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PLANAR SUBLATTICES OF A FREE LATTICE. I

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1. Introduction. There are three lattice-theoretic properties that are generally used to open a discussion on sublattices of a free lattice:

(**W**) for all a, b, c, d, $a \wedge b \leqq c \vee d$ implies $a \wedge b \leqq c$, $a \wedge b \leqq d$, $a \leqq c \vee d$, or $b \leqq c \vee d$;

($\mathbf{SD}_\vee$) for all a, b, c, $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$;

($\mathbf{SD}_\wedge$) for all a, b, c, $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$.

(**W**) is one of the conditions present in P. M. Whitman's solution [**17**] of the word problem for lattices while ($\mathbf{SD}_\vee$) and ($\mathbf{SD}_\wedge$) were originated by B. Jónsson [**8**] (cf. R. A. Dean [**2**]). It is well known that each of these conditions holds in every sublattice of a free lattice.

In the late 1950's Jónsson posed a conjecture (cf. B. Jónsson and J. E. Kiefer [**9**]) that has in the intervening period attracted considerable attention.

CONJECTURE. *A finite lattice is a sublattice of a free lattice if and only if it satisfies* ($\mathbf{SD}_\vee$), ($\mathbf{SD}_\wedge$) *and* (**W**).

What is known?

F. Galvin and B. Jónsson [**3**] considered a special case of the conjecture in 1961: they showed that any finite distributive lattice that satisfies ($\mathbf{SD}_\vee$), ($\mathbf{SD}_\wedge$), and (**W**) is a sublattice of a free lattice. What is more, they could essentially display all finite distributive sublattices of a free lattice: *a finite distributive lattice is a sublattice of a free lattice if and only if it is a linear sum of lattices, each of which is isomorphic to* $\mathbf{1}$, $\mathbf{2}^3$, *or* $\mathbf{2} \times \boldsymbol{n}$ *for some* n. (For a positive integer m, $\boldsymbol{m}$ denotes the m-element chain.)

Call a lattice *semidistributive* if it satisfies ($\mathbf{SD}_\vee$) and ($\mathbf{SD}_\wedge$). Of course, every distributive lattice is semidistributive while a modular semidistributive lattice is distributive. Hence, Galvin and Jónsson had really settled the conjecture for all finite modular lattices.

In 1962, B. Jónsson and J. E. Kiefer [**9**] proved that finite sublattices of a free lattice have breadth at most *four*; moreover, they showed that the conjecture will hold for all finite lattices if it holds for finite lattices of breadth at most *three*. Still, all efforts to settle even the breadth *two* case have been unsuccessful.

The purpose of this paper is to settle the conjecture in the affirmative for a rather extensive class of breadth two lattices; namely, *planar* lattices, that is, finite lattices with planar (Hasse) diagrams.

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THEOREM. *A finite planar lattice is a sublattice of a free lattice if and only if it satisfies* ($\mathbf{SD}_\vee$), ($\mathbf{SD}_\wedge$), *and* ($\mathbf{W}$).

In a companion paper [**15**] we construct a *minimum* list $\mathscr{L}$ of lattices such that *a finite lattice is a planar sublattice of a free lattice if and only if it contains no sublattice isomorphic to a lattice in* $\mathscr{L}$.

Some words about the plan of this paper are in order.

In § 2 we shall elaborate on some basic facts concerning breadth, especially in the context of semidistributive lattices of breadth at most two.

One of the most interesting by-products of our investigations stems from the importance of the "finiteness" condition in the conjecture. R. Freese and J. B. Nation (cf. B. Jónsson and J. B. Nation [**10**]) have already exhibited a finitely generated (infinite) lattice that satisfies ($\mathbf{SD}_\vee$), ($\mathbf{SD}_\wedge$), and ($\mathbf{W}$) yet is not a sublattice of a free lattice. In § 3 we shall construct a small family of finite partially ordered sets none of which can appear as a subset in a *finite* breadth two lattice satisfying ($\mathbf{SD}_\vee$), ($\mathbf{SD}_\wedge$), and ($\mathbf{W}$). This result contrasts sharply with the problem of characterizing those finite partially ordered sets that generate a finite free lattice (cf. Yu. I. Sorkin [**16**] and R. Wille [**18**]). The prospect of enumerating, or at least characterizing, those finite partially ordered sets that cannot appear as subsets in a finite sublattice of a free lattice, may be as illuminating about the structure of free lattices as the solution to the current conjecture itself.

In § 4 we shall recall some elementary facts about planarity for lattices.

Our basic approach to the problem originates in the thesis of H. S. Gaskill [**4**] (cf. [**5**]) where the conditions ($\mathbf{T}_\vee$) and ($\mathbf{T}_\wedge$) were first formulated. A paper of H. S. Gaskill, G. Grätzer, and C. R. Platt [**6**] continued the study of ($\mathbf{T}_\vee$) and ($\mathbf{T}_\wedge$) along the lines initiated by Gaskill. R. McKenzie had shown in [**13**] that a finite lattice is a sublattice of a free lattice if and only if it is a bounded homomorphic image of a free lattice and satisfies ($\mathbf{W}$). Gaskill and Platt [**7**] combined these results to prove that *a finite lattice is a sublattice of a free lattice if and only if it satisfies* ($\mathbf{T}_\vee$), ($\mathbf{T}_\wedge$), *and* ($\mathbf{W}$). This criterion is likely the most practical characterization known for actually determining whether or not a particular finite lattice is a sublattice of a free lattice; it is the one that we shall use. The central ideas in this approach are "minimal pairs" and "cycles". These ideas will be developed in § 5 and § 6. (The substance of these ideas was also suggested by B. Jónsson in the early 1960's in private notes which were not widely circulated (cf. [**10**]).)

The main body of the proof occurs in § 7 and § 8.

Finally, for an obvious reason we prefer to include in the introduction some retrospective remarks about our proof. The proof of the main theorem of this paper is long and arduous. In some respects the proof amounts to an accumulation of techniques which in concert enable us to carry a straightforward approach through to the end. We maintain a guarded optimism that this same approach may yet be extended to settle the complete breadth two case of the conjecture. In any case, several of the techniques in our repertoire seem to be of quite independent interest.

2. Breadth. The *breadth* $b(L)$ of a lattice L is the smallest integer b such that every join $\bigvee_{i=1}^{b+1} x_i$ of elements of L is equal to a join of b of the x_i's. Of course, a lattice has breadth one if and only if it is a chain.

What would happen if we were to use meets instead of joins in the definition of $b(L)$? Notice that this definition does not *appear* to be self-dual: actually, it is.

LEMMA 2.1. *Let L be a lattice with $b(L) = b$, and let b' be the smallest integer such that every meet $\bigwedge_{i=1}^{b'+1} x_i$ of elements of L is equal to a meet of b' of the x_i's. Then $b' = b$.*

Proof. Suppose $b' > b$. It follows that there are elements x_i, $1 \leqq i \leqq b + 1$, of L such that $\bigwedge_{i=1}^{b+1} x_i$ is not equal to the meet of any b of the x_i's. Let $y_j = \bigwedge \{x_i | 1 \leqq i \leqq b + 1, i \neq j\}$ for each $j \in \{1, \ldots, b + 1\}$; we have that $y_j \nleqq x_j$, for each j. Now let $z_k = \bigvee \{y_j | 1 \leqq j \leqq b + 1, j \neq k\}$ for each $k \in \{1, \ldots, b + 1\}$. Note that $z_k \leqq x_k$ for each k but that $\bigvee_{j=1}^{b+1} y_j \nleqq x_k$ for any k; thus, $z_k \neq \bigvee_{j=1}^{b+1} y_j$ for any k, contradicting $b(L) = b$. Hence $b' \leqq b$, and a dual argument shows equality.

For any integer $n \geqq 3$, a *crown* [**1**] of order $2n$ is a subset

$$C = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$$

of a partially ordered set in which $x_1 < y_1$, $y_1 > x_2$, $x_2 < y_2, \ldots, x_n < y_n$, and $y_n > x_1$ are the only comparability relations that hold in C (see Figure 1). The next lemma illustrates the usefulness of crowns of order six in a discussion of breadth two lattices.

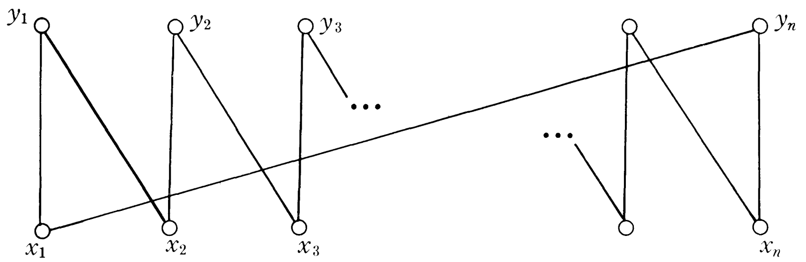


FIGURE 1. A crown of order $2n$.

LEMMA 2.2. *A lattice has breadth at most two if and only if it contains no crown of order six.*

Call a finite lattice L *dismantlable* if we can write $L = \{x_1, x_2, \ldots, x_n\}$ such that $\{x_1, x_2, \ldots, x_i\}$ is a sublattice of L for each $i = 1, 2, \ldots, n$. The notion of dismantlability was first introduced by I. Rival in [**14**] and it has since played a major role particularly in the study of planar lattices. Dismantlable lattices themselves have been characterized by D. Kelly and I. Rival [**11**].

LEMMA 2.3. *A finite lattice is dismantlable if and only if it contains no crown.*

As a consequence, every dismantlable lattice has breadth at most two. It turns out that, in the presence of $(\mathbf{SD}_\vee)$, the converse is true as well.

LEMMA 2.4. *Let L be a finite lattice satisfying $(\mathbf{SD}_\vee)$. Then L is dismantlable if and only if $b(L) \leqq 2$.*

Proof. Let $b(L) \leqq 2$, and suppose that L is not dismantlable. By Lemma 2.3, L contains a crown; let

$$\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$$

be a crown of minimum order in L. We may assume that $x_i \vee x_{i+1} = y_i$ and $y_i \wedge y_{i+1} = x_{i+1}$, $i = 1, \ldots, n-1$, and $x_n \vee x_1 = y_n$, $y_n \wedge y_1 = x_1$ hold in L. Furthermore, since $b(L) \leqq 2$, Lemma 2.2 implies that $n \geqq 4$. Now, if $x_1 \vee x_3 \not> x_2$, $\{x_1, y_1, x_2, y_2, x_3, x_1 \vee x_3\}$ is a crown of order 6, which is impossible; hence $x_1 \vee x_3 > x_2$ and by duality $y_1 \wedge y_3 < y_2$. Also, if $x_1 \vee x_3 \not> x_4$, it is easy to see that $\{x_1, x_1 \vee x_3, x_3, y_3, x_4, y_4, \ldots, x_n, y_n\}$ will contain a crown of order $< 2n$, which is a contradiction. Therefore $x_1 \vee x_3 > x_4$, and we have by symmetry that

$$x_1 \vee x_3 \geqq x_2 \vee x_4 \geqq x_3 \vee x_5 \geqq \ldots \geqq x_n \vee x_2 \geqq x_1 \vee x_3.$$

Now

$$y_1 \vee y_2 = x_1 \vee x_2 \vee x_3 = x_1 \vee x_3 = x_2 \vee x_4 = x_2 \vee x_3 \vee x_4 = y_2 \vee y_3,$$

and hence $(\mathbf{SD}_\vee)$ implies that $y_1 \vee y_2 = y_2 \vee (y_1 \wedge y_3) = y_2$, a contradiction. Thus L is dismantlable, and the lemma is established.

In this paper we shall be primarily interested in finite lattices of breadth at most two. Still, it seems worthwhile to prove the next two results in the framework of finite lattices of arbitrary breadth.

Recall that for elements x and y of a lattice L, with $x > y$, *x covers y* (or *y is covered by x*, or x is an *upper cover* of y, or y is a *lower cover* of x) if $x > z \geqq y$ implies $z = y$. We write $x \succ y$ or $y \prec x$.

LEMMA 2.5. *Let n_* be the maximum number of elements covered by any element of the finite lattice L. Then $n_* \geqq b(L)$.*

Proof. Let $b = b(L)$. There are elements $x_1, \ldots, x_b$ of L such that

$$y_j = \bigvee \{x_i | 1 \leqq i \leqq b, i \neq j\} < \bigvee_{i=1}^{b} x_i$$

for each $j \in \{1, \ldots, b\}$. Hence for each j there exists $w_j \prec \bigvee_{i=1}^{b} x_i$ such that

$y_j \leqq w_j$. Moreover $y_j \vee y_k = \bigvee_{i=1}^{b} x_i$ if $j \neq k$; thus all the w_j's are distinct, showing that $\bigvee_{i=1}^{b} x_i$ has at least b lower covers.

