Z}/3\mathbb{Z}$.
- Representatives of junior elements: $g_1, g_k := g_2^{k-1}$ $(k=2,\dots,6)$.
- x (resp. y) is a i (resp. i)-eigenvector of g_1.
- $x+(\sqrt{2}\omega\zeta^3-1)y$ (resp. $x+(\sqrt{2}\omega\zeta^7-\zeta^2)y$) is a $(-\omega)^{k-1}$(resp. $(-\omega)^{-k+1}$)-eigenvector of g_k, $(k=2,\dots,6)$, where $\omega=\exp(2\pi i/3)$.

We can take $x^4+y^4+2\sqrt{-3}x^2y^2$, $x^4+y^4-2\sqrt{-3}x^2y^2$ and x^5y-xy^5 as homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. The information of the valuations are summarized as follows.

k	1	2	3	4	5	6
$\nu_k(x^4+y^4+2\sqrt{-3}x^2y^2)$	4	4	4	4	5	8
$\nu_k(x^4+y^4-2\sqrt{-3}x^2y^2)$	4	8	5	4	4	4
$\nu_k(x^5y-xy^5)$	8	6	6	6	6	6
$\sharp\langle g_k\rangle$	4	6	3	2	3	6

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By applying the algorithm to $S = \{x^4 + y^4 + 2\sqrt{-3}x^2y^2,\ x^4 + y^4 - 2\sqrt{-3}x^2y^2,\ x^5y - xy^5\}$, one sees that

$$(x^4 + y^4 + 2\sqrt{-3}x^2y^2)t_1^4t_2^8t_3^5t_4^4t_5^4t_6^4,\ (x^4 + y^4 - 2\sqrt{-3}x^2y^2)t_1^4t_2^4t_3^4t_4^4t_5^5t_6^8,$$
$$(x^5y - xy^5)t_1^8t_2^6t_3^6t_4^6t_5^6t_6^6,\ t_1^{-4},\ t_2^{-6},\ t_3^{-3},\ t_4^{-2},\ t_5^{-3},\ t_6^{-6}$$

in R are generators of the Cox ring.

If we rename these elements as $X_1, X_2, X_3, Y_1, \dots, Y_6$ in order, then they have a single relation

$$X_1^3Y_5Y_6^2 - X_2^3Y_2^2Y_3 - 12\sqrt{-3}X_3^2Y_1 = 0.$$

Case 4. E_7-singularity

- Binary octahedral group $G = \left\langle g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, g_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta & \zeta \\ \zeta^3 & \zeta^7 \end{pmatrix} \right\rangle$, $\zeta = \exp(2\pi i/8)$,
  $[G, G] = \langle g_1^2, g_5\rangle$(=binary tetrahedral group), $Ab(G) \cong \mathbb{Z}/2\mathbb{Z}$.
- Representatives of junior elements: $g_1, g_2 := g_1^2, g_3 := g_1^3, g_4 := g_1^4, g_5, g_6 := g_5^2$, $g_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$.
- x (resp. y) is a ζ^k (resp. ζ^{-k})-eigenvector of g_k, $(k = 1, \dots, 4)$.
- $x + (\sqrt{2}\omega\zeta^3 - 1)y$ and $x + (\sqrt{2}\omega\zeta^7 - \zeta^2)y$ are $(-\omega)^{k-4}$(resp. $(-\omega)^{-k+4}$)-eigenvectors of g_5 and g_6, where $\omega = \exp(2\pi i/3)$.
- $x - (1 - \sqrt{2})iy$ (resp. $x - (1 + \sqrt{2})iy$) is a i (resp. $-i$)-eigenvector of g_7.

We can take $x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$, $x^8 + 14x^4y^4 + y^8$ and $x^5y - xy^5$ as homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. The information of the valuations are summarized as follows.

k	1	2	3	4	5	6	7
$\nu_k(x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12})$	12	12	36	12	12	12	14
$\nu_k(x^8 + 14x^4y^4 + y^8)$	8	8	24	8	12	9	8
$\nu_k(x^5y - xy^5)$	12	8	20	6	6	6	6
$\sharp\langle g_k\rangle$	8	4	8	2	6	3	4

By applying the algorithm to $S = \{x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12},\ x^8 + 14x^4y^4 + y^8,\ x^5y - xy^5\}$, one sees that

$$(x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12})t_1^{12}t_2^{12}t_3^{36}t_4^{12}t_5^{12}t_6^{12}t_7^{14},\ (x^8 + 14x^4y^4 + y^8)t_1^8t_2^8t_3^{24}t_4^8t_5^{12}t_6^9t_7^8,$$
$$(x^5y - xy^5)t_1^{12}t_2^8t_3^{20}t_4^6t_5^6t_6^6t_7^6,\ t_1^{-8},\ t_2^{-4},\ t_3^{-8},\ t_4^{-2},\ t_5^{-6},\ t_6^{-3},\ t_7^{-4}$$

in R are generators of the Cox ring.

If we rename these elements as $X_1, X_2, X_3, Y_1, \dots, Y_7$ in order, then they have a single relation

$$X_1^2Y_7 - X_2^3Y_5^2Y_6 + 108X_3^4Y_1^3Y_2^2Y_3 = 0.$$

Case 5. E_8-singularity

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- Binary icosahedral group
  $G = \left\langle g_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, h = \frac{1}{\sqrt{5}} \begin{pmatrix} \epsilon^2 - \epsilon^8 & \epsilon^4 - \epsilon^6 \\ \epsilon^4 - \epsilon^6 & \epsilon^8 - \epsilon^2 \end{pmatrix}, g_8 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$,
  $\epsilon = \exp(2\pi i/10)$, $[G, G] = G$.
- Representatives of junior elements: $g_1, g_2 := g_1^2, g_3 := g_1^3, g_4 := g_1^4, g_5 := g_1^5, g_6 := -g_1 h$,
  $g_7 := g_6^2, g_8$.
- x (resp. y) is a ϵ^k (resp. ϵ^{-k})-eigenvector of g_k, $(k = 1, \dots, 5)$.
- $x + (\epsilon^3 - \omega\epsilon^2 - \epsilon^2 - \omega\epsilon - 1)y$ (resp. $x + (\epsilon^3 + \omega\epsilon^2 + \omega\epsilon + \epsilon - 1)y$) is a $(-\omega)^{k-5}$ (resp. $(-\omega)^{-k+5}$)-eigenvector of g_k, $(k = 6, 7)$, where $\omega = \exp(2\pi i/3)$.
- $x + iy$ (resp. $x - iy$) is a i (resp. $-i$)-eigenvector of g_8.

