Canad. Math. Bull. Vol. 18 (5), 1975

SOME RESULTS ON SPIRAL-LIKE FUNCTIONS

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We begin with the following definitions.

DEFINITION. Let $f(z)$ be regular near $z=0$ and let $f(0)=0$, $f'(0)\neq 0$. Let α and λ be two real numbers such that $|\alpha|<\pi/2$ and $0\leq\lambda<1$. Then $R_{\alpha,\lambda}$ is the largest value of r such that the following conditions are satisfied for $|z|<r$:

(i) $f(z)$ is regular,

(ii) $f(z)\neq 0$ for $z\neq 0$,

(iii) $\mathscr{R}\left[\exp(i\alpha)\dfrac{zf'(z)}{f(z)}\right]>\lambda$.

In particular with $\lambda=0$, $R_{\alpha,0}$ coincides with the radius of spiral-likeness; with $\alpha=0$, $R_{0,\lambda}$ gives the radius of starlikeness of order λ; and with $\alpha=\lambda=0$, $R_{0,0}$ gives the radius of starlikeness.

DEFINITION. Let $f(z)$ be regular near $z=0$ and let $f(0)=0$, $f'(0)\neq 0$. Let α and λ be two real numbers such that $|\alpha|<\pi/2$ and $0\leq\lambda<1$. Then $r_{\alpha,\lambda}$ is the largest value of r such that the following conditions are satisfied for $|z|<r$:

(i) $f(z)$ is regular,

(ii) $f'(z)\neq 0$,

(iii) $\mathscr{R}\left[\exp(i\alpha)\left(1+\dfrac{zf''(z)}{f'(z)}\right)\right]>\lambda$.

For $\alpha=0$, $r_{0,\lambda}$ gives the radius of convexity of order λ and for $\alpha=\lambda=0$, $r_{0,0}$ gives the radius of convexity.

The purpose of this note is to determine $R_{\alpha,\lambda}$ and $r_{\alpha,\lambda}$ for certain classes of analytic functions. To do this we will require the following lemmas.

LEMMA A[(1)]. *Let $z=r\exp(i\theta)$, $z_1=R\exp(i\phi)$ where $0<r<R$, and let α be a real number. Then*

$$-\frac{r(R+r\cos\alpha)}{R^2-r^2}\leq\mathscr{R}\left[\frac{\exp(i\alpha)z}{z-z_1}\right]\leq\frac{r(R-r\cos\alpha)}{R^2-r^2} \tag{1}$$

Received by the editors October 28, 1974.

(1) This result is an extension of the Lemma we obtained in [2, p. 140]. Without the conditions under which the equality signs hold in (1) it is first proved in [1, p. 8–9]. The proof is simplified in the above form by Professor F. R. Keogh.

Equality holds in the first inequality of (1) *if and only if*

$$z=\frac{r}{R}z_1\frac{R+r\exp(i\alpha)}{r+R\exp(i\alpha)}$$

and in the second inequality if and only if

$$z=\frac{r}{R}z_1\frac{R-r\exp(i\alpha)}{r-R\exp(i\alpha)}$$

Proof. The transformation $w=\exp(i\alpha)z/(z-z_1)$ maps the circle $|z|=r$ onto a circle in the w plane with centre $(-r^2\cos\alpha/(R^2-r^2), -r^2\sin\alpha/(R^2-r^2))$ and radius $rR/(R^2-r^2)$, which gives the required result.

In our proofs we also use the following two inequalities [1, p. 10], which we state as lemma B.

LEMMA B. *For* $0<r<1\leq R$, *we have*

(i) $\dfrac{R+r\cos\alpha}{R^2-r^2}\leq\dfrac{1+r\cos\alpha}{1-r^2}$,

(ii) $\dfrac{R-r\cos\alpha}{R^2-r^2}\leq\dfrac{1-r\cos\alpha}{1-r^2}$.

Equality holds in both inequalities if and only if $R=1$.

Now we can prove the following theorem.

THEOREM 1. *Let* $P(z)$ *be a polynomial of degree* $n>0$ *all of whose zeros are outside or on the unit circle. Then for* $f(z)=z[P(z)]^{\beta/n}$, *where* β *is real and non-zero, and for* $\alpha\neq 0$ *we have*

$$R_{\alpha,0}\geq\cos\alpha \quad \textit{if} \quad \beta=-1,$$

$$R_{\alpha,0}\geq\frac{-|\beta|+[\beta^2+4(1+\beta)\cos^2\alpha]^{1/2}}{2(1+\beta)\cos\alpha} \quad \textit{otherwise.}$$

Equality holds in both inequalities if and only if all the zeros of $P(z)$ *are concentrated at the same point on the unit circle.*[2]

Proof. Let $P(z)=a_0\prod_{k=1}^n(z-z_k)$; then

$$\frac{P'(z)}{P(z)}=\sum_{k=1}^n\frac{1}{z-z_k},$$

and since

$$\frac{zf'(z)}{f(z)}=1+\frac{\beta}{n}z\frac{P'(z)}{P(z)}$$

we have

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right]=\cos\alpha+\frac{\beta}{n}\mathscr{R}\left[\sum_{k=1}^n\frac{\exp(i\alpha)z}{z-z_k}\right] \tag{2}$$

[2] For $\alpha=0$ the inequalities given in Theorem 1 hold, but the distributions of zeros which give the equalities are different and for those distributions $R_{0,0}$ gives the radius of univalence [2, Theorem 1].