The self-dual nature of breadth has already been demonstrated in Lemma 2.1. Letting n^* be the maximum number of elements covering any element of L, we have $n^* \geqq b(L)$ by duality.

LEMMA 2.6. *Let L be a finite lattice satisfying* $(\mathbf{SD}_\wedge)$ *and let n^* be the maximum number of elements covering any element of L. Then $n^* = b(L)$.*

Proof. Let $a \in L$ and let $x_1, \ldots, x_{n^*}$ be distinct elements covering a. Since $x_i \wedge x_j = a$ whenever $i \neq j$, successive applications of $(\mathbf{SD}_\wedge)$ show that $x_i \wedge \bigvee \{x_j \mid 1 \leqq j \leqq n^*, \ j \neq i\} = a$ for each $i \in \{1, \ldots, n^*\}$; that is, $x_i \nleqq \bigvee \{x_j \mid 1 \leqq j \leqq n^*, j \neq i\}$ for each i. It follows that

$$\bigvee \{x_j \mid 1 \leqq j \leqq n^*, j \neq i\} < \bigvee_{j=1}^{n^*} x_j$$

for each i, and so $b(L) \geqq n^*$. From Lemma 2.5 we conclude $b(L) = n^*$.

We conclude this section with some further results concerning breadth two lattices.

LEMMA 2.7. *Let L be a finite lattice satisfying* $(\mathbf{SD}_\wedge)$ *such that $b(L) \leqq 2$. Then*

(i) *each element of L has at most two upper covers;*

(ii) *if x, y, z are distinct elements of L and $u = x \wedge y = x \wedge z = y \wedge z$ then $u \in \{x, y, z\}$, that is, $\{x, y, z\}$ is not an antichain.*

Proof. (i) This is immediate from Lemma 2.6.

(ii) Let $u = x \wedge y = x \wedge z = y \wedge z$, and suppose $\{x, y, z\}$ is an antichain. Then x, y, and z are all greater than u, and we may choose upper covers x_1, y_1, z_1 of u such that $x \geqq x_1$, $y \geqq y_1$, and $z \geqq z_1$; since $x \wedge y = x \wedge z = y \wedge z = u$, x_1, y_1, and z_1 are distinct, contradicting (i).

LEMMA 2.8. (i) *Let L be a finite breadth two lattice, and let $\{a, b, c\}$ be a three-element antichain in L with $a \vee b$* and *$a \vee c$ noncomparable. Then $a < b \vee c$.*

(ii) *If in addition L satisfies* $(\mathbf{SD}_\wedge)$ *then, letting a' be a lower cover of a, either $a' \vee b > a$ or $a' \vee c > a$.*

Proof. (i) If $a \nless b \vee c$ then $\{b, a \vee b, a, a \vee c, c, b \vee c\}$ is a crown of order six, contradicting Lemma 2.2.

(ii) Let us suppose that $a' \vee b \ngtr a$ and $a' \vee c \ngtr a$. Then $a \wedge (a' \vee b) = a' = a \wedge (a' \vee c)$ whence, by $(\mathbf{SD}_\wedge)$ $a' = a \wedge (a' \vee b \vee c)$. From (i), $b \vee c > a$ so that $a' = a$, which is impossible.

3. Finiteness. The purpose of this section is to establish a result that provides us with a powerful technique in applying the finiteness assumption of the conjecture. The result, which is due to I. Rival, also seems to be of considerable independent interest.

For noncomparable elements a and b of a partially ordered set we write $a||b$. An element a of a lattice L, $a \neq 0$, is *join irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$; $a \neq 1$ is *meet irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$. $J(L)$, respectively $M(L)$, shall denote the subset of all join irreducible elements, respectively meet irreducible elements, of L.

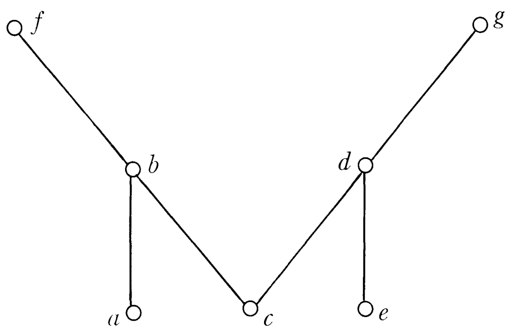
If P and Q are partially ordered sets, recall that a one-to-one map $\eta : P \rightarrow Q$ is a *weak embedding* if both η and η^{-1} are order-preserving. Consider the partially ordered set $M0$ of Figure 2, and construct partially ordered sets $M1$ and $M2$ (Figure 2) as follows:

$M1 = M0 \cup \{h\}$, where $h < f$, $h||b$, c, d, e, and g, and h may or may not be comparable with a;

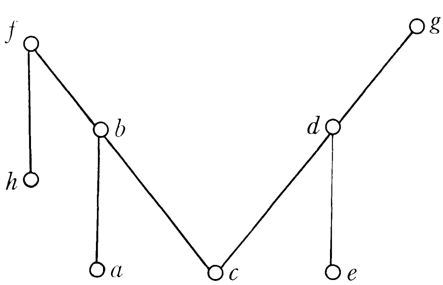
$M2 = M1 \cup \{i\}$, where $i < g$, $i||a$, b, c, d, f, and h, and i may or may not be comparable with e.

Proposition 3.1. *Let L be a semidistributive lattice satisfying* (**W**) *such that* $b(L) \leqq 2$. *Suppose that*

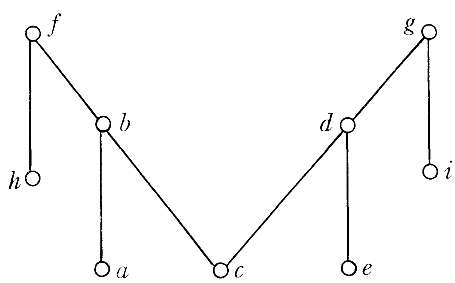
(i) *there is a weak embedding $\eta : M0 \rightarrow L$ such that $f\eta$ and $g\eta$ are join irreducible elements of L, or*



$M0$



$M1$



$M2$

Figure 2

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(ii) *there is a weak embedding* $\eta : M1 \rightarrow L$ *such that* $g\eta$ *is a join irreducible element of* L, *or*

(iii) *there is a weak embedding* $\eta : M2 \rightarrow L$.

Then L *is infinite.*

Proof. We will give the proof, assuming that (ii) holds; it will be easy to see that similar proofs exist in the other two cases.

Suppose (ii), and consider the set

$$C = \{c\eta | \eta : M1 \rightarrow L \text{ is a weak embedding, } g\eta \in J(L)\} \subseteq L.$$

By assumption C is nonempty. Suppose that L is finite; then we can choose η satisfying (ii) such that $c\eta$ is a maximal element of C. Furthermore, letting $x\eta = x'$ for $x \in M1$, we may assume that $a' \vee c' = b'$, $c' \vee e' = d'$, $b' \wedge d' = c'$, and $h' \vee b' = f'$. Let $c_* \prec c'$. We have from Lemma 2.8(ii) that either $c_* \vee a' = b'$ or $c_* \vee e' = d'$.

Suppose $c_* \vee a' = b'$. By (**W**) we must have $f' \wedge d' \not\leqq b'$. If $(f' \wedge d') \vee b' < f'$ then

$$\{a', (f' \wedge d') \vee b', f' \wedge d', d', e', f', g', h'\} \cong M1,$$

and $f' \wedge d' > c'$, contradicting the maximality of $c\eta = c'$. So $(f' \wedge d') \vee b' \nless f'$, which implies that $(f' \wedge d') \vee b' = f'$. Since $h' \vee b' = f'$ and $\{h', b', f' \wedge d'\}$ is an antichain, the dual of Lemma 2.7(ii) implies that $(f' \wedge d') \vee h' < f'$. But now $(f' \wedge d') \vee h' \ngeqq b'$, and hence

$$\{h', (f' \wedge d') \vee h', f' \wedge d', d', e', f', g', b'\} \cong M1,$$

and $f' \wedge d' > c'$, again contradicting the maximality of c'.

Hence $c_* \vee e' = d'$, and by (**W**) we must have $b' \wedge g' \not\leqq d'$. Now $(b' \wedge g') \vee d' \leqq g'$, and since $g' \in J(L)$ we in fact have $(b' \wedge g') \vee d' < g'$. Thus

$$\{a', b', b' \wedge g', (b' \wedge g') \vee d', e', f', g', h'\} \cong M1,$$

and $b' \wedge g' > c'$, contradicting the maximality of c'.

For readers of a statistical bent, we remark that Proposition 3.1 will be applied no less than 18 times in the proof of the main theorem.

4. Planarity. Planar lattices were investigated and characterized by D. Kelly and I. Rival [**12**]. Their characterization, while not needed here, will play a central role in a companion paper [**15**]. We shall assume familiarity with some of the more transparent concepts concerning planar lattices.

Let $e(L)$ be a planar embedding of the planar lattice L, and let $x \in L$. It is intuitively obvious that the lower covers of x are linearly ordered from left to right. Hence we can define a relation λ on the elements of L as follows: $x\lambda y$ if and only if $x||y$ and there are lower covers x' and y' of $x \vee y$ such that $x \leqq x'$, $y \leqq y'$, and x' is to the left of y' (with respect to $e(L)$).

For the remainder of this section we let L be a planar (finite) lattice, and we suppose that λ has been defined with respect to some planar embedding of L.

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The first lemma, due to J. Zilber, sets out the basic properties of λ (for a proof, see [**12**]).

LEMMA 4.1. *λ is a strict partial order on L. Moreover, if $x||y$, then $x\lambda y$ or $y\lambda x$.*

It follows that for $x, y \in L$, exactly one of the following holds: $x = y$, $x < y$, $x > y$, $x\lambda y$, $y\lambda x$. Consequently, the expression "$x\lambda y$" can be read, and thought of, as "x is to the left of y".

LEMMA 4.2(i) *If $x\lambda y$ and $y < z$, then $x\lambda z$ or $x < z$.*
(ii) *If $x\lambda y$ and $x < z$, then $z\lambda y$ or $z > y$.*

Proof. (i) If $x \geqq z$ then $x > y$, contradicting $x\lambda y$. If $z\lambda x$ then by transitivity $z\lambda y$, contradicting $y < z$. The conclusion follows from Lemma 4.1. The proof of (ii) is similar.

Remark. Clearly there is a kind of duality at work here. Corresponding to λ, we may define a partial order λ' on L by: $x\lambda' y$ if and only if $y\lambda x$ (the expression "$x\lambda' y$" of course could stand for "x is to the right of y"). Then given any statement S valid for a particular planar embedding of a planar lattice L, we may replace "left" with "right" and λ with λ' in S without affecting its validity. The resulting statement will be called the *reflection* of S. With this terminology, Lemma 4.2(ii) is the reflection of Lemma 4.2(i).

While we are on the subject, we may as well point out that the next lemma is "self-reflective".

LEMMA 4.3. *If $x\lambda y\lambda z$ then $x \wedge z < y < x \vee z$.*

Proof. Suppose $y \nless x \vee z$. Since $y\lambda z$ and $z < x \vee z$, Lemma 4.2 implies that $y\lambda x \vee z$. But since $x\lambda y$ and $x \vee z > x$, Lemma 4.2 also implies that $x \vee z\lambda y$, a contradiction. Hence $y < x \vee z$ and, dually, $x \wedge z < y$.

COROLLARY 4.4. *If $x\lambda y\lambda z$ then $x \vee y \vee z = x \vee z$.*

It follows, of course, that a planar lattice has breadth at most two.

For a join irreducible element a of L we denote its unique lower cover by a_*. Similarly, if $a \in M(L)$ we denote its unique upper cover by a^*. (Covering elements shall be separated in the figures by double lines when we wish to emphasize the fact.)

LEMMA 4.5. *Let L satisfy ($\mathbf{SD}_\vee$) and let x, y, z be elements of L such that $x\lambda y\lambda z$, $y \in J(L)$, and $y_* < z$. Then $x\lambda y \vee z$.*

Proof. If $x < y \vee z$ then by Corollary 4.4 $x \vee z = y \vee z$. By ($\mathbf{SD}_\vee$), $y \vee z = (x \wedge y) \vee z$; but since $y \in J(L)$ and $x||y$, $x \wedge y \leqq y_* < z$ and hence $y \vee z = z$, a contradiction. Therefore $x \nless y \vee z$, and by Lemma 4.2 (i) we conclude that $x\lambda y \vee z$.