We can take $x^{30} + y^{30} - 522(x^{25}y^5 - x^5y^{25}) - 10,005(x^{20}y^{10} + x^{10}y^{20})$, $x^{20} + y^{20} + 228(x^{15}y^5 - x^5y^{15}) + 494x^{10}y^{10}$ and $xy(x^{10} - 11x^5y^5 - y^{10})$ as homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. The information of the valuations are summarized as follows.

k	1	2	3	4	5	6	7	8
$\nu_k(x^{30} + y^{30} - 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}))$	30	30	90	60	30	30	30	32
$\nu_k(x^{20} + y^{20} + 228(x^{15}y^5 - x^5y^{15}) + 494x^{10}y^{10})$	20	20	60	40	20	24	21	20
$\nu_k(xy(x^{10} - 11x^5y^5 - y^{10}))$	20	15	40	25	12	12	12	12
$\sharp\langle g_k\rangle$	10	5	10	5	2	6	3	4

By applying the algorithm to $S = \{x^{30} + y^{30} - 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}),\ x^{20} + y^{20} + 228(x^{15}y^5 - x^5y^{15}) + 494x^{10}y^{10},\ xy(x^{10} - 11x^5y^5 - y^{10})\}$, one sees that

$$(x^{30} + y^{30} - 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}))t_1^{30}t_2^{30}t_3^{90}t_4^{60}t_5^{30}t_6^{30}t_7^{30}t_8^{32},$$
$$(x^{20} + y^{20} + 228(x^{15}y^5 - x^5y^{15}) + 494x^{10}y^{10})t_1^{20}t_2^{20}t_3^{60}t_4^{40}t_5^{20}t_6^{24}t_7^{21}t_8^{20},$$
$$xy(x^{10} - 11x^5y^5 - y^{10})t_1^{20}t_2^{15}t_3^{40}t_4^{25}t_5^{12}t_6^{12}t_7^{12}t_8^{12}, t_1^{-10}, t_2^{-5}, t_3^{-10}, t_4^{-5}, t_5^{-2}, t_6^{-6}, t_7^{-3}, t_8^{-4}$$

in R are generators of the Cox ring.

If we rename these elements as $X_1, X_2, X_3, Y_1, \dots, Y_8$ in order, then they have a single relation

$$X_1^2 Y_8 - X_2^3 Y_6^2 Y_7 + 1728 X_3^5 Y_1^4 Y_2^3 Y_3^2 Y_4 = 0.$$

REMARK. By taking linear changes of coordinates, one can check that these results agree with the results in [**16**] and [**13**].

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EXAMPLE 2. A group of order 32 acting on a four-dimensional vector space cf. [**10**],[**15**]

$$\bullet\ G = \left\langle g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ g_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},\ g_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\left. g_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle, [G, G] = \langle -\mathrm{Id} \rangle\ ,\ Ab(G) \cong (\mathbb{Z}/2\mathbb{Z})^{\times 4}$$

- Representatives of junior elements: $g_1, g_2, g_3, g_4, g_5 := g_1 g_2 g_3 g_4$

We can take $\phi_{12} = -2(xw + yz)$, $\phi_{13} = 2i(-xw + yz)$, $\phi_{14} = 2i(xy + zw)$, $\phi_{15} = 2(-xy + zw)$, $\phi_{23} = 2(xz - yw)$, $\phi_{24} = -x^2 - y^2 + z^2 + w^2$, $\phi_{25} = i(x^2 + y^2 + z^2 + w^2)$, $\phi_{34} = i(-x^2 + y^2 - z^2 + w^2)$, $\phi_{35} = x^2 - y^2 - z^2 + w^2$ and $\phi_{45} = 2(xz + yw)$ as homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. The information of the valuations are summarized as follows (cf. [**15**, 3.13]).

k	1	2	3	4	5
$\nu_k(\phi_{12})$	1	1	0	0	0
$\nu_k(\phi_{13})$	1	0	1	0	0
$\nu_k(\phi_{14})$	1	0	0	1	0
$\nu_k(\phi_{15})$	1	0	0	0	1
$\nu_k(\phi_{23})$	0	1	1	0	0
$\nu_k(\phi_{24})$	0	1	0	1	0
$\nu_k(\phi_{25})$	0	1	0	0	1
$\nu_k(\phi_{34})$	0	0	1	1	0
$\nu_k(\phi_{35})$	0	0	1	0	1
$\nu_k(\phi_{45})$	0	0	0	1	1
$\sharp\langle g_k \rangle$	2	2	2	2	2

By applying the algorithm to $S = \{\phi_{12}, \ldots, \phi_{45}\}$, one sees that each step ends at one try and thus

$$\{\phi_{i,j} t_i t_j\}_{1 \le i < j \le 5} \cup \{t_i^{-2}\}_{i=1,\ldots,5}$$

in R are generators of the Cox ring as stated in [**15**]. The relations of these elements are calculated in [**15**, Proposition 3.17].

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EXAMPLE 3. The complex reflection group G_4 cf. [**5**], [**24**]

- $$G = \left\langle g_1 = -\frac{1}{2}\begin{pmatrix} (1+i)\omega & (1+i)\omega & 0 & 0 \\ (-1+i)\omega & (1-i)\omega & 0 & 0 \\ 0 & 0 & (1-i)\omega^2 & (1-i)\omega^2 \\ 0 & 0 & (-1-i)\omega^2 & (1+i)\omega^2 \end{pmatrix}, \right.$$
$$\left. g_2 = -\frac{1}{2}\begin{pmatrix} (1+i)\omega & (1-i)\omega & 0 & 0 \\ (-1-i)\omega & (1-i)\omega & 0 & 0 \\ 0 & 0 & (1-i)\omega^2 & (1+i)\omega^2 \\ 0 & 0 & (-1-i)\omega^2 & (-1+i)\omega^2 \end{pmatrix} \right\rangle,$$
$$\omega = \exp(2\pi i/3),$$
$$[G, G] = \left\langle \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\rangle : \text{the quaternion group}$$