Case 1. $\beta>0$. Let $z=r\exp(i\theta)$, $z_k=R_k\exp(i\phi_k)$, $k=1, 2, \ldots, n$. By Lemma A and by part (i) of Lemma B from (2) we obtain

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] \geq \cos\alpha-\frac{\beta}{n}\sum_{k=1}^{n}\frac{r(R_k+r\cos\alpha)}{R_k^2-r^2}$$
$$\geq \cos\alpha-\frac{\beta(r+r^2\cos\alpha)}{1-r^2},$$

and this gives $R_{\alpha,0}\geq(-\beta+[\beta^2+4(1+\beta)\cos^2\alpha]^{1/2})/2(1+\beta)\cos\alpha$, with equality if and only if all the zeros of $P(z)$ are concentrated at the same point on the unit circle.

Case 2. $\beta<0$. By Lemma A and by part (ii) of Lemma B fro m(2) we obtain

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] \geq \cos\alpha+\frac{\beta}{n}\sum_{k=1}^{n}\frac{r(R_k-r\cos\alpha)}{R_k^2-r^2}$$
$$\geq \cos\alpha+\beta\frac{r(1-r\cos\alpha)}{1-r^2} = B(r),$$

say. The condition that $B(r)>0$ gives the required results.

The following theorem reduces to the theorem given in [3, p. 16] for $\alpha=\lambda=0$ and to Theorem 3 in [2] for $\alpha=0$.

THEOREM 2. *Let $P(z)$ be a polynomial of degree $n>0$ all of whose zeros are outside or on the unit circle. If* $\cos\alpha>\lambda$, *then for $f(z)=zP(z)$ we have*

$$R_{\alpha,\lambda} \geq \frac{n-[4\lambda^2+n^2+(4n+4)\cos^2\alpha-(4n\lambda+8\lambda)\cos\alpha]^{1/2}}{2[\lambda-(n+1)\cos\alpha]}$$

where equality holds if and only if all the zeros of $P(z)$ are concentrated at the same point on the unit circle.

Proof. Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$. Then

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] = \cos\alpha+\mathscr{R}\left[\sum_{k=1}^{n}\frac{\exp(i\alpha)z}{z-z_k}\right]. \tag{3}$$

If $z=r\exp(i\theta)$, $z_k=R_k\exp(i\phi_k)$ then by Lemma A and by part (i) of Lemma B from (3) we obtain

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] \geq \cos\alpha-\sum_{k=1}^{n}\frac{r(R_k+r\cos\alpha)}{R_k^2-r^2}$$
$$\geq \cos\alpha-n\,\frac{r(1+r\cos\alpha)}{1-r^2};$$

hence

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] \geq \lambda \quad\text{if}\quad r \leq \frac{n-[4\lambda^2+n^2+(4n+4)\cos^2\alpha-(4n\lambda+8\lambda)\cos\alpha]^{1/2}}{2[\lambda-(n+1)\cos\alpha]}$$

with equality if and only if all the z_k are concentrated at one point on the unit circle.

The next theorem reduces to the theorem given in [3, p. 16] for $n=\alpha=0$ and to Theorem 4 in [2] for $\alpha=0$.

THEOREM 3. *Let $f(z)=zM(z)/N(z)$, where M, N are polynomials of degree $m\geq 1$, $n\geq 0$ respectively, all of whose zeros are outside or on the unit circle. If $n-m-1>0$ then $R_{\alpha,0}\geq r_0$, where r_0 is the smaller root of the equation*

$$F(r) = (n-m-1)r^2\cos\alpha-(n+m)r+\cos\alpha = 0, \tag{4}$$

If $n-m-1\leq 0$ then $R_{\alpha,0}\geq r_0$, where r_0 is now the positive root of this equation. In both cases we have $R_{\alpha,0}=r_0$ if and only if the zeros of M are concentrated at a point $\exp(i\theta_1)$ *and those of N are concentrated at a point* $\exp(i\theta_2)$ *such that* $\exp(i(\theta_1-\theta_2))=(1-r_0\exp(i\alpha))(r_0+\exp(i\alpha))/(1+r_0\exp(i\alpha))(r_0-\exp(i\alpha))$.