LEMMA 4.6. *Let L satisfy $\mathbf{SD}_\wedge$.*

(i) *If* x, y, y_1, y_2 *are elements of* L *such that* $x\lambda y$, $y_1 > y$, $y_2 > y$, *and* $y_1\lambda y_2$, *then* $x \vee y = x \vee y_1$.

(ii) *If* $x\lambda y\lambda z$ *then* $x \wedge y \neq y \wedge z$.

Proof. (i) It is clear that $x \vee y > y$ implies $x \vee y \geqq u$ for some $u > y$; from Lemma 2.7(i), $u = y_1$ or $u \leqq y_2$. In the first case, $x \vee y \geqq x \vee u = x \vee y_1 \geqq x \vee y$, so that $x \vee y = x \vee y_1$ as desired. If $u \leqq y_2$ then $y_1\lambda u$ from Lemma 4.2, and so Lemma 4.3 implies that $x \vee y \geqq x \vee u \geqq x \vee y_1 \geqq x \vee y$, showing that $x \vee y = x \vee y_1$ in either case.

(ii) If $x \wedge y = y \wedge z$ then $x \wedge y = y \wedge z = x \wedge z$ from the dual of Corollary 4.4. But this is impossible by Lemma 2.7(ii).

Consider the right boundary B of L (of course, with respect to the planar embedding $e(L)$). B certainly contains at least one element of $M(L)$, if $|L| > 1$; for instance B will contain a dual atom of L. Therefore we can speak of the minimal meet irreducible element of B. The first part of the following lemma is due to K. A. Baker, P. C. Fishburn, and F. S. Roberts [**1**], and the second part (actually, its reflection) is Proposition 2.6 of [**12**].

LEMMA 4.7. *Let* B *be the right boundary of* L *and let* a *be the minimal meet irreducible element of* B. *Then*

(i) a *is doubly irreducible;*

(ii) *if* $x \in L\backslash B$ *there exists* $y \in B$ *such that* y *is doubly irreducible and* $x\lambda y$;

(iii) $x < a$ *implies that* $x \in B$.

Proof of (iii). If $x \notin B$, choose y as in (ii). From Lemma 4.2, either $a\lambda y$ or $a > y$; since $a \in B$, we have $a > y$. But $y \in M(L)$, contradicting the choice of a.

This last result will reemerge to play an important role in §6 and beyond.

5. Minimal pairs. Let L be a lattice and let $X, Y \subseteq L$. We shall write $X \ll Y$ if, for every $x \in X$, there exists $y \in Y$ such that $x \leqq y$.

Let $p \in L$ and let J be a finite subset of L. The pair $\langle p, J\rangle$ is called a *minimal pair* if the following three conditions hold:

(i) $p \notin J$;

(ii) $p \leqq \bigvee J$;

(iii) if J' is a finite subset of L such that $p \leqq \bigvee J'$ and $J' \ll J$, then $J' \supseteq J$.

We begin this section with some well-known properties of minimal pairs.

LEMMA 5.1. *Let* $\langle p, J\rangle$ *be a minimal pair. Then*

(i) J *is an antichain;*

(ii) $p \nleqq x$ *for any* $x \in J$;

(iii) $J \subseteq J(L)$;

(iv) $x \in J$ *implies that* $x_* \notin M(L)$;

(v) *if* $\langle p, J\rangle$ *is a minimal pair and* $p < p' \leqq \bigvee J$ *for* $p' \in L$, $\langle p', J\rangle$ *is also a minimal pair;*

(vi) $b(L) = n$ *implies* $|J| \leqq n$.

Proof. (i) and (vi) are true because otherwise $\bigvee J = \bigvee J'$ for some proper subset J' of J; since $J' \ll J$, this is a contradiction.

(ii) If $p < x$ for $x \in J$ then, letting $x_* \in L$ be such that $p \leqq x_* \prec x$ and setting $J' = (J \backslash \{x\}) \cup \{x_*\}$, we have that $J' \ll J$, $p \leqq \bigvee J'$, and $J' \not\supseteq J$, contradicting the assumption that $\langle p, J\rangle$ is a minimal pair.

(iii) Let $x = y \vee z \in J$, and set $J' = (J\backslash\{x\}) \cup \{y, z\}$. Then $p \leqq \bigvee J'$ and $J' \ll J$, so we must have that $J \subseteq J'$. This implies that $x = y$ or $x = z$, as desired.

(iv) Since $|J| \geqq 2$ from (ii), we can find $x' \in J$ with $x' \neq x$. If $x_* \in M(L)$, then $x' \vee x_* = x' \vee x$; hence, letting $J' = (J\backslash\{x\}) \cup \{x_*\}$, we have that $J' \ll J$, $\bigvee J' = \bigvee J$, and $J' \not\supseteq J$, contradicting the assumption that $\langle p, J\rangle$ is a minimal pair.

(v) This is obvious.

As a consequence, there is an alternate description of minimal pairs in a lattice of breadth two.

LEMMA 5.2. *Let* $p \in L$ *and* $J \subseteq L$ *with* J *finite, and assume* $b(L) = 2$. *Then* $\langle p, J\rangle$ *is a minimal pair if and only if the following conditions hold:*

(i) $J = \{a, b\}$ *for some* $a, b \in L$;

(ii) $p \leqq a \vee b$ *while* $p \nleqq a$, $p \nleqq b$;

(iii) *if* $a_1 \leqq a$, $b_1 \leqq b$ *are such that* $p \leqq a_1 \vee b_1$, *then* $a_1 = a$ *and* $b_1 = b$.

We shall require some further techniques to assist us in our investigation of minimal pairs.

LEMMA 5.3. *Let* $b(L) = 2$, *let* $x \in J(L)$, *and let* y *and* p *be elements of* L *such that* $x||y$, $y < p \leqq x \vee y$, *and* $p \nleqq x_* \vee y$. *Then there exists* $y' \leqq y$ *such that* $\langle p, \{x, y'\}\rangle$ *is a minimal pair.*

Proof. Let $Y = \{u \in L | u \leqq y,\ p \leqq x \vee u\}$. Since $y \in Y$, Y is nonempty, and hence we can find a minimal element y' of Y. If $x_1 \leqq x$ and $y_1 \leqq y'$ are such that $p \leqq x_1 \vee y_1$, then $x_1 = x$; for otherwise $x_1 \leqq x_*$ and so $x_1 \vee y_1 \leqq x_* \vee y < p$. But now $y_1 = y'$ by the minimality of y', hence by Lemma 5.2 we are done.

LEMMA 5.4. *Let* L *be planar, and let* $\langle p, \{x, y\}\rangle$ *be a minimal pair with* $x < p$ *and* $p\lambda y$. *Let* $p' \in L$ *be such that* $x < p' < x \vee y$ *and* $p'\lambda p$. *Then* $\langle p', \{x, y\}\rangle$ *is a minimal pair.*

Proof. We need only check that (iii) of Lemma 5.2 holds. Let $x' \leqq x$ and $y' \leqq y$ with $p' \leqq x' \vee y'$. If $y' < p$ then $p' \leqq x' \vee y' \leqq p$, which is impossible. Since $p\lambda y$, the dual of Lemma 4.2(i) implies that $p'\lambda p\lambda y'$. Now Lemma 4.3 implies $p < y' \vee p' \leqq x' \vee y'$, and we conclude that $x' = x$ and $y' = y$.

LEMMA 5.5. *Let L be planar and let $\langle p_1, \{x_1, y_1\}\rangle$, $\langle p_2, \{x_2, y_2\}\rangle$ be minimal pairs with $x_1 \lambda y_1$, $y_1 < p_1$, and $p_1 = y_2$. Then either $x_2 < x_1$ or $x_1 \lambda x_2$.*

Proof. Let $x_2 \lambda x_1$ or $x_2 \geqq x_1$. Then $x_2 \vee y_2 = x_2 \vee y_1$ although $\langle p_2, \{x_2, y_2\}\rangle$ is a minimal pair.

Now, let L have breadth two and satisfy $(\mathbf{SD}_\vee)$ as well. We proceed to show that there are, in this case, just two kinds of minimal pairs, as illustrated in Figure 3. Let $\langle p, \{x, y\}\rangle$ be a minimal pair. If $x < p$, this is Figure 3(a) (except that p may equal $x \vee y$). If $y < p$ a similar diagram results. Suppose, then, that x, y, and p form an antichain. Since $x \vee y \geqq p$ we have $x \vee y = x \vee y \vee p$. Suppose that $x \vee p \geqq y$; then $x \vee p = x \vee y \vee p = x \vee y$, so by $(\mathbf{SD}_\vee)$ $x \vee (p \wedge y) = x \vee y \geqq p$. But since $p \wedge y < y$ this contradicts the assumption that $\langle p, \{x, y\}\rangle$ is a minimal pair. Hence $x \vee p \ngeqq y$ and by symmetry $y \vee p \ngeqq x$. Thus we arrive at Figure 3(b) (of course, p need not equal

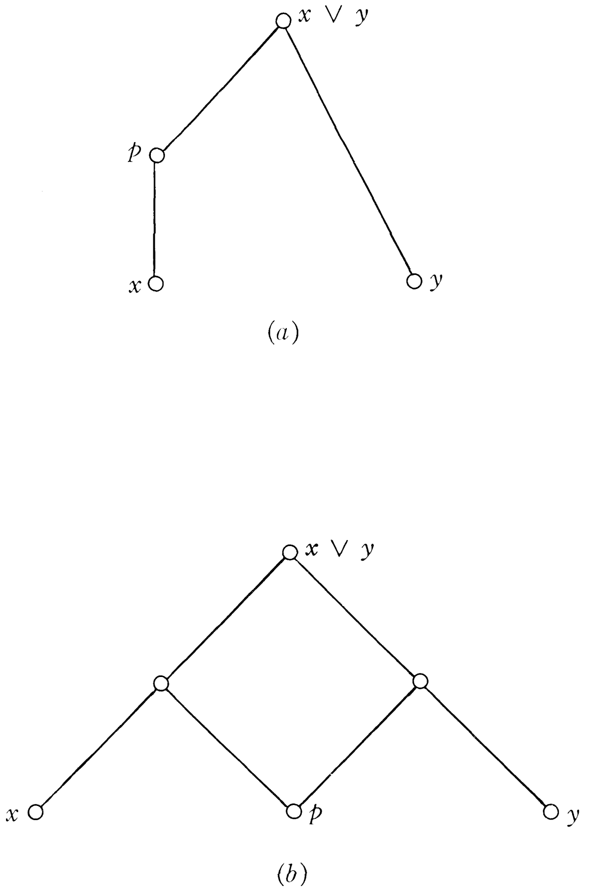

FIGURE 3. Two kinds of minimal pairs.

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the meet of $x \vee p$ and $y \vee p$). Minimal pairs corresponding to Figures 3(a) and 3(b) will be said to be of *type (a)* and *type (b)*, respectively. It is important to note that the minimal pairs constructed in Lemmas 5.3 and 5.4 are of type (a).

We have shown that in a lattice of breadth two and satisfying ($\mathbf{SD}_\vee$), every minimal pair is of type (a) or type (b). Indeed, even minimal pairs of type (b), as we shall shortly see, may be "replaced" by minimal pairs of type (a).

PROPOSITION 5.6. *Let L be a finite semidistributive lattice of breadth two and satisfying* ($\mathbf{W}$). *Let $\langle p, \{x, y\} \rangle$ be a minimal pair of type* (*b*) *with $p \in J(L)$. Then either*

(i) *there is $y' \in L$ such that $y' < p$ and $\langle p, \{x, y'\} \rangle$ is a minimal pair of type* (*a*), *or*

(ii) *there are elements x', x'' of L such that $x'' < x' < p$, and $\langle p, \{x', y\} \rangle$, $\langle x', \{x, x''\} \rangle$ are minimal pairs of type* (*a*).

Remarks. In this proposition the pair $\{x, y\}$ is to be read as an *ordered* pair, so the minimal pairs $\langle p, \{x, y\} \rangle$ and $\langle p, \{y, x\} \rangle$ are different. For example, the lattice of Figure 4, with the minimal pair $\langle p, \{x, y\} \rangle$ as indicated, satisfies (ii) but not (i).