- Representatives of junior elements: g_1, g_2

The homogeneous minimal generators of the invariant ring $\mathbb{C}[x, y, z, w]^{[G,G]}$ are listed as follows: $\phi_1 = xz + yw$,
$\phi_2 = x^5y - xy^5$,
$\phi_3 = z^5w - zw^5$,
$\phi_4 = x^4 + (4a-2)x^2y^2 + y^4$,
$\phi_5 = z^4 - (4a-2)z^2w^2 + w^4$,
$\phi_6 = xw^3 + (-2a+1)yzw^2 + (2a-1)xz^2w - yz^3$,
$\phi_7 = x^3w - (2a-1)xy^2w + (2a-1)x^2yz - y^3z$,
$\phi_8 = x^2yw^3 - x^3zw^2 - y^3z^2w + xy^2z^3$,
$\phi_9 = 3ax^2w^2 - (a-2)x^2z^2 + (4a-8)xyzw - (a-2)y^2w^2 + 3ay^2z^2$,
$\phi_{10} = z^4 + (4a-2)z^2w^2 + w^4$,
$\phi_{11} = x^3w + (2a-1)xy^2w - (2a-1)x^2yz - y^3z$,
$\phi_{12} = 5x^4yw - x^5z - y^5w + 5xy^4z$,
$\phi_{13} = xyz^4 + 2x^2zw^3 - 2y^2z^3w - xyw^4$,
$\phi_{14} = 3(a-1)x^2w^2 - (a+1)y^2w^2 + 4(a+1)xyzw - (a+1)x^2z^2 + 3(a-1)y^2z^2$,
$\phi_{15} = x^4 - (4a-2)x^2y^2 + y^4$,
$\phi_{16} = xw^3 + (2a-1)yzw^2 - (2a-1)xz^2w - yz^3$,
$\phi_{17} = xz^5 - 5xzw^4 - 5yz^4w + yw^5$,
$\phi_{18} = 2x^3yw^2 - x^4zw + y^4zw - 2xy^3z^2$,
where a denotes $\exp(2\pi i/6)$.

The information of the valuations are summarized as follows.

$$v_1(\phi_i) = \begin{cases} 0 & \text{if } 1 \le i \le 8 \\ 2 & \text{if } 9 \le i \le 13 \\ 1 & \text{if } 14 \le i \le 18 \end{cases}$$

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$$\nu_2(\phi_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq 8 \\ 1 & \text{if } 9 \leq i \leq 13 \\ 2 & \text{if } 14 \leq i \leq 18 \end{cases}$$

$$\sharp \langle g_1 \rangle = \sharp \langle g_2 \rangle = 3.$$

We apply the algorithm to $S = \{\phi_1, \ldots, \phi_{18}\}$. In this case, $\min_1(J)$ is strictly bigger than $\min_1(I)$ and one can check that there is $h = X_1^3 + (-6a+3)X_8 \in \mathbb{C}[X_1, \ldots, X_{18}]$ such that

$$\underline{\min}_1(J) = \min_1(I) + (h).$$

Thus, we should add

$$\begin{aligned}\phi_{19} := \alpha(h) =& x^3z^3 + (-6a+3)xy^2z^3 + 3x^2yz^2w + (6a-3)y^3z^2w + (6a-3)x^3zw^2 \\ &+ 3xy^2zw^2 + (-6a+3)x^2yw^3 + y^3w^3,\end{aligned}$$

to S and try *Step* (0, 1) again.

One can check that each step ends at one try for $S = \{\phi_1, \ldots, \phi_{19}\}$ and thus

$$\phi_1, \ldots, \phi_8, \phi_9 t_1^2 t_2, \ldots, \phi_{13} t_1^2 t_2, \phi_{14} t_1 t_2^2, \ldots, \phi_{18} t_1 t_2^2, \phi_{19} t_1^3 t_2^3, t_1^{-3}, t_2^{-3}$$

are the generators of the Cox ring.

REMARK. This result shows that the conjecture in [**14**, Section 6] is negative and one more generator is necessary. Note that we have a relation $(6a-3)\phi_8 = \phi_1^3 - \phi_{19} \cdot t_1^{-3} \cdot t_2^{-3}$ in R. This shows that, in general, the resulting generating system of the Cox ring by the algorithm is not minimal even if we take a minimal generating system of $\mathbb{C}[V]^{[G,G]}$ as the original S and choose minimal generators in each step of the algorithm.

5. GIT chambers and ample cones. In this section, we summarize the basic notions and results about GIT for the case of the Cox ring of a minimal model X of the quotient singularity V/G.

As stated in Section 2, the algebraic group $T = \mathrm{Hom}(\mathrm{Cl(X)}, \mathbb{C}^*)$ (which is a torus when $\mathrm{Cl}(X)$ is free) acts on the spectrum $\mathfrak{X} = \mathrm{Spec(Cox(X))}$, and every divisor class $D \in \mathrm{Cl}(X)$ can be considered as a character of T. Now, we introduce the notion of (semi)stability. Consider the following vector space

$$R(D) := \{f \in H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \mid t \cdot f = D(t)f \text{ for all } t \in T\} (= H^0(X, \mathcal{O}_X(D))).$$

DEFINITION 5.1. We say that a point $x \in \mathfrak{X}$ is *D-semistable* if there exist $i \in \mathbb{Z}_{>0}$ and $f \in R(iD)$, such that $f(x) \neq 0$. If moreover x has a finite stabilizer and the T-orbit of x is closed in $\{x \in \mathfrak{X} | f(x) \neq 0\}$, we say that x is *D-stable*. $\mathfrak{X}^{D-ss}$ (resp. $\mathfrak{X}^{D-s}$) denotes the subset of D-semistable (resp. D-stable) points in $\mathfrak{X}$. We call a divisor class D in $\mathrm{Cl}(X)$ *generic* if $\mathfrak{X}^{D-ss} = \mathfrak{X}^{D-s}$.

We define the *GIT quotient* of $\mathfrak{X}$ by T with respect to D as

$$\mathfrak{X}//_D T := \operatorname{Proj} \bigoplus_{i=0}^{\infty} R(iD).$$

Note that there is a natural map from $\mathfrak{X}//_D T$ to $\mathfrak{X}//_0 T = V/G$ for each D. GIT quotients of $\mathfrak{X}$ by T have the following property.