Proof. Denoting the zeros of M by $z_1, z_2, \ldots, z_m$ and the zeros of N by $z_{m+1}, z_{m+2}, \ldots, z_{m+n}$, we have

$$\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] = \cos\alpha+\mathscr{R}\left[\sum_{k=1}^{m}\frac{\exp(i\alpha)z}{z-z_k}\right]-\mathscr{R}\left[\sum_{k=m+1}^{m+n}\frac{\exp(i\alpha)z}{z-z_k}\right]. \tag{5}$$

If $z=r\exp(i\theta)$, $z_k=R_k\exp(i\phi_k)$ then, by Lemma A and Lemma B from (5) we obtain

$$\begin{aligned}\mathscr{R}\left[\exp(i\alpha)\frac{zf'(z)}{f(z)}\right] &\geq \cos\alpha-\sum_{k=1}^{m}\frac{r(R_k+r\cos\alpha)}{R_k^2-r^2}-\sum_{k=m+1}^{m+n}\frac{r(R_k-r\cos\alpha)}{R_k^2-r^2}\\ &\geq \cos\alpha-\frac{mr(1+r\cos\alpha)}{1-r^2}-\frac{nr(1-r\cos\alpha)}{1-r^2} = G(r),\end{aligned}$$

say. The condition that $G(r)>0$ is equivalent to $F(r)>0$, so $R_{\alpha,0}\geq r_0$. Now, if $R_{\alpha,0}=r_0$ then $z_1, z_2, \ldots, z_m$ are concentrated at a point $\exp(i\theta_1)$ and $z_{m+1}, z_{m+2}, \ldots, z_{m+n}$ are concentrated at a point $\exp(i\theta_2)$ such that

$$\exp(i(\theta_1-\theta_2)) = \frac{(1-r_0\exp(i\alpha))(r_0+\exp(i\alpha))}{(1+r_0\exp(i\alpha))(r_0-\exp(i\alpha))} \tag{6}$$

and the converse is also true. If $\alpha=0$ then by (6), $R_{0,0}=r_0$ if and only if $z_1, z_2, \ldots, z_m$ are concentrated at one end of a diameter of the unit circle and $z_{m+1}, z_{m+2}, \ldots, z_{m+n}$ are concentrated at the opposite end of this diameter. When $R_{0,0}=r_0$, $f'(z)$ has a zero on $|z|=r_0$, so $R_{0,0}$ gives also the radius of univalence.

Now let $f(z)=zg'(z)$ then $r_{\alpha,\lambda}$ for $g(z)$ is the same as $R_{\alpha,\lambda}$ for $f(z)$. Therefore from the above theorems the following results can be deduced.

THEOREM 1′. *Let $P(z)$ be a polynomial of degree $n>0$ all of whose zeros are outside or on the unit circle, and let $f(z)$ be the function such that $f(0)=0$ and $f'(z)=[P(z)]^{\beta/n}$, where β is real and non-zero. Then for $f(z)$ and $\alpha\neq 0$, we have*

$$r_{\alpha,0} \geq \cos\alpha \quad \text{if} \quad \beta=-1,$$

$$r_{\alpha,0} \geq \frac{-|\beta|+[\beta^2+4(1+\beta)\cos^2\alpha]^{1/2}}{2(1+\beta)\cos\alpha} \text{ otherwise.}$$

Equality holds in both inequalities if and only if all the zeros of $P(z)$ are concentrated at the same point on the unit circle.

THEOREM 2′. *Let $f'(z)$ be a polynomial of degree $n>0$, $f(0)=0$, and suppose that all the zeros of $f'(z)$ are outside or on the unit circle. If $\cos\alpha>\lambda$, then for $f(z)$ we have*

$$r_{\alpha,\lambda} \geq \frac{n-[4\lambda^2+n^2+(4n+4)\cos^2\alpha-(4n\lambda+8\lambda)\cos\alpha]^{1/2}}{2[\lambda-(n+1)\cos\alpha]},$$

with equality if and only if the zeros of $f'(z)$ are concentrated at the same point on the unit circle.

THEOREM 3′. *Let $f(z)$ be a function such that $f(0)=0$ and $f'(z)=M(z)/N(z)$, where M, N are polynomials of degree $m\geq 1$, $n\geq 0$ respectively, all of whose zeros are outside or on the unit circle. If $n-m-1>0$ then $r_{\alpha,0}\geq r_0$, where r_0 is the smaller root of the equation* (4). *If $n-m-1\leq 0$ then $r_{\alpha,0}\geq r_0$, where r_0 is the positive root of the equation. In both cases we have equality under the same conditions on M, N as in Theorem* 3.

By using arguments similar to those in the proofs of the preceding theorems we can easily obtain results for $R_{\alpha,\lambda}$ and $r_{\alpha,\lambda}$ for other cases considered in [1] and [2].

REFERENCES

1. A. F. Barr, *The radius of univalence of certain classes of analytic functions* Ph.D. Dissertation. The University of Mississippi, (1971).

2. T. Başgöze, *On the univalence of certain classes of analytic functions*, J. London Math. Soc. (2), **1** (1969), p. 140–144.

3. J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), p. 12–22.

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