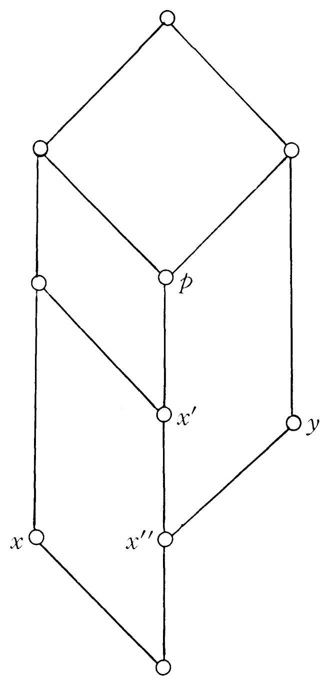

FIGURE 4

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The proof of Proposition 5.7 is comparatively easy under the additional hypothesis that L is planar. Indeed, let L be a planar lattice satisfying the hypotheses of Proposition 5.7. By Corollary 4.4, either $x\lambda p\lambda y$ or $y\lambda p\lambda x$, say the former. Since $p \not\leqq x_* \vee y$, Corollary 4.4 implies that $x_* < p$; symmetrically, $y_* < p$. Now, either $p \leqq p_* \vee x$ or $p \leqq p_* \vee y$. If $p \leqq p_* \vee x$ we readily obtain a minimal pair $\langle p, \{x, y'\}\rangle$ with $y' < p$, whence (i) holds. Otherwise, $p \leqq p_* \vee y$ and we obtain a minimal pair $\langle p, \{x', y\}\rangle$ with $x' < p$. If $x_*\lambda x'$ then $x_*\lambda x'\lambda y$ and $p < x' \vee y \leqq x_* \vee y$, which is impossible; hence, $x_* < x'$. According to the minimality of x', $x'_* \vee y \not\geqq x'$, where $x'_* \prec x'$. By Lemma 2.8(ii), $x'_* \vee x > x'$. Finally, we choose x'' minimal so that $x'' \leqq x'_*$ and $x'' \vee x > x'$, whence $\langle x', \{x, x''\}\rangle$ is a minimal pair of type (a).

Proof. Let $x_* \prec x$ and let us suppose that $p \not> x_*$. If $x_* \not< p \vee y$ then $p \leqq x_* \vee p \vee y = x_* \vee y$ by Lemma 2.8(i); hence, $x_* < p \vee y$. Now, let $a = (x \vee p) \wedge (x_* \vee y)$ and $b = (x \vee p) \wedge (y \vee p)$. Then $x_* \leqq a \leqq b$. As $\langle p, \{x, y\}\rangle$ is a minimal pair and $p \not\geqq x_*$, it follows that $p||a$. In addition, $x \vee a \leqq x \vee b = x \vee p$. If $x \vee a = x \vee p$ then, according to $(\mathbf{SD}_\vee)$, $x \vee p = x \vee (a \wedge p)$. From Lemma 5.3 there exists $y' \leqq a \wedge p$ such that $\langle p, \{x, y'\}\rangle$ is a minimal pair of type (a). On the other hand, if $x \vee a < x \vee p$ then $x \vee a \not\geqq p$. Furthermore, $x \wedge b = x_* = x \wedge (x_* \vee y)$ so that from Lemma 2.7(ii) $a = b \wedge (x_* \vee y) > x_*$ and $a||x$. Applying Lemma 2.8(ii) we have that either $a_* \vee x > a$ or $a_* \vee p > a$, where $a_* \prec a$. Either case is a violation of $(\mathbf{W})$. We conclude that $p > x_*$.

Let p_* be the unique lower cover of p. Then $p_* \geqq x_*$. By Lemma 2.8(ii) either $x \vee p_* > p$ or $y \vee p_* > p$. If $x \vee p_* > p$, then by Lemma 5.3 there exists $y' \leqq p_*$ such that $\langle p, \{x, y'\}\rangle$ is a minimal pair of type (a), as claimed. Otherwise, $x \vee p_* \not\geqq p$ while $y \vee p_* > p$. Let $y_* \prec y$. If $y_* \vee p_* > p$ then $x \vee p_*||y_* \vee p_*$, and by Lemma 2.8(i), $p_* < y_* \vee x$; hence, $p < y_* \vee p_* \leqq x \vee y_*$, contradicting the fact that $\langle p, \{x, y\}\rangle$ is a minimal pair. Thus, $y_* \vee p_* \not\geqq p$, and by Lemma 5.3, there exists $x' \leqq p_*$ such that $\langle p, \{x', y\}\rangle$ is a minimal pair of type (a).

Let $x'||x_*$. Set $c = p \wedge (x_* \vee y)$; then $x_* \leqq c < p_*$. Since $x \wedge p = x_* = x \wedge (x_* \vee y)$ we have that $c > x_*$; that is, $c||x$. Also, $c||x'$ since $x' \vee y > p$. As $x \vee x' \leqq x \vee p_*$, $(\mathbf{W})$ implies that $x \vee x' \not\geqq c$. Let $d = (x \vee x') \wedge (x_* \vee y)$; then, by the dual of Lemma 2.8(i), $d < p$ and so $d < c$. We know that $x \vee d \leqq x \vee x'$. If $x \vee d < x \vee x'$ then $x \vee d \not\geqq x'$. Since $(x \vee d) \wedge (x_* \vee y) = d = (x' \vee d) \wedge (x_* \vee y)$ we have that $e = (x \vee d) \wedge (x' \vee d) > d$ and $e \not\leqq x_* \vee y$. Since $d \vee x' \leqq p_*$ we have $e \vee y \leqq p_* \vee y = x' \vee y$. If $e \vee y = x' \vee y$ then $x' \vee y = (e \wedge x') \vee y$ and $e \wedge x' < x'$, which contradicts the minimality of x'. Thus, $e \vee y < x' \vee y$ and $e \vee y \not\geqq x'$. It follows that

$$\{x', x' \vee d, e, e \vee c, c, x \vee x', e \vee y, x, y\} \cong M2$$

(see Figure 5). By virtue of Proposition 3.1 L must be infinite which is a contradiction. It follows that $x \vee d = x \vee x'$ whence, by $(\mathbf{SD}_\vee)$ $x \vee (x' \wedge d) > x'$.

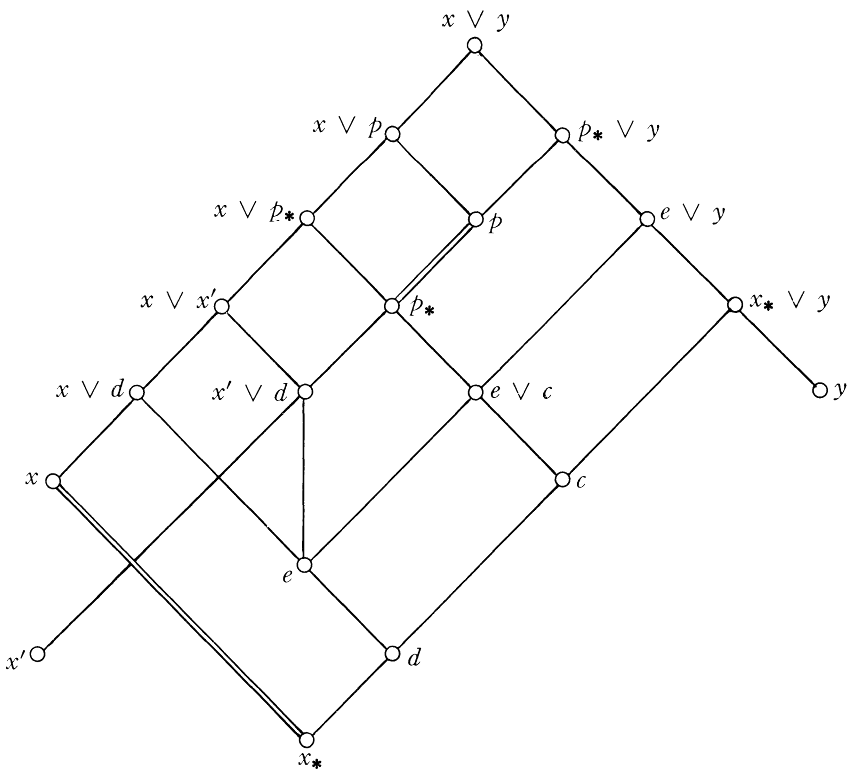

FIGURE 5

Since $x_* \vee (x' \wedge d) \leqq d$, we may apply Lemma 5.3 to find $x'' \leqq x' \wedge d$ such that $\langle x', \{x, x''\}\rangle$ is a minimal pair of type (a).

Finally, let $x' > x_*$. In view of the minimality of x', $x'_* \vee y \ngeq x'$, where $x'_* \prec x'$. Now, Lemma 2.8(ii) implies that $x'_* \vee x > x'$. By Lemma 5.3 there exists $x'' \leqq x'_*$ such that $\langle x', \{x, x''\}\rangle$ is a minimal pair of type (a).

6. Cycles. A finite lattice L is said to satisfy $(\mathbf{T}_\vee)$ if there is a linear ordering $\{x_1, \ldots, x_n\}$ of all the elements of L such that if $\langle x_i, J\rangle$ is a minimal pair and $x_j \in J$, then $i < j$. L is said to satisfy $(\mathbf{T}_\wedge)$ if the dual of L satisfies $(\mathbf{T}_\vee)$.

For us, the significance of these concepts lies in the following result.

THEOREM 6.1. (H. Gaskill and C. R. Platt [**7**]). *A finite lattice is a sublattice of a free lattice if and only if it satisfies* $(\mathbf{T}_\vee)$, $(\mathbf{T}_\wedge)$, *and* $(\mathbf{W})$.

It is clear that a finite lattice L will fail to satisfy $(\mathbf{T}_\vee)$ if and only if there is an integer $n > 1$ and minimal pairs $\langle p_i, J_i\rangle$, $i = 1, 2, \ldots, n$, such that

$p_{i+1} \in J_i$ for $1 \leqq i < n$ and $p_1 \in J_n$; in this case we call $C = \{\langle p_1, J_1\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$ a *cycle* of length n in L, and we write $l(C) = n$.

The next result is essentially a corollary of Proposition 5.6.

PROPOSITION 6.2. *Let L be a semidistributive lattice of breadth two, and let L contain a cycle of minimal pairs. Then L contains a cycle, all of whose minimal pairs are of type* (*a*).

Proof. For each cycle $C = \{\langle p_i, J_i\rangle \mid i = 1, \ldots, n\}$ in L, define

$$\beta(C) = |\{i \mid 1 \leqq i \leqq n, \langle p_i, J_i\rangle \text{ is of type } (b)\}|,$$

and let $\beta_0 = \min\{\beta(C) \mid C \text{ is a cycle in } L\}$. Choose a cycle $C = \{\langle p_i, J_i\rangle \mid i = 1, \ldots, n\}$ such that $\beta(C) = \beta_0$. If $\beta_0 = 0$, we have nothing to do. Suppose $\beta_0 > 0$, and, without loss of generality, assume that $\langle p_1, J_1\rangle$ is a minimal pair of type (b). Letting $J_1 = \{x_1, y_1\}$ and $p_2 = x_1$, we apply Proposition 5.6 with $x = x_1$, $y = y_1$, and $p = p_1$. If (i) holds, then

$$C_1 = \{\langle p_1, \{x_1, y'\}\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in L with $\beta(C_1) < \beta(C) = \beta_0$, contrary to assumption. Similarly, if (ii) holds, then

$$C_2 = \{\langle p_1, \{x', y_1\}\rangle, \langle x', \{x_1, x''\}\rangle, \langle p_2, J_2\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in L with $\beta(C_2) < \beta(C) = \beta_0$, again contrary to assumption.

Remark. Proposition 6.2 has also been proved by B. Jónsson and J. B. Nation [**10**, Corollary 6.4], their approach being quite different from ours. Immediate use will be made of this result, but we will eventually require the full force of Proposition 5.6.

We are ready to state our main theorem and begin its proof.

THEOREM 6.3. *A planar lattice L is a sublattice of a free lattice if and only if L is semidistributive and satisfies* (**W**).

Proof. The "only if" direction, of course, is well-known.

The proof of the converse will be by induction on the order of L. Let L be a planar semidistributive lattice satisfying (**W**); by Theorem 6.1 it is enough to show L satisfies ($\mathbf{T}_\vee$) and ($\mathbf{T}_\wedge$). We assume we are given a fixed planar embedding of L, and any ensuing references to λ or other concepts associated with planarity will be with respect to this planar embedding.

Let a be the minimal meet irreducible on the right boundary of L. By Lemma 4.7, a is doubly irreducible; hence $L\backslash\{a\}$ is a sublattice of L, and thus is a planar semidistributive lattice satisfying (**W**) whose order is smaller than that of L. By the induction hypothesis, $L\backslash\{a\}$ satisfies Theorem 6.3.