PROPOSITION 5.2 [**26**, Theorem 1.10]. *The morphism $q : \mathfrak{X}^{D-ss} \to \mathfrak{X}//_D T$ induced by the inclusion $R(iD) \hookrightarrow H^0(\mathcal{O}_{\mathfrak{X}})$ is a categorical quotient. Moreover, there is an open subset U of $\mathfrak{X}//_D T$ such that $q^{-1}(U) = \mathfrak{X}^{D-s}$ and $q|_{\mathfrak{X}^{D-s}} : \mathfrak{X}^{D-s} \to U$ is a geometric quotient (i.e. each fibre is a single T-orbit).*

If two divisor classes in $\mathrm{Cl}(X)$ give the same semistable loci in $\mathfrak{X}$ and hence give the canonically isomorphic GIT quotients, we call them *GIT equivalent*. It is known that GIT equivalence classes give a chamber structure on the finite dimensional real vector space $\mathrm{Cl}(X)_{\mathbb{R}} := \mathrm{Cl}(X) \otimes \mathbb{R}$ (cf. [**29**, 2.3]) i.e.

(i) there are only finitely many GIT equivalence classes

(ii) for every GIT equivalence class C, the closure $\overline{C}$ is a rational polyhedral cone in $\mathrm{Cl}(X)_{\mathbb{R}}$ and C is a relative interior of $\overline{C}$.

We call C a *GIT chamber* if C is not contained in any hyperplane in $\mathrm{Cl}(X)_{\mathbb{R}}$. The fan on $\mathrm{Cl}(X)_{\mathbb{R}}$ given by the closures of all GIT chambers is called the *GIT fan*. It is known that $D \in \mathrm{Cl}(X)_{\mathbb{R}}$ is generic if and only if D is in a GIT chamber.

GIT chambers in $\mathrm{Cl}(X)_{\mathbb{R}}$ are closely related with the birational geometry of X. To see this, we introduce (π-)movable line bundles for the morphism $\pi : X \to V/G$.

DEFINITION 5.3. A line bundle L on X is (π-)*movable* if $\operatorname{codim} \operatorname{Supp}(\operatorname{Coker} \alpha) \geq 2$ where $\alpha : \pi^*\pi_* L \to L$ is the natural map of sheaves on X. The (π-)*movable* cone $\mathrm{Mov}(\pi)$ in $\mathrm{Pic}(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is the cone generated by the classes of π-movable line bundles.

Let $\pi' : X' \to V/G$ be another minimal model. Then the birational map $\pi' \circ \pi^{-1} : X \dashrightarrow X'$ is an isomorphism in codimension 1 over V/G (see e.g. [**33**, Lemma 3.3]). Therefore there is a natural isomorphism $\mathrm{Pic}(X')_{\mathbb{R}} \to \mathrm{Pic}(X)_{\mathbb{R}}$, and we call the image of $\mathrm{Amp}(X')$ by this map the *ample cone* of π'. The fact that the Cox ring of X is finitely generated implies that X is a (relative) Mori dream space introduced by Hu and Keel [**19**, 1.10, 2.9]. Although Mori dream spaces were originally defined for projective varieties in [**19**], the definition can naturally extend to varieties which are projective over affine varieties (see e.g. [**3**, 2.5]). Since X is a Mori dream space, X has finitely many ample cones and they satisfy the following properties.

(1) $\mathrm{Amp}(X')$ and $\mathrm{Amp}(X'')$ are disjoint for different minimal models X' and X'', and their closures are rational polyhedral cones.
(2) The movable cone is also a rational polyhedral cone and is covered with the closures of all ample cones

$$\mathrm{Mov}(\pi) = \bigcup_{X'} \overline{\mathrm{Amp}}(X'),$$

where X' runs through all minimal models.

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Note that we can regard the movable cone and the ample cones as the subset of $\mathrm{Cl}(X)_{\mathbb{R}}$ since minimal models are $\mathbb{Q}$-factorial. In [**19**, Theorem 2.3], it is proved that each ample cone $\mathrm{Amp}(X')$ coincides with some GIT chamber and that the corresponding GIT quotient is X'. Therefore, every minimal model is realized as the GIT quotient $\mathfrak{X}//_D T$ for some divisor D on X.

To describe the on structure of the GIT fan on $\mathrm{Cl}(X)_{\mathbb{R}}$, we introduce results in [**7**] and [**8**]. Let $\psi_1, \ldots, \psi_k$ be homogeneous generators of the Cox ring. Consider the $\mathbb{R}$-linear map $Q : \mathbb{R}^k \to \mathrm{Cl}(X)_{\mathbb{R}}$ defined by $e_i \mapsto \deg(\psi_i)$ where e_i is the ith canonical vector. For a face γ of the positive orthant $\mathbb{R}^k_{\geq 0}$, let $\mathfrak{X}(\gamma) \subset \mathfrak{X}$ be the open subset $\{x \in \mathfrak{X} \mid \psi_i(x) \neq 0$ for any i such that $e_i \in \gamma\}$. Let Ω be the set of faces γ of $\mathbb{R}^k_{\geq 0}$ such that there is a point $x \in \mathfrak{X}$ satisfying $e_i \in \gamma \iff \psi_i(x) \neq 0$. Berchtold and Hausen proved the following.

THEOREM 5.4 ([**7**, Lemma 2.7 and Proposition 2.9] and [**8**, Proposition 7.2]). *For $D \in \mathrm{Cl}(X)_{\mathbb{R}}$, the GIT equivalence class containing D is described as*

$$\bigcap_{D \in \gamma \in \Omega} Q(\gamma) \subset \mathrm{Cl}(X)_{\mathbb{R}}$$

and the D-semistable locus $\mathfrak{X}^{D-ss}$ is described as

$$\bigcup_{D \in \gamma \in \Omega} \mathfrak{X}(\gamma) \subset \mathfrak{X}.$$

Moreover, the movable cone of X is described as

$$\bigcap_{\gamma \subset \mathbb{R}^k_{\geq 0}: facet} Q(\gamma) \subset \mathrm{Cl}(X)_{\mathbb{R}}.$$

From this theorem, we can determine the location of the movable cone in the case of minimal models of quotient singularities. Let $\phi_1, \ldots, \phi_k \in \mathbb{C}[V]^{[G,G]}$ be elements which are homogeneous with respect to $Ab(G)$-action such that $\{\phi_j \otimes t^{\bar{D}_{\phi_j}}\}_{j=1,\ldots,k} \cup \{t^{\bar{E}_1}, \ldots, t^{\bar{E}_m}\} \subset \mathbb{C}[V]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ is a generating system of the Cox ring of X (cf. Proposition 4.4). Taking the forms of the generators into account, one sees that the movable cone is generated by $\{\bar{D}_{\phi_j}\}_{j=1,\ldots,k} \subset \mathrm{Cl}(X)_{\mathbb{R}}$ by Theorem 5.4.