Suppose L violates ($\mathbf{T}_\vee$) or ($\mathbf{T}_\wedge$); by dualizing if necessary, we may assume that L violates ($\mathbf{T}_\vee$). (Recently, A. Day has shown that for a finite semidistributive lattice ($\mathbf{T}_\vee$) is equivalent to ($\mathbf{T}_\wedge$) (Can. J. Math. (1978)).) Thus L

contains a cycle $\{\langle p_i, J_i\rangle \mid i = 1, 2, \ldots, n\}$ of minimal pairs, and from Proposition 6.2 we may assume that every $\langle p_i, J_i\rangle$ is of type (a). Clearly $a \in \bigcup_{i=1}^n J_i$, for otherwise $\{\langle p_i, J_i\rangle \mid i = 1, 2, \ldots, n\}$ will be a cycle in $L\backslash\{a\}$, contradicting the induction hypothesis.

Aside. By definition, $p_{i+1} \in J_i$ for $1 \leqq i \leqq n - 1$, and $p_1 \in J_n$. For convenience, we will simply write that $p_{i+1} \in J_i$ for $1 \leqq i \leqq n$; that is, the subscripts are to be read modulo n. A similar convention will be adopted, usually without comment, throughout this paper.

For each $i \in \{1, 2, \ldots, n\}$, define

$$p_i' = \begin{cases} p_i & \text{if } p_i \neq a \\ a^* & \text{if } p_i = a \end{cases}$$

and

$$J_i' = \begin{cases} J_i & \text{if } a \notin J_i \\ (J_i\backslash\{a\}) \cup \{a^*\} & \text{if } a \in J_i \end{cases}.$$

We first claim that $p_{i+1}' \in J_i'$ for each $i \in \{1, 2, \ldots, n\}$. If $p_{i+1} = a$, then since $p_{i+1} \in J_i$ we have $p_{i+1}' = a^* \in J_i'$. Suppose $p_{i+1} \neq a$; then clearly $p_{i+1} \notin J_i\backslash J_i'$, and since $p_{i+1}' = p_{i+1} \in J_i$ we conclude $p_{i+1}' \in J_i'$, as claimed.

Notice also that $\bigcup_{i=1}^n J_i' \subseteq L\backslash\{a\}$; therefore, if each $\langle p_i', J_i'\rangle$ were a minimal pair in $L\backslash\{a\}$, $\{\langle p_i', J_i'\rangle \mid i = 1, 2, \ldots, n\}$ would be a cycle of minimal pairs in $L\backslash\{a\}$, contrary to the induction hypothesis. Consequently we choose k such that $\langle p_k', J_k'\rangle$ is not a minimal pair in $L\backslash\{a\}$. In particular, $\langle p_k', J_k'\rangle \neq \langle p_k, J_k\rangle$, and hence $a = p_k$ or $a \in J_k$. If $a = p_k$, then $a \notin J_k$, and $a \leqq \bigvee J_k$; but since a is doubly irreducible and $a \prec a^* = p_k'$, it follows that $p_k' \leqq \bigvee J_k = \bigvee J_k'$, and from Lemma 5.1 (v) $\langle p_k', J_k'\rangle$ is a minimal pair in $L\backslash\{a\}$, contrary to assumption. Therefore let $a \in J_k$, and let $J_k = \{x_k, y_k\}$ where $y_k < p_k$.

Suppose $a = y_k$. Then $p_k \geqq a^*$, and since $p_k \in J_{k-1}$, Lemma 5.1 (iv) implies that $p_k > a^*$. We have $x_k \vee a^* = x_k \vee y_k > p_k$; also, if $b \leqq x_k$ is such that $b \vee a^* > p_k$, then since $b \vee a^* = b \vee y_k$ and $\langle p_k, J_k\rangle$ is a minimal pair, $b = x_k$. Since $\langle p_k', J_k'\rangle = \langle p_k, \{a^*, x_k\}\rangle$ is not a minimal pair in $L\backslash\{a\}$, we can find $c < a^*$, $c \neq a$, such that $x_k \vee c > p_k$. But now $x_k \vee c = x_k \vee a^* = x_k \vee a$, and by $(\mathbf{SD}_\vee)$ we have $x_k \vee a = x_k \vee (c \wedge a)$, contradicting the fact that $\langle p_k, J_k\rangle$ is a minimal pair. Hence $a = x_k$.

Let $\mathscr{C}$ denote the set of all cycles in L whose minimal pairs are all of type (a). For each $C \in \mathscr{C}$ we have seen that we may choose a minimal pair $\langle p_C, J_C\rangle$ of C such that $\langle p_C', J_C'\rangle$ (defined above) is not a minimal pair in $L\backslash\{a\}$; furthermore, $a \in J_C$ and $a \| p_C$. For $C = \{\langle p_i, J_i\rangle \mid i = 1, 2, \ldots, n\} \in \mathscr{C}$ we define

$$\alpha(C) = |\{i | a \in J_i, 1 \leqq i \leqq n\}|,$$

and we set $\alpha_0 = \min\{\alpha(C) | C \in \mathscr{C}\}$.

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We now show that there exists a cycle $\bar{C} \in \mathscr{C}$ and a choice for $\langle p_{\bar{C}}, J_{\bar{C}}\rangle$ such that $\alpha(\bar{C}) = \alpha_0$ and $a^* > p_C$. Choose $C = \{\langle p_i, J_i\rangle \mid i = 1, 2, \ldots, n\} \in \mathscr{C}$ (where $\langle p_C, J_C\rangle = \langle p_k, J_k\rangle$, say) such that $\alpha(C) = \alpha_0$, and assume that $a^* \not> p_k$. Let $J_k = \{x_k, y_k\}$ where $y_k < p_k$ and $x_k = a$. If $u \leqq y_k$ is such that $u \vee a^* > p_k$, then since $u \vee a^* = u \vee x_k$ and $\langle p_k, J_k\rangle$ is a minimal pair, $u = y_k$. Since $\langle p_k', J_k'\rangle = \langle p_k, \{y_k, a^*\}\rangle$ is not a minimal pair in $L\backslash\{a\}$, by Lemma 5.3 we can find $x' < a^*$, $x' \neq a$, such that $\langle p_k, \{y_k, x'\}\rangle$ is a minimal pair of type (a). If $x'\lambda a_*$ then $y_k \vee a_* \geqq y_k \vee x' > p_k$, contradicting the fact that $\langle p_k, J_k\rangle$ is a minimal pair; since $x'\lambda a$ it follows that $x' > a_*$. Let $x'_* \prec x'$. By the choice of x', $y_k \vee x'_* \ngeqq p_k$, implying $y_k \vee x'_* \ngeqq x'$, and so $(y_k \vee x'_*) \wedge x' = x'_*$. If $x'_* \vee a \ngeqq x'$, then $(x'_* \vee a) \wedge x' = x'_* = (y_k \vee x'_*) \wedge x'$, and also $y_k \vee x'_*\lambda x'\lambda x'_* \vee a$, contradicting Lemma 4.6(ii). Thus $x'_* \vee a > x'$, and by Lemma 5.3 there exists $x'' \leqq x'_*$ such that $\langle x', \{x'', a\}\rangle$ is a minimal pair of type (a). Hence

$$\bar{C} = \{\langle p_1, J_1\rangle, \ldots, \langle p_{k-1}, J_{k-1}\rangle, \langle p_k, \{y_k, x'\}\rangle, \langle x', \{x'', x_k\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_n, J_n\rangle\}$$

is a cycle in $\mathscr{C}$ with $\alpha(\bar{C}) = \alpha_0$, $\langle p_{\bar{C}}, J_{\bar{C}}\rangle = \langle x', \{x'', x_k\}\rangle$, and $a^* > p_{\bar{C}}$, as desired. For future reference, we summarize this result in a lemma.

LEMMA 6.4. *If $C \in \mathscr{C}$ and $\langle p_C, J_C\rangle$ are such that $\alpha(C) = \alpha_0$ and $a^* \not> p_C$, then there exists $\bar{C} \in \mathscr{C}$ and $\langle p_{\bar{C}}, J_{\bar{C}}\rangle$ such that $\alpha(\bar{C}) = \alpha_0$, $l(\bar{C}) = l(C) + 1$, and $a^* > p_{\bar{C}}$.*

We are now able to choose a specific cycle in L to be studied throughout the remainder of the proof. Let

$$\overline{\mathscr{C}} = \{C \in \mathscr{C} \mid \alpha(C) = \alpha_0 \text{ and there is a choice of } \langle p_C, J_C\rangle \text{ such that } a^* > p_C\}$$

and set $l_0 = \min\{l(C) \mid C \in \overline{\mathscr{C}}\}$. Choose $C_0 = \{\langle p_i, J_i\rangle \mid i = 1, 2, \ldots, l_0\} \in \overline{\mathscr{C}}$, where $l(C_0) = l_0$, and suppose that $\langle p_{C_0}, J_{C_0}\rangle = \langle p_k, J_k\rangle$. Let $J_i = \{x_i, y_i\}$ and $z_i = x_i \vee y_i$ for $i = 1, 2, \ldots, l_0$. Then we have that $y_k < p_k < z_k = a^*$ and $x_k = a$. Furthermore, let $y_{k*} \prec y_k$; then if $y_{k*} \| x_k$, we see that $y_{k*} \vee x_k = a^* > p_k$, contradicting the fact that $\langle p_k, J_k\rangle$ is a minimal pair. Therefore $y_{k*} < x_k = a$, which implies $y_{k*} \leqq a_*$. Figure 6 illustrates the minimal pair $\langle p_k, J_k\rangle$, with the understanding that y_{k*} may equal a_*.

We continue our explorations by locating p_{k+1} which, by assumption, is in $J_k = \{x_k, y_k\}$. Our goal now is to prove that $p_{k+1} = x_k$.

Suppose $p_{k+1} = y_k$. Letting $x_{k+1} < p_{k+1}$, we have $x_{k+1} \leqq y_{k*}$ and hence $y_{k+1}\lambda x_{k+1}$, since x_{k+1} is on the right boundary by Lemma 4.7. If $z_{k+1} = x_{k+1} \vee y_{k+1} \geqq p_k$, then by Lemma 5.1(v) $\langle p_k, J_{k+1}\rangle$ is a minimal pair of type (a), and

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_{k-1}, J_{k-1}\rangle, \langle p_k, J_{k+1}\rangle, \langle p_{k+2}, J_{k+2}\rangle, \ldots, \langle p_{l_0}, J_{l_0}\rangle\}$$

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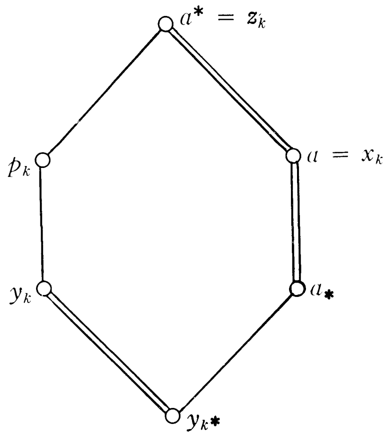

FIGURE 6

is a cycle in $\mathscr{C}$ for which $\alpha(C) < \alpha(C_0) = \alpha_0$, an impossibility. Also, since $z_k = a^*$, Lemma 4.5 shows that $y_{k+1} \nleqq a^*$. Hence $z_{k+1}||p_k$ and $z_{k+1}||a^*$. Since $y_{k+1}\lambda x_{k+1}$, we deduce from Lemma 4.2 that $y_{k+1}\lambda p_k$ and hence $z_{k+1}\lambda p_k$ and $z_{k+1}\lambda a^*$. Consequently $z_{k+1} < y_{k+1} \vee a$.

We claim that $x_{k+1} = y_{k*}$. Otherwise, $x_{k+1} < y_{k*}$, and (**W**) implies that $z_{k+1} = x_{k+1} \vee y_{k+1} \ngeqq p_k \wedge a$. Also, x_{k+1} is meet reducible by the choice of a, and hence there exists $u > x_{k+1}$ such that $u\lambda a_*$. Since $y_k \in J(L)$ we have $u\lambda y_k$, and by Lemma 4.5 $u||a^*$. However, $y_{k+1}\lambda x_{k+1}$ implies $y_{k+1}\lambda u\lambda y_k$, and so by Lemma 4.6(i) $y_{k+1} \vee u = z_{k+1} = y_{k+1} \vee y_k$. By Lemma 2.7(ii), $u \vee y_k < z_{k+1}$. Therefore, as in Figure 7,

$$\{u, u \vee y_k, y_k, p_k, p_k \wedge a, z_{k+1}, a^*, y_{k+1}, a\} \cong M2.$$

By Proposition 3.1 this is a contradiction, and hence $x_{k+1} = y_{k*}$.

Now let $y_{k+1*} \prec y_{k+1}$, and suppose that $y_{k+1*}\lambda x_{k+1}$. From Lemma 4.6(i), $y_{k+1*} \vee x_{k+1} \geqq p_{k+1}$, contradicting the fact that $\langle p_{k+1}, J_{k+1}\rangle$ is a minimal pair. Thus $y_{k+1*} < x_{k+1}$.

We use the same arguments to investigate $\langle p_{k+2}, J_{k+2}\rangle$. Suppose $x_{k+1} = p_{k+2}$. If $z_{k+2} \geqq p_{k+1}$, then by Lemma 5.1(v) $\langle p_{k+1}, J_{k+2}\rangle$ is a minimal pair of type (a), and

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_k, J_k\rangle, \langle p_{k+1}, J_{k+2}\rangle, \langle p_{k+3}, J_{k+3}\rangle, \ldots, \langle p_{l_0}, J_{l_0}\rangle\}$$

is a cycle in $\overline{\mathscr{C}}$ for which $l(C) < l(C_0) = l_0$, an impossibility. Therefore $z_{k+2} \ngeqq p_{k+1}$, and hence $z_{k+2} \ngeqq a$. But now $z_{k+2} = x_{k+2} \vee y_{k+2} \geqq p_{k+2} = x_{k+1} = p_{k+1} \wedge a$, which is a violation of (**W**). Consequently we must have $p_{k+2} = y_{k+1}$, and we may assume $x_{k+2} \leqq y_{k+1*}$ and $y_{k+2}\lambda y_{k+1}$. By Lemma 4.5, $y_{k+2}||y_{k+1} \vee a$, and $z_{k+2} \ngeqq p_{k+1}$ as before. Suppose that $x_{k+2} < y_{k+1*}$; then there

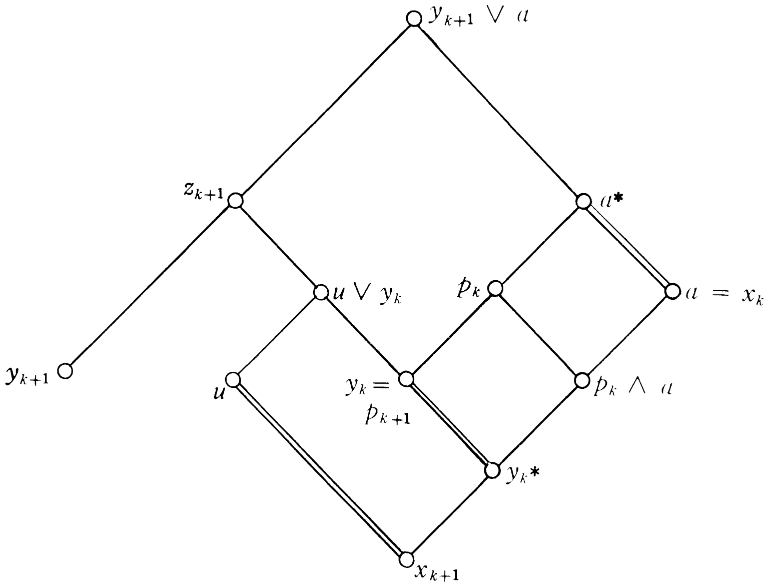

FIGURE 7

exists $u' > x_{k+2}$ such that $u'\lambda a_*$. It follows that $y_{k+2}\lambda u'\lambda y_{k+1}$ and $u' \vee y_{k+1} < z_{k+2}$. Also $u' \not\leqq y_{k+1} \vee a$, and therefore

$$\{u', u' \vee y_{k+1}, y_{k+1}, z_{k+1}, x_{k+1}, z_{k+2}, y_{k+1} \vee a, y_{k+2}, a\} \cong M2$$

which is impossible. Hence $x_{k+2} = y_{k+1*}$. Letting $y_{k+2*} \prec y_{k+2}$, we see that $y_{k+2*} < x_{k+2}$, since $\langle p_{k+2}, J_{k+2}\rangle$ is a minimal pair.

If $p_{k+3} = x_{k+2}$ then, as before, $z_{k+3} \not\geqq p_{k+2}$ and $z_{k+3} \not\geqq p_{k+1}$. But $z_{k+3} = x_{k+3} \vee y_{k+3} \geqq p_{k+3} = x_{k+2} = p_{k+2} \wedge p_{k+1}$ is a violation of (**W**); hence $p_{k+3} = y_{k+2}$. By Lemma 4.6(i), $y_{k+2} \vee x_{k+1} > p_{k+1}$. By Lemma 5.3, there exists $x' \leqq x_{k+1}$ such that $\langle p_{k+1}, \{y_{k+2}, x'\}\rangle$ is a minimal pair of type (a). Thus

$$C = \{\langle p_1, J_1\rangle, \ldots, \langle p_k, J_k\rangle, \langle p_{k+1}, \{y_{k+2}, x'\}\rangle, \langle p_{k+3}, J_{k+3}\rangle, \ldots, \langle p_{l_0}, J_{l_0}\rangle\}$$

is a cycle in $\overline{\mathscr{C}}$ with $l(C) < l(C_0) = l_0$, which is impossible.

We conclude that $p_{k+1} = x_k = a$, as claimed. An important consequence is:

LEMMA 6.5. (i) $a \in J_i$ *implies that* $i = k$; *in other words*, $\alpha_0 = 1$.
(ii) y_k *does not equal* p_t *for any* $t \in \{1, \ldots, l_0\}$.

Proof. (i) Suppose $a \in J_t$, $t \neq k$; then

$$C = \{\langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_t, J_t\rangle\}$$

is a cycle in $\overline{\mathscr{C}}$ with $\alpha(C) < \alpha(C_0) = \alpha_0$, which is impossible.

(ii) Suppose that $y_k = p_t$ for some t; then $t \neq k + 1$, and so

$$C = \{\langle p_t, J_t\rangle, \ldots, \langle p_k, J_k\rangle\}$$

is a cycle in $\mathscr{C}$ with $l(C) < l_0$, which is impossible.

The following two lemmas will be useful in asserting the nonexistence of certain minimal pairs, and will be applied many times in conjunction with Lemma 5.3.

LEMMA 6.6. *Let* $1 \leqq t \leqq l_0$ *and let* $x' \in L$ *be such that* $\langle p_t, \{a, x'\}\rangle$ *is a minimal pair of type* (*a*). *Then* $t = k$ *or* $t = k - 1$ (*modulo* l_0).

Proof. Suppose $t \neq k - 1$ or k and let

$$C = \{\langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle, \langle p_t, \{a, x'\}\rangle\}.$$

Then $C \in \mathscr{C}$ with $\alpha(C) = \alpha_0 = 1$, $l(C) < l_0 - 1$. It follows that $C \notin \overline{\mathscr{C}}$; that is, $a^* \nleqq p_t$. But from Lemma 6.4 we can construct $\bar{C} \in \overline{\mathscr{C}}$ such that $l(\bar{C}) = l(C) + 1 < l_0$; since this is impossible, we are done.

LEMMA 6.7. *Let* $1 \leqq t \leqq l_0$ *and let* $z \in L$ *be such that* $\langle p_t, \{z, a\}\rangle$ *is a minimal pair. Then* $t = k$, $k - 1$, *or* $k - 2$ (*modulo* l_0).

Proof. If $\langle p_t, \{z, a\}\rangle$ is of type (a), we are done by Lemma 6.6. Therefore let $\langle p_t, \{z, a\}\rangle$ be of type (b), and let

$$C = \{\langle p_t, \{z, a\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}.$$

Assume that $t \neq k$, $k - 1$, or $k - 2$, and apply Proposition 5.6 with $x = a$, $y = z$, and $p = p_t$. If (i) holds, then there exists $y' \in L$ such that $\langle p_t, \{a, y'\}\rangle$ is a minimal pair of type (a), and

$$C_1 = \{\langle p_t, \{a, y'\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}$$

is a cycle in $\mathscr{C}$ such that $\alpha(C_1) = 1$ and $l(C_1) < l_0 - 2$. On the other hand, if (ii) holds then there exist $x', x'' \in L$ such that $\langle p_t, \{x', z\}\rangle$ and $\langle x', \{a, x''\}\rangle$ are minimal pairs of type (a), and

$$C_2 = \{\langle p_t, \{x', z\}\rangle, \langle x', \{a, x''\}\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{t-1}, J_{t-1}\rangle\}$$

is a cycle in $\mathscr{C}$ such that $\alpha(C_2) = 1$ and $l(C_2) < l_0 - 1$. In either case, Lemma 6.4 again implies that there exists $\bar{C} \in \overline{\mathscr{C}}$ such that $l(\bar{C}) < l_0$, which is impossible.

7. The case $a_* \nleqq p_k$.

Before proceeding to the substance of this section, we prove a simple lemma.

LEMMA 7.1. *Let* K *be a lattice satisfying* $(\mathbf{SD}_\wedge)$, *and let* u, v, a, b, *and* c *be elements of* K *such that* $u \prec a < v$, $u < b \prec v$, $a \| b$, *and* $u < c < v$. *Then either* $c \leqq b$ *or* $c \geqq a$.

Proof. If $c \nleqq b$ and $c \ngeqq a$, then $c \vee b = v$ and $c \wedge a = u = b \wedge a$. By $(\mathbf{SD}_\wedge)$, $u = (c \vee b) \wedge a = v \wedge a = a$, a contradiction.

Without loss of generality, we may let $p_k = y_{k-1}$. Assume that $a_* \nleqq p_k$; we will eventually arrive at a contradiction. For convenience, this will be accomplished in two parts.

A. *Claim.* $y_{k-1}\lambda x_{k-1}$.

Suppose that $x_{k-1}\lambda y_{k-1}$; then $x_{k-1}\lambda y_k$ or $x_{k-1} > y_k$. If $x_{k-1} < a^*$, Lemma 4.5 implies that $x_{k-1} > y_k$; but by Lemma 5.1(v) and Lemma 5.4 this would mean that $\langle p_{k-1}, J_k\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \overline{\mathscr{C}}$ with $l(C) < l_0$. This is a contradiction, and therefore $x_{k-1}\lambda a^*$. If $z_{k-1} > a_*$, then by Lemma 4.3 $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-1} \vee a_*$, and by $(\mathbf{SD}_\vee)$ $z_{k-1} = x_{k-1} \vee (y_{k-1} \wedge a_*)$, contradicting the fact that $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair. Hence $z_{k-1}\lambda a_*$ and $z_{k-1}\lambda a^*$ both hold. As a result, $z_{k-1} < x_{k-1} \vee a$ (see Figure 8).

Let $y_{k-1*} \prec y_{k-1}$ and set $u = x_{k-1} \vee y_{k-1*}$. Since $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair, $u \ngeqq y_{k-1}$ and hence $u\lambda y_{k-1}$. Since $\langle p_k, J_k\rangle$ is a minimal pair, $y_k \vee a_* \ngeqq p_k = y_{k-1}$ and hence $y_{k-1}\lambda y_k \vee a_*\lambda a$. Now since $y_{k-1} \vee a = a^* = (y_k \vee a_*) \vee a$, the dual of Lemma 2.7(ii) implies that $y_{k-1} \vee a_* = y_{k-1} \vee (y_k \vee a_*) < a^*$, and so $y_{k-1} \vee a_* \ngeqq a$. Moreover, from Lemma 4.6 (ii) $y_{k-1} \wedge (y_k \vee a_*) \neq y_{k-1*}$. It follows that $y_k < y_{k-1*}$. Now, letting $v = z_{k-1} \wedge a^*$, we can choose z such that $v \prec z \leqq z_{k-1}$. Of course $z \nleqq a^*$. If $z < z_{k-1}$, then by $(\mathbf{W})$ we deduce

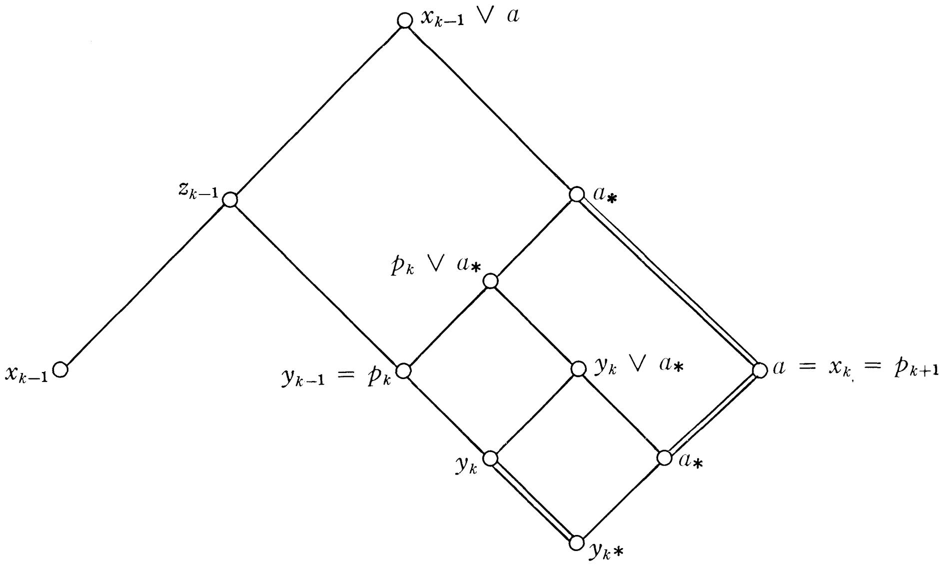

FIGURE 8

$z \wedge u \nleqq a^* = y_k \vee a$; hence

$$\{z \wedge u, z, y_{k-1}, y_{k-1} \vee a_*, a_*, z_{k-1}, a^*, x_{k-1}, a\} \cong M2,$$

which is impossible. Therefore $v \prec z_{k-1}$.