Now, we apply the above results to a minimal model X of V/G, where G is the binary tetrahedral group treated in Section 4, Example 3. According to the results in the previous section, the degrees of the generators of the Cox ring of X are (0, 0), (2, 1), (1, 2), (3, 3), (−3, 0) and (0, −3). Thus, Figure 1 gives a refinement of the GIT fan on $\mathrm{Cl}(X)_{\mathbb{R}} \cong \mathbb{R}^2$.

One can check that this fan itself is the GIT fan by Theorem 5.4. One also knows that the cone generated by (2,1) and (1,2) is the movable cone.

We can also investigate the smoothness of X. It was already proven by Bellamy [**5**] that V/G admits a symplectic resolution. We now try to prove the same thing using the Cox ring. To do this, let us consider two $\mathbb{C}^*$-actions on V^* defined by $(x, y, z, w) \mapsto (tx, ty, tz, tw)$ and $(x, y, z, w) \mapsto (x, y, tz, tw)$ for $t \in \mathbb{C}^*$. Since these actions are compatible with the G-action on V, they induce the actions of $\mathbb{C}^*$ on V/G. Kaledin showed that these actions on V/G lift to X and the common fixed point sets consists of finitely many points [**22**, Proposition 6.3]. In our case, these actions

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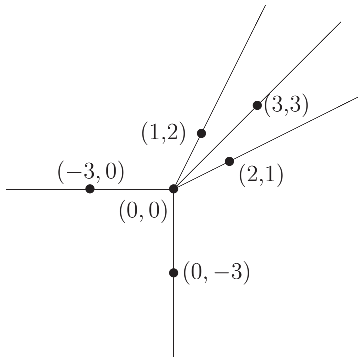

Figure 1. Fan structure on $\mathrm{Cl}(X)_{\mathbb{R}}$.

also lift to $\mathfrak{X} := \operatorname{Spec}\operatorname{Cox}(X)$, and the fixed point set on the semistable locus of $\mathfrak{X}$ with respect to the movable chambers is a single $T(= (\mathbb{C}^*)^2)$-orbit

$$F = \{\psi_1 = \cdots = \psi_{14} = \psi_{16} = \cdots = \psi_{19} = \psi_{21} = 0, \psi_{15} \neq 0, \psi_{20} \neq 0\} \subset \mathfrak{X},$$

where ψ_i is the ith generator of the Cox ring in the previous section regarded as the coordinate of $\mathfrak{X}$. If the semistable locus of $\mathfrak{X}$ has a singular point p, we see that the limit q of p obtained by $t \to 0$ with respect to the first $\mathbb{C}^*$-action is also singular in $\mathfrak{X}$ and contained in the central fibre $\pi^{-1}(0)$. Moreover, since the second $\mathbb{C}^*$-action preserves $\pi^{-1}(0)$, we also see that the limit of q obtained by $t \to 0$ with respect to the second $\mathbb{C}^*$-action is singular in $\mathfrak{X}$ and contained in F. Therefore, to prove the smoothness of X, it suffices to show that X has no singular points contained in F.

As we already know the explicit generators of the Cox ring, we can obtain their relations by a computer calculation, see Appendix 7.2. Then, the Jacobian criterion shows that $\mathfrak{X}$ is nonsingular at any point in F. One can check that each point of the semistable locus has a nontrivial stabilizer subgroup $T' \subset T$ of order 3 and that the quotient torus T/T' acts freely on it. Therefore, by Luna's étale slice theorem, one can conclude that X is also smooth.

6. (Non)smoothness of the minimal models of some symplectically imprimitive quotient singularities. In this section, we investigate the smoothness of the minimal models for several cases. Now, we are particularly interested in symplectic cases. By Verbitsky's result, we only have to check the groups that are generated by symplectic reflections. Such groups are classified by Cohen [**12**]. In his original paper, he considered quaternion reflection groups rather than symplectic reflection groups, but one sees that these two kinds of groups can be identified.

To explain the classification, we introduce some terminology. Let V be a finite dimensional symplectic $\mathbb{C}$-vector space, and let ω be its symplectic form. Let $Sp(V, \omega)$ (or simply $Sp(V)$) be the group of linear automorphisms of V that preserve ω and let G be a finite subgroup of $Sp(V)$. We say that the subgroup (or the representation) $G \subset Sp(V)$ is *irreducible* if there are no nontrivial decomposition of V into G-invariant symplectic vector subspaces. Note that, for every representation $G \subset Sp(V)$ which is generated by symplectic reflections, the quotient singularity V/G is the product of quotient singularities for irreducible representations (see [**11**, Section 1]). Thus, we will only consider irreducible ones from now on.

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An irreducible G is called *improper* if there is a G-invariant Lagrangian subspace L of V with respect to ω and otherwise we call G *proper*. If G is improper with $L \subset V$, the symplectic reflection group G can be regarded as a complex reflection group via the natural inclusion $GL(L) \subset Sp(V)$. Complex reflection groups are classified by Shepherd-Todd [**28**] into three infinite families and 34 exceptional groups $G_4, \ldots, G_{37}$.

Proper groups are also divided into two classes. We say that G is *symplectically imprimitive* if there is a nontrivial decomposition

$$V = V_1 \oplus \cdots \oplus V_k$$

into symplectic subspaces such that for any $g \in G$ and $i \in \{1, \ldots, k\}$ there is j such that $g(V_i) \subset V_j$. Otherwise G is called *symplectically primitive*.

Bellamy and Schedler studied when quotient singularities by symplectically imprimitive groups have projective symplectic resolutions [**11**]. They showed there that if dim $V > 4$, then V/G has a symplectic resolution if and only if G is the wreath product of a finite subgroup of $SL(2, \mathbb{C})$ and a symmetric group. Four dimensional (irreducible) proper symplectically imprimitive representations are classified up to conjugacy by Cohen and listed in table I in [**12**]. However, one should note that Cohen's list contains mistakes. The list includes some improper groups and mutually conjugate groups as we will see later. We call the groups in the Cohen's list type (A), (B),..., (V) as in [**11**]. Bellamy and Schedler also determined which V/G has a symplectic resolution except 6 cases: type (G), (K), (P), (Q), (U) and (V). The aim of this section is to complete their work by studying these remaining cases.