Either $p_{k-1} > x_{k-1}$ or $p_{k-1} > y_{k-1}$. If $y_{k-1} < p_{k-1} < z_{k-1}$ then $y_k < p_{k-1} < a^* = z_k$; but then $\langle p_{k-1}, J_k\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \overline{\mathscr{C}}$ with $l(C) < l_0$. Hence we must have $x_{k-1} < p_{k-1} < z_{k-1}$.

Assume that $x_{k-1} \neq u$. Since $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair, $p_{k-1} \nleq u$. If $p_{k-1} || u$ then $p_{k-1} \ngeq y_{k-1*}$, and

$$\{x_{k-1}, u, y_{k-1*}, y_{k-1} \vee a_*, a_*, p_{k-1} \vee u, a^*, p_{k-1}, a\} \cong M2,$$

which is impossible. On the other hand, if $p_{k-1} > u$ then (recalling that $p_{k-1} \in J(L)$) we have

$$\{x_{k-1}, u, y_{k-1*}, y_{k-1} \vee a_*, a_*, p_{k-1}, a^*, a\} \cong M1,$$

also an impossibility. Thus $x_{k-1} = u$, that is, $x_{k-1} > y_{k-1*}$.

Let $x_{k-1*} \prec x_{k-1}$; either $x_{k-1*} \lambda v$ or $x_{k-1*} < v$. If $x_{k-1*} \lambda v$ then, since $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair, Lemma 4.2 implies that $x_{k-1} \lambda x_{k-1*} \vee y_{k-1} \lambda v$; but now $(x_{k-1*} \vee y_{k-1}) \vee v = z_{k-1} = x_{k-1} \vee (x_{k-1*} \vee y_{k-1})$, contradicting the dual of Lemma 4.6(ii). Hence $x_{k-1*} < v$.

Since $p_{k-1} \in J_{k-2}$, we may let $p_{k-1} = y_{k-2}$. Suppose that $x_{k-2} \lambda y_{k-2}$; we proceed to show this is impossible.

First assume that $x_{k-2} < z_{k-1}$. Either $x_{k-2} \lambda x_{k-1}$ or $x_{k-2} > x_{k-1}$; Lemma 4.5 shows that $x_{k-2} > x_{k-1}$. But now $\langle p_{k-2}, J_{k-1}\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \mathscr{C}$ with $l(C) < l_0$, which is impossible. Therefore $x_{k-2} \lambda z_{k-1}$. Since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair, $x_{k-2} \vee x_{k-1} \ngeq y_{k-2}$, and so $x_{k-2} \vee x_{k-1} \ngeq y_{k-1}$. Hence $y_{k-1} \wedge y_{k-2} = y_{k-1*} = y_{k-1} \wedge (x_{k-2} \vee x_{k-1})$, and by $(\mathbf{SD}_\wedge)$ we get that $z_{k-2} = (x_{k-2} \vee x_{k-1}) \vee y_{k-2} \ngeq y_{k-1}$.

Now since $y_{k-1} \vee a_* < a^*$, $(\mathbf{W})$ implies that $y_{k-1*} \vee a_* \ngeq v = z_{k-1} \wedge a^*$. Also, since $x_{k-1} \lambda y_{k-1}$ and $y_{k-1*} \lambda a_*$, the reflection of Lemma 4.6(i) shows that $y_{k-1*} \vee a_* = y_{k-1} \vee a_*$. Consequently, $y_{k-1} < v$, and

$$\{z_{k-2}, y_{k-2}, z_{k-1}, v, a^*, x_{k-1}, y_{k-1}, x_{k-2} \vee x_{k-1}, y_{k-1} \vee a_*\} \cong (M2)^d.$$

Thus we conclude that $y_{k-2} \lambda x_{k-2}$.

Since $y_{k-1*} < x_{k-1} < y_{k-2}$, it is clear that $x_{k-2} \nless y_{k-1}$. From Lemma 5.5 we have that $x_{k-2} \lambda y_{k-1}$, and so $z_{k-2} \leqq z_{k-1}$ by Lemma 4.3. If $x_{k-2} > x_{k-1}$, then $\langle p_{k-2}, J_{k-1}\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \mathscr{C}$ with $l(C) < l_0$, which is impossible. Therefore, since $y_{k-2} \lambda x_{k-2}$, we have $x_{k-1} \lambda x_{k-2} \lambda y_{k-1}$. If $x_{k-2} \lambda v$, then by Lemma 4.3 $x_{k-1*} = x_{k-1} \wedge v < x_{k-2} < x_{k-1} \vee v$, contradicting Lemma 7.1. Thus $x_{k-2} < v$. Also, $y_{k-2} \lambda x_{k-1} \vee x_{k-2}$ since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair, and $(x_{k-1} \vee x_{k-2}) \vee v = z_{k-1} = y_{k-2} \vee v$; Lemma 2.7(ii) implies that $z_{k-2} = (x_{k-1} \vee x_{k-2}) \vee y_{k-2} < z_{k-1}$. We now have

$$\{x_{k-1}, x_{k-1} \vee x_{k-2}, x_{k-2}, v, y_{k-1}, z_{k-2}, a^*, y_{k-2}, a\} \cong M2.$$

This contradiction establishes that $y_{k-1} \lambda x_{k-1}$, as advertised.

B. *Claim. For all $j < k$, $y_{k*} < p_j < y_k \vee a_*$ and $y_{k*} < x_j$, $y_j < p_k \vee a_*$.*

First notice that $y_{k*} < x_{k-1} < a^*$ and $y_{k*} < p_{k-1} < z_{k-1} \leqq a^*$, since y_{k*} and a^* are both on the right boundary. Also, it is easy to see that neither x_{k-1} nor p_{k-1} can equal a. By Lemma 7.1, either $p_{k-1} > y_k$ or $p_{k-1} \leqq a_*$. If $p_{k-1} \nless y_k$

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$\vee a_*$, then $\langle p_{k-1}, J_k\rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \overline{\mathscr{C}}$ with $l(C) < l_0$, which is impossible. Hence $p_{k-1} < y_k \vee a_*$, and it follows that $x_{k-1} < p_{k-1}$. We have shown that the claim is true for $j = k - 1$.

Proceeding by induction, we assume that $y_{k*} < p_j < y_k \vee a_*$ and $y_{k*} < x_j$, $y_j < p_k \vee a_*$ for all j such that $i \leqq j < k$, and we consider $\langle p_{i-1}, J_{i-1}\rangle$. Without loss of generality, let $x_i < p_i = y_{i-1}$ (this means that we get $y_{k*} < y_{i-1} < p_k \vee a_*$ *gratis*). Notice that, since x_i, y_i, and p_i are all distinct from a, Lemma 7.1 implies that each of x_i, y_i and p_i are either less than or equal to a_*, or greater than or equal to y_k.

Assume that $x_{i-1} < p_k \vee a_*$. Certainly $x_{i-1} \nleqq y_{k*}$, since $y_{k*} < y_{i-1}$. If $x_{i-1}\lambda y_{k*}$ then $x_{i-1}\lambda y_k$, and from Lemma 4.5 this is impossible. y_{k*} is on the right boundary, so $x_{i-1} > y_{k*}$. Now p_{i-1} is greater than one of x_{i-1} and y_{i-1}, and in either case $y_{k*} < p_{i-1} < z_{i-1} \leqq p_k \vee a_*$ and $p_{i-1} \neq y_k$. If $p_{i-1} \nless y_k \vee a_*$, then $p_{i-1} \ngeqq a$, and by Lemma 7.1 we have that $p_{i-1} > y_k$. Thus $\langle p_{i-1}, J_k\rangle$ is a minimal pair of type (a), and neither y_{i-1} nor x_{i-1} equals $a = x_k$, showing that $i - 1 \neq k$ (modulo l_0). Hence

$$C = \{\langle p_{i-1}, J_k\rangle, \langle p_{k+1}, J_{k+1}\rangle, \ldots, \langle p_{i-2}, J_{i-2}\rangle\}$$

is a cycle in $\overline{\mathscr{C}}$ with $l(C) < l_0$, which is impossible. Therefore $p_{i-1} < y_k \vee a_*$, establishing the claim for $j = i - 1$.

We now assume that $x_{i-1} \nless p_k \vee a_*$. Since $y_{i-1} < y_k \vee a_* < p_k \vee a_*$, we have $x_{i-1} \ngeqq p_k \vee a_*$. If $p_k \vee a_* \lambda x_{i-1}$, then Lemma 4.3 and $x_{i-1}\lambda a$ imply $y_{k*} < a_* = (p_k \vee a_*) \wedge a < x_{i-1} < (p_k \vee a_*) \vee a = a^*$. Hence from Lemma 7.1 we infer $x_{i-1} \geqq y_k$, so $x_{i-1} \geqq y_k \vee a_* > y_{i-1}$, a contradiction. Consequently we let $x_{i-1}\lambda p_k \vee a_*$. It follows that $x_{i-1}\lambda y_{i-1} = p_i > x_i$, that either $x_{i-1}\lambda x_i$ or $x_{i-1} > x_i$, and that either $x_{i-1}\lambda y_i$ or $x_{i-1} > y_i$. From Lemma 5.5 we infer $x_i\lambda y_i$, and so $x_{i-1}\lambda y_{i-1}\lambda y_i$. Also $y_i < y_k \vee a_*$, for otherwise $y_k \vee a_*\lambda y_i$ which implies $(y_k \vee a_*) \wedge y_i = a_* = y_i \wedge a$, contradicting Lemma 4.6(ii). If $x_i \leqq a_*$ then $y_i \leqq a_*$ also, contradicting the fact that $x_i || y_i$. Hence by Lemma 7.1, $y_k \leqq x_i < p_i = y_{i-1}$. If $z_{i-1} > y_i$, by Corollary 4.4 $z_{i-1} = x_{i-1} \vee y_{i-1} = x_{i-1} \vee y_i$; by $(\mathbf{SD}_\vee)$, $z_{i-1} = x_{i-1} \vee (y_{i-1} \wedge y_i)$, contradicting the fact that $\langle p_{i-1}, J_{i-1}\rangle$ is a minimal pair. Thus $z_{i-1}\lambda y_i$ and $z_{i-1}\lambda a_*$.

Let $y_{i-1*} \prec y_{i-1}$, and set $b = x_{i-1} \vee y_{i-1*}$. Since $\langle p_{i-1}, J_{i-1}\rangle$ is a minimal pair, $b \ngeqq y_{i-1}$ and so $b\lambda y_{i-1}$. We claim that $y_{i-1*} > y_k$. If $y_{i-1*} = y_k$ then $x_i = y_k$, and so from Lemma 6.5(ii) $y_i = p_{i+1} \leqq a_*$. Let $u \succ y_{k*}$ such that $u \leqq a_*$. From the reflection of Lemma 4.6(i), $x_i \vee u \geqq y_{i-1} = p_i$; since $\langle p_i, J_i\rangle$ is a minimal pair, $u = y_i = p_{i+1}$. Assuming $x_{i+1} < p_{i+1}$, we have $x_{i+1} \leqq y_{k*}$. As i is not $k - 1$ (modulo l_0), we have a contradiction to the induction hypothesis. Thus $y_{i-1*} > y_k$, as claimed.