Let $V = \mathbb{C}^4$ and $\omega = dx \wedge dy + dz \wedge dw$, where x, y, z and w is the standard coordinate on $\mathbb{C}^4$. Then any of the six groups is of the following form

$$G(K, \alpha) = \left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^i \right\}_{x \in K,\, i=1,2},$$

where K is a finite subgroup of $SL(2, \mathbb{C})$ and $\alpha \in \mathrm{Aut}(K)$ is an involution.

The six cases are listed as follows.

type $(\mathrm{G})_{l,r}$ ($l, r \in \mathbb{N}$ such that $r \leq l$, r is odd and $l = \gcd(l, \frac{r+1}{2})\gcd(l, \frac{r-1}{2})$):
$K = \left\langle g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$ ($\zeta = \exp(2\pi i/2l)$) is a binary dihedral group and α is defined by $\alpha(g_1) = g_1^r$, $\alpha(g_2) = -g_2$.

type (K): $K = \left\langle g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, g_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^5 & \zeta^5 \\ \zeta^7 & \zeta^3 \end{pmatrix} \right\rangle$ ($\zeta = \exp(2\pi i/8)$) is a binary tetrahedral group and α is defined by $\alpha(g_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\alpha(g_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta \\ \zeta^3 & \zeta^5 \end{pmatrix}$.

type (P): $K = \left\langle g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, g_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^5 & \zeta^5 \\ \zeta^7 & \zeta^3 \end{pmatrix} \right\rangle$ ($\zeta = \exp(2\pi i/8)$) is a binary octahedral group and α is defined by $\alpha(g_1) = g_1^{-1}$ and $\alpha(g_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^7 \\ \zeta^5 & \zeta^5 \end{pmatrix}$.

type (Q): $K = \left\langle g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, g_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^5 & \zeta^5 \\ \zeta^7 & \zeta^3 \end{pmatrix} \right\rangle$ ($\zeta = \exp(2\pi i/8)$) is a binary octahedral group and α is defined by $\alpha(g_1) = -g_1$ and $\alpha(g_2) = g_2$.

type (U): $K = \left\langle g_1 = \frac{1}{2}\begin{pmatrix} \phi + i\phi^{-1} & 1 \\ -1 & \phi - i\phi^{-1} \end{pmatrix}, g_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta^5 & \zeta^5 \\ \zeta^7 & \zeta^3 \end{pmatrix} \right\rangle$ ($\phi = \frac{1+\sqrt{5}}{2}$, $\zeta = \exp(2\pi i/8)$) is a binary icosahedral group and α is defined by $\alpha(g_1) = g_1^{-1}$ and $\alpha(g_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta^3 & \zeta^7 \\ \zeta^5 & \zeta^5 \end{pmatrix}$.

type (V): $K = \left\langle g_1 = \frac{1}{2}\begin{pmatrix} \phi + i\phi^{-1} & 1 \\ -1 & \phi - i\phi^{-1} \end{pmatrix}, g_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$ ($\phi = \frac{1+\sqrt{5}}{2}$) is a binary icosahedral group and α is defined by
$\alpha(g_1) = g_1^3 g_2 g_1$ and $\alpha(g_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

Our main result in this section is the following.

THEOREM 6.1. *Let G be one of the six types above. Then, the quotient singularity V/G has a projective symplectic resolution if and only if G is of type $(G)_{1,1}$.*

Proof. We prove the claim by case-by-case analysis. First, consider type (P) and (U). These groups are improper groups (with respect to ω). Indeed, one can easily check that the Lagrangian subspace $L = \{x - w = y - z = 0\}$ of V is preserved by the actions of the two groups. The corresponding complex reflection groups to type (P) and (U) are G_{13} and G_{22} in the Shepherd-Todd classification [**28**], respectively. By Bellamy's result [**5**], we know that V/G for each of G_{13} and G_{22} dose not have projective symplectic resolutions.

Next, we consider type (K) and (V). To cope with these groups, we consider the Cox rings of minimal models. Since direct computer calculations of the Cox rings could not be done in a reasonable amount of time, we adopt another approach.

Let G be the group of type (K) and G' the group of type (J). Then G' is generated by G and $g_2 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$, and G is a normal subgroup of G' of index 2. Let g_1 be a representative of the unique junior conjugacy class in G. The commutator groups $[G, G]$ and $[G', G']$ are the same, and we let H denote this subgroup. By the results of Section 4, the Cox ring of a minimal model X for type (K) and that of a minimal model X' for type (J) are realized as subrings of $R_1 := \mathbb{C}[V]^H[t_1^{\pm 1}]$ and $R_2 := \mathbb{C}[V]^H[t_1^{\pm 1}, t_2^{\pm 1}]$ respectively. Let

$$\psi_1 = \phi_1 t_1^{\nu_1(\phi_1)} t_2^{\nu_2(\phi_1)}, \ldots, \psi_k = \phi_k t_1^{\nu_1(\phi_k)} t_2^{\nu_2(\phi_k)}, T_1 = t_1^{-2}, T_2 = t_2^{-2} \in R_2$$

be the generators of $\mathrm{Cox}(X')$ (see Proposition 4.4) that are homogeneous with respect to $G'/H (\cong \mathbb{Z}/4\mathbb{Z})$-action, where $\phi_1 \ldots, \phi_k \in \mathbb{C}[V]^H$. By Proposition 4.4, we see that $\psi_1|_{t_2=1}, \ldots, \psi_k|_{t_2=1}, T_1 \in R_1$ are generators of $\mathrm{Cox}(X)$. Since ψ_i's are homogeneous with respect to $\langle g_2 \rangle$-action, the G'/G-action on V/G lifts to $\mathrm{Spec}\,\mathrm{Cox}(X)$. This action descends to one on the GIT quotient X since the semistable locus is defined by the homogeneous elements by Theorem 5.4. Note that the fixed point set of this action on X is the common zero locus of $\psi_i|_{t_2=1}$'s such that $\nu_2(\phi_i)$ is odd.

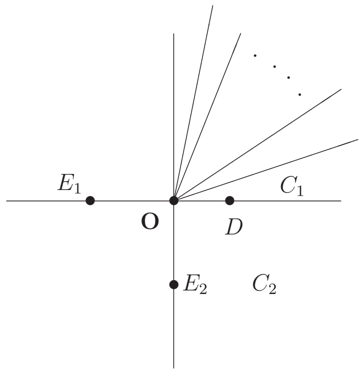

Figure 2. The GIT fan for type (K).

By Theorem 5.4, the GIT chambers on $\mathrm{Cl}(X')_{\mathbb{R}} \cong \mathbb{R}^2$ are described as in Figure 2.

The divisor $D = -\frac{1}{2}E_1$ in the figure is obtained, for example, as the degree of the element in the Cox ring associated to the semi-invariant $x^5y - xy^5 - z^5w + zw^5 \in \mathbb{C}[V]^H$. The half-line $\mathbb{R}_{\geq 0}D$ lies on the boundary of the movable cone. Note that the GIT quotient of the spectrum $\mathfrak{X}' := \mathrm{Spec}\,\mathrm{Cox}(X')$ with respect to $D \in \mathrm{Cl}(X')_{\mathbb{R}}$ is the quotient of X by the $\langle g_2 \rangle$ ($\cong \mathbb{Z}/2\mathbb{Z}$)-action. The GIT quotient of $\mathfrak{X}'$ with respect to the open chamber C_2 in Figure 2 is the same as $\mathfrak{X}'//_D T$. This is shown as follows. Since the movable part of any divisor D' in C_2 coincides with D up to positive multiple, D and D' induce isomorphic rational contractions from X' (see [**19**, Lemma 1.6]). These contractions are the same as the ones induced by GIT (see the last part of [**19**, Theorem 2.3]).

We may assume that X' is the minimal model which corresponds to the open chamber C_1 in Figure 2. The semistable loci on $\mathfrak{X}'$ with respect to C_1, C_2 and D have the following inclusions:

$$\mathfrak{X}'^{C_1-ss} \subset \mathfrak{X}'^{D-ss} \supset \mathfrak{X}'^{C_2-ss}.$$

By the definition of a stable point, we also see that $\mathfrak{X}'^{D-s} = \mathfrak{X}'^{C_1-s} \cap \mathfrak{X}'^{C_2-s}$. Recall that the morphism $\pi : \mathfrak{X}'//_{C_1} T \to \mathfrak{X}'//_D T = X/\langle g_2 \rangle$ and the isomorphism $\mathfrak{X}'//_{C_2} T \to \mathfrak{X}'//_D T$ of GIT quotients are induced from the inclusions of the semistable loci on $\mathfrak{X}'$. Therefore, we see that π is an isomorphism on the image of $\mathfrak{X}'^{D-s}$ in $\mathfrak{X}'//_{C_1} T$. One can directly check by Theorem 5.4 that $\mathfrak{X}'^{C_1-s} \setminus \mathfrak{X}'^{D-s} = \mathfrak{X}'^{C_1-s} \cap \{T_2 = 0\}$ and $\mathfrak{X}'^{C_2-s} \setminus \mathfrak{X}'^{D-s} = \mathfrak{X}'^{C_2-s} \cap \{\psi_i = 0 \mid \nu_2(\phi_i) > 0\}$. Therefore, π contracts the unique irreducible exceptional divisor E_2 defined by $\{T_2 = 0\}$ onto the set $F \subset X/\langle g_2 \rangle$, which is defined by ψ_i's such that $\nu_2(\phi_i) > 0$.

Now, we assume that X is smooth. Since the $\langle g_2 \rangle$-action is symplectic, the singularities of $X/\langle g_2 \rangle$ is analytically locally isomorphic to $\mathbb{C}^2 \times (\mathbb{C}^2/\{\pm 1\})$ or $\mathbb{C}^4/\{\pm 1\}$. Since the isolated singularity $\mathbb{C}^4/\{\pm 1\}$ is already terminal, the singularity of $X/\langle g_2 \rangle$ along F is isomorphic to $\mathbb{C}^2 \times (\mathbb{C}^2/\{\pm 1\})$. (Note that $X/\langle g_2 \rangle$ may have singularities of type $\mathbb{C}^4/\{\pm 1\}$ outside F.) Thus, the blowing-up X' of $X/\langle g_2 \rangle$ along F must be smooth in a neighbourhood of E_2. Therefore, in order to prove that X is singular, it suffices to show that X' has singularities in E_2.

In [**11**], the authors consider the minimal resolution Y of $\mathbb{C}^2/\{\pm 1\}$ and show that some minimal model $X'' \to V/G'$ factors through the quotient $(Y \times Y)/H'$ for

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$H' = G'/\left\langle g_2, \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \right\rangle$. Note that the exceptional divisor of $(Y \times Y)/H' \to V/G'$ is associated to the symplectic reflection g_2. One can directly check that $(Y \times Y)/H'$ has a singularity that is isolated (and hence terminal) using the argument in [**11**, 5.3]. Thus, X'' has a singular point in the exceptional divisor associated to g_2. Since X' and X'' are connected by a sequence of Mukai flops [**32**, Theorem 1.2], X' also has a singular point in E_2. Therefore, X is singular. Note that the minimal model X is unique since $\mathrm{Cl}(X)_{\mathbb{R}}$ is 1-dimensional.

For type (V), we can use the exactly same argument as one for type (K) by replacing type (J) by type (T).

For type (Q), the group G is $Sp(V, \omega)$-conjugate to another group in Cohen's list. Indeed, one can easily check that the matrix

$$g = \frac{i}{2}\begin{pmatrix} \zeta & -\zeta^3 & -\zeta & \zeta^3 \\ \zeta & \zeta^3 & -\zeta & -\zeta^3 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

is in $Sp(V, \omega)$ and that the group of type (J) is the g-conjugate of G where $\zeta = \exp(2\pi i/8)$. Therefore, V/G does not admit projective symplectic resolutions by [**11**].

Finally, we treat type $(\mathrm{G})_{l,r}$. When r is 1, one can check that G preserves the Lagrangian subspace $L = \{x + iw = y - iz = 0\}$ of V and thus G is improper (with respect to ω). The corresponding complex reflection group is $G(2l, l, 2)$ in the Shepherd-Todd classification [**28**]. By the result of Bellamy [**5**], we know that V/G with $G = G(2l, l, 2)$ admits a projective symplectic resolution if and only if $l = 1$.

When $r \neq 1$, we use the similar method as type (K) and (V). The group $G' = \left\langle G, g = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \right\rangle$ coincides with the group of type (B) (resp. type (F)) in Cohen's list if l is even (resp. odd). By the same argument above using the description of Cox rings, one sees that g-action on V/G lifts to its minimal model X and that a minimal model X' of V/G' is obtained as the blowing-up of $X/\langle g \rangle$. Similarly to the cases type (K) and (V), it suffices to show that X' has a singular point in the exceptional divisor E that corresponds to the symplectic reflection g.

Consider the same Y as above and $H' = G'/\left\langle g, \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \right\rangle$. In this case, $Y \times Y$ has no isolated fixed points by the H'-action but one can find a point $x \in Y \times Y$ in the exceptional divisor such that the stabilizer subgroup $\mathrm{Stab}_{H'}(x) \subset H'$ is not generated by symplectic reflections by using the argument in [**11**, 5.3]. Since elements in $\mathrm{Stab}_{H'}(x)$ preserves the exceptional divisor of $Y \times Y \to V/\left\langle g, \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \right\rangle$, the minimal model X' has a singular point in E. □

7. Appendix.

7.1. How to calculate $\min_i(I)$, I_A and $\underline{\min}_i(J)$. In this subsection, we give a concrete method to perform the algorithm. We will usually need computer calculations in practice.

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Let $G \subset SL(V)$ be a finite subgroup, and let $g_1, \ldots, g_m$ be a complete system of representatives of the conjugacy classes of the junior elements in G. Assume that we are given the generators $\phi_1, \ldots, \phi_k$ of the invariant ring $\mathbb{C}[V]^{[G,G]}$ that are homogeneous with respect to $Ab(G)$-action.

Let I be the kernel of $\alpha : \mathbb{C}[X_1, \ldots, X_k] \to \mathbb{C}[V]^{[G,G]}$ (see Section 4). Recall that for each $i \in \{1, \ldots, m\}$, the variable X_j has degree $\nu_i(\phi_j)$. Let $\mathbb{C}[X_1, \ldots, X_k, t]$ be the new graded ring with one more variable t whose degree is -1. The ideal $\min_i(I)$ $(i = 1, \ldots, m)$ is calculated by taking the following steps:

(1) Consider the ideal generated by the image of I by the inclusion $\mathbb{C}[X_1, \ldots, X_k] \hookrightarrow \mathbb{C}[X_1, \ldots, X_k, t]$, and let this ideal also be denoted by I by abuse of notation.
(2) Homogenize the generators of I with respect to the variable t, and let I_i be the ideal generated by these elements.
(3) Compute the saturation $\widetilde{I}_i := \bigcup_{l=0}^{\infty} I_i : (t^l)$ of I_i with respect to t.
(4) Evaluate $t = 0$ in $\widetilde{I}_i$.

Then, the resulting ideal (regarded as the ideal of $\mathbb{C}[X_1, \ldots, X_k]$) is $\min_i(I)$. Note that just homogenizing the generators is not enough and the saturation is necessary in general.

In order to obtain $I_A \subset R_A$ for a subset $A \subset \{1, \ldots, m\}$, one should consider $R_A = \mathbb{C}[X_1, \ldots, X_k, \{t_i\}_{i \in A}]$ and perform the steps from 1. to 3. for each t_i $(i \in A)$.

Next, let us consider $\underline{\min}_i(J)$ $(i = 1, \ldots, m)$. Let $g \in G$ and let r be the order of g in $Ab(G)$. As each ϕ_j is homogeneous with respect to $Ab(G)$-action, there is an integer $0 \leq a_j < r$ such that g acts on ϕ_j by multiplication of $\exp(2\pi i a_j / r)$. Let $\mathbb{C}[X_1, \ldots, X_k, s]$ be the graded polynomial ring where $\deg(X_j) = a_i$ and $\deg(s) = 1$. Let J be the kernel of β_i (see Section 4). We can calculate the ideal J_g generated by homogeneous elements of J with respect to g by taking the following steps:

(1) Consider the ideal generated by the image of J by the inclusion $\mathbb{C}[X_1, \ldots, X_k] \hookrightarrow \mathbb{C}[X_1, \ldots, X_k, s]$, and let this ideal also be denoted by J by abuse of notation.
(2) Homogenize the generators of J with respect to the variable s, and let J_i be the ideal generated by these elements.
(3) Compute the saturation $\widetilde{J}_i := \bigcup_{l=0}^{\infty} J_i : (s^l)$ of J_i with respect to s.
Then, the preimage of $\widetilde{J}_i + (s^r - 1)$ by the inclusion $\mathbb{C}[X_1, \ldots, X_k] \hookrightarrow \mathbb{C}[X_1, \ldots, X_k, s]$ is J_g.

If we apply the above procedure to J_g and another $g' \in Ab(G)$ instead of J and g, respectively, we obtain the ideal generated by homogeneous elements of J with respect to both g and g'. Thus, by repeating the same procedures over g's which generate $Ab(G)$, we finally obtain $\underline{\min}_i(J)$.

7.2. How to calculate the relations of generators of a Cox ring. In this subsection, we give a relatively easy way of calculation of the relations of the generators of the Cox ring.

By the algorithm, we know that the generators of the Cox ring of the minimal model X of V/G is of the following form:

$$\psi_1 := \phi_1 \prod_{i=1}^{m} t_1^{\nu_i(\phi_1)}, \ldots, \psi_k := \phi_1 \prod_{i=1}^{m} t_k^{\nu_i(\phi_k)}, T_1 := t_1^{-r_1}, \ldots, T_m := t_m^{-r_m}$$

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where ϕ_i's are the homogeneous generators of $\mathbb{C}[V]^{[G,G]}$, $g_1, \ldots, g_m$ are the representatives of the junior elements in G and $r_i := \sharp \langle g_i \rangle$. Assume that we are already given the ideal $I \subset \mathbb{C}[X_1, \ldots, X_k]$ of the relations of ϕ_i's. Then, the ideal $\widetilde{I} \subset \mathbb{C}[X_1, \ldots, X_k, Y_1, \ldots, Y_m]$ of the relations of ψ_i's and T_i's are calculated as follows:

(1) Compute $I_{\{1,\ldots,m\}} \subset R_{\{1,\ldots,m\}} = \mathbb{C}[X_1, \ldots, X_k, t_1, \ldots, t_m]$ (see 7.1).
(2) Replace every $t_i^{r_i}$ by Y_i in the $Ab(G)^\vee$-homogeneous generators of $I_{\{1,\ldots,m\}}$ for each i. (This is possible since homogeneity implies that t_i's appear only with powers of multiples of r_i.)

The resulting ideal is $\widetilde{I}$. Indeed, the resulting ideal is clearly contained in $\widetilde{I}$, and conversely any $\mathrm{Cl}(X)^{\mathrm{free}}$-homogeneous element f in $\widetilde{I}$ is obtained (up to multiplication by Y_i's) by applying the homogenization to $f|_{Y=1} \in I$.

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