Now let $c = z_{i-1} \wedge (p_k \vee a_*)$; by $(\mathbf{W})$, $c \nleqq y_k \vee a_*$. Also by $(\mathbf{W})$, $c \wedge b \nleqq y_k \vee a_*$. Suppose that $y_i \neq a_*$. By Lemma 7.1, $a_* \prec y_k \vee a_*$, so $y_i \ngeqq a_*$. By $(\mathbf{W})$ again, $z_i = x_i \vee y_i \ngeqq a_* = (p_k \vee a_*) \wedge a$, while $z_i > p_i = y_{i-1}$. Hence

$$\{b \wedge c, c, y_{i-1}, z_i, y_i, z_{i-1}, y_k \vee a_*, x_{i-1}, a_*\} \cong M2.$$

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Therefore we must have $y_i = a_*$. Since $\langle p_i, J_i \rangle$ is a minimal pair and $z_i = y_k \vee a_*$, we have $x_i = y_k$. Let $v \prec a_*$; then $y_{i-1} \lambda v$ or $y_{i-1} > v$. If $y_{i-1} \lambda v$, then $x_i \vee v \ngeq p_i$ since $\langle p_i, J_i \rangle$ is a minimal pair; hence $p_i \lambda x_i \vee v \lambda a_*$. Now $p_i \vee a_* = y_k \vee a_* = (x_i \vee v) \vee a_*$, and from Lemma 2.7(ii) $p_i \vee v = p_i \vee (x_i \vee v) < y_k \vee a_*$. Also, $v < z_{i-1}$ would imply $z_{i-1} = x_{i-1} \vee y_{i-1} = x_{i-1} \vee v$ by Lemma 4.3, and hence $z_{i-1} = x_{i-1} \vee (y_{i-1} \wedge v)$ by $(\mathbf{SD}_\vee)$, contradicting the fact that $\langle p_{i-1}, J_{i-1} \rangle$ is a minimal pair. Hence $z_{i-1} \ngeq v$, and

$$\{b \wedge c, c, y_{i-1}, p_i \vee v, v, z_{i-1}, y_k \vee a_*, x_{i-1}, a_*\} \cong M2.$$

We conclude that $v < y_{i-1}$. From Lemma 6.5(ii), $x_i = y_k \neq p_{i+1}$, and so $a_* = y_i = p_{i+1}$. Letting $x_{i+1} < p_{i+1}$, we get $x_{i+1} < y_{i-1} < c$; also $y_{i+1} \lambda x_{i+1}$, since x_{i+1} is on the right boundary. Certainly $i + 1 \neq k$ (modulo l_0), so by the induction hypothesis $y_{k*} < y_{i+1} < p_k \vee a_*$. It follows that $y_{i+1} < z_{i-1}$, implying $y_{i+1} \leqq c$. But this means $a_* = p_{i+1} < z_{i+1} \leqq c$, a contradiction. We have shown that the claim is true for $j = i - 1$, and the proof of **B** is complete.

Now, by going completely around the cycle C_0, we conclude from **B** that $y_{k*} < p_k < y_k \vee a_*$. This contradiction completes the case $a_* \nleq p_k$, and this section.

8. The case $a_* \leqq p_k$. Gamely continuing with the proof of Theorem 6.3, we may now assume that $a_* \leqq p_k$ in L. First we rapidly show that $x_{k-1} \lambda y_{k-1}$, where $y_{k-1} = p_k$. Suppose $y_{k-1} \lambda x_{k-1}$. Since a_* and a^* are on the right boundary, we know that $a_* < x_{k-1} < a^*$. But $x_{k-1} \neq a$, and hence $x_{k-1} \geqq y_k$ by Lemma 7.1. Thus $y_k \vee a_* < x_{k-1} < a^*$, and so $y_k \vee a_* < p_{k-1} < z_{k-1} \leqq a^* = z_k$. It follows that $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), and we can construct a cycle $C \in \mathscr{C}$ such that $l(C) < l_0$, an impossibility. Therefore $x_{k-1} \lambda y_{k-1}$.

If $x_{k-1} < a^*$, then $x_{k-1} \geqq y_k$ from Lemma 4.5, and $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a). This is impossible, so $x_{k-1} \lambda a^*$. If $z_{k-1} \geqq a^*$, then $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-1} \vee a$ from Lemma 4.3, and hence by $(\mathbf{SD}_\vee)$ $z_{k-1} = x_{k-1} \vee (y_{k-1} \wedge a) = x_{k-1} \vee a_*$, contradicting the fact that $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair. Thus $z_{k-1} \lambda a^*$.

Set $w = x_{k-1} \vee a_*$. Since $\langle p_{k-1}, J_{k-1} \rangle$ is a minimal pair, $w \ngeq p_{k-1}$ and hence $w \ngeq y_{k-1}$. However, we claim that $w > y_k$. If $y_k \ngeq a_*$ then $y_k \lambda a_*$, and since $x_{k-1} \lambda y_k$ or $x_{k-1} > y_k$ we have $w = x_{k-1} \vee a_* > y_k$ by Lemma 4.3. On the other hand, if $y_k > a_*$ then $y_{k*} = a_*$, and Lemma 4.6(i) implies that $w = x_{k-1} \vee a_* > y_k$.

A. In this part we assume that $p_{k-1} > x_{k-1}$ (later we will deal with the other case, $p_{k-1} > y_{k-1}$). If $w \lambda p_{k-1}$ then $a_* < p_{k-1}$, implying that $w = x_{k-1} \vee a_* \leqq p_{k-1}$. It follows that either $p_{k-1} \lambda w$ or $p_{k-1} > w$. In either case, $p_{k-1} \vee y_{k-1} = z_{k-1} = w \vee y_{k-1}$, so $p_{k-1} \vee w < z_{k-1}$ by the dual of Lemma 2.7 (ii). Since $p_{k-1} \in J_{k-2}$, we may assume that $p_{k-1} = y_{k-2}$.

Case (i). Let $y_{k-2} \lambda x_{k-2}$.

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From Lemma 5.5 we know that $x_{k-2} \lambda y_{k-1}$ or $x_{k-2} < y_{k-1}$. In either case $z_{k-2} = x_{k-2} \vee y_{k-2} \leqq z_{k-1}$ follows. Since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $x_{k-1} \vee x_{k-2} < z_{k-2}$ and $y_{k-2} \lambda x_{k-1} \vee x_{k-2} \lambda y_{k-1}$. Since $y_{k-2} \vee y_{k-1} = z_{k-1} = (x_{k-1} \vee x_{k-2}) \vee y_{k-1}$, the dual of Lemma 2.7 (ii) implies that $z_{k-2} = y_{k-2} \vee (x_{k-1} \vee x_{k-2}) < z_{k-1}$.

We claim that $z_{k-2} > a_*$. Otherwise $z_{k-2} \ngeqq a_*$, from which we get $y_{k-2} \lambda w$ and $z_{k-2} \lambda w$. Hence either $p_{k-2} \lambda w$ or $p_{k-2} < w$. If $p_{k-2} \lambda w$ then from Lemma 4.3 $p_{k-2} > y_{k-2} \wedge w \geqq x_{k-1}$, whence $x_{k-1} < p_{k-2} < x_{k-1} \vee a$ and $p_{k-2} \nleqq w = x_{k-1} \vee a_*$. Thus from Lemma 5.3 there exists $x' \leqq x_{k-1}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contrary to Lemma 6.6. Therefore $p_{k-2} < w$. Since $p_{k-2} \vee x_{k-1} \nleqq y_{k-2}$ we have

$$\{z_{k-2}, p_{k-2} \vee x_{k-1}, w, y_k \vee a_*, y_{k-1}, x_{k-1}, a_*, y_{k-2}, a\} \cong (M2)^d.$$

Hence $z_{k-2} > a_*$.

Now either $x_{k-2} \geqq a_*$ or $y_{k-2} > a_*$, since otherwise $x_{k-2} \vee y_{k-2} = z_{k-2} > a_* = z_{k-1} \wedge a$ is a violation of (**W**). Since $y_{k-2} \lambda x_{k-2}$, we have $x_{k-2} \geqq a_*$ in any case, and so $x_{k-1} \vee x_{k-2} \geqq w$. If $p_{k-2} < w$ then $x_{k-1} \vee x_{k-2} > p_{k-2}$, contradicting the fact that $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair; thus $p_{k-2} \nleqq w$. If $p_{k-2} > x_{k-1}$ then by Lemma 5.3 there exists $x' \leqq x_{k-1}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contrary to Lemma 6.6. Therefore $p_{k-2} \ngeqq x_{k-1}$, and so $x_{k-1} \lambda p_{k-2}$, $w \lambda p_{k-2}$, and $p_{k-2} > x_{k-2}$. Since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $y_{k-2} \lambda x_{k-1} \vee x_{k-2} \lambda p_{k-2}$. Since $y_{k-2} \vee (x_{k-1} \vee x_{k-2}) = z_{k-2} = y_{k-2} \vee p_{k-2}$, the dual of Lemma 2.6(ii) implies that $x_{k-1} \vee p_{k-2} = (x_{k-1} \vee x_{k-2}) \vee p_{k-2} < z_{k-2}$, implying $x_{k-1} \vee p_{k-2} \ngeqq y_{k-2}$.

Next we claim that $p_{k-2} \vee a \ngeqq x_{k-1}$. Otherwise, $p_{k-2} \vee a = x_{k-1} \vee a$, and by $(\mathbf{SD}_\vee)$ $p_{k-2} \vee a = (x_{k-1} \wedge p_{k-2}) \vee a$. However, $(x_{k-1} \wedge p_{k-2}) \vee a_* \leqq w$ which means $p_{k-2} \nleqq (x_{k-1} \wedge p_{k-2}) \vee a_*$. Thus by Lemma 5.3 there exists $x' \leqq x_{k-1} \wedge p_{k-2}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contradicting Lemma 6.6.

Therefore $p_{k-2} \vee a \ngeqq x_{k-1}$ and so

$$\{x_{k-1}, x_{k-1} \vee p_{k-2}, p_{k-2}, p_{k-2} \vee y_{k-1}, y_{k-1}, z_{k-2}, p_{k-2} \vee a, y_{k-2}, a\} \cong M2.$$

We conclude that Case (i) is impossible.

For the remaining two cases, recall that either $y_{k-2} > w$ or $y_{k-2} \lambda w$.

Case (ii). Let $x_{k-2} \lambda y_{k-2}$ and $y_{k-2} > w$.

Since $x_{k-2} \lambda y_{k-2} \lambda a$, we have $p_{k-2} < z_{k-2} \leqq x_{k-2} \vee a$. Let $v = x_{k-2} \vee w$; then $p_{k-2} \nleqq v$ and $y_{k-2} \nleqq v$, since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair. If $p_{k-2} > x_{k-2}$, it follows from Lemma 5.3 that there exists $x' \leqq x_{k-2}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a). This contradicts Lemma 6.6, and hence $p_{k-2} > y_{k-2}$.

Suppose that $p_{k-2} \leqq x_{k-1} \vee a$. Then, since $x_{k-1} \vee a_* = w < p_{k-1}$, we may apply Lemma 5.3 again to find $x'' \leqq x_{k-1}$ such that $\langle p_{k-2}, \{x'', a\} \rangle$ is a minimal pair of type (a), which is another contradiction of Lemma 6.6. Therefore

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$p_{k-2}\lambda x_{k-1} \vee a$. Next, suppose that $z_{k-2} \geqq z_{k-1}$. Since $x_{k-2}\lambda y_{k-2}\lambda y_{k-1}$ we have $z_{k-2} = x_{k-2} \vee y_{k-2} = x_{k-2} \vee y_{k-1}$ by Lemma 4.3; but now $z_{k-2} = x_{k-2} \vee (y_{k-2} \wedge y_{k-1})$ by $(\mathbf{SD}_\vee)$, and $y_{k-2} \wedge y_{k-1} < y_{k-2}$, contradicting the fact that $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair. Thus we have shown that $z_{k-2}\lambda x_{k-1} \vee a$ and $z_{k-2}\lambda z_{k-1}$.

If $v \wedge p_{k-2} > x_{k-1}$, then $v \wedge p_{k-2} \nleqq x_{k-1} \vee a$ by $(\mathbf{W})$, and

$$\{v \wedge p_{k-2}, p_{k-2}, y_{k-2}, z_{k-1}, y_{k-1}, z_{k-2}, x_{k-1} \vee a, x_{k-2}, a^*\} \cong M2.$$

Therefore $v \wedge p_{k-2} = x_{k-1}$, implying that $x_{k-1} = w > a_*$. Also, we now have that $v \wedge p_{k-2} = v \wedge (x_{k-1} \vee a)$, and so $p_{k-2} \vee a = p_{k-2} \vee (x_{k-1} \vee a) \ngeqq v$ by $(\mathbf{SD}_\wedge)$.

We may let $p_{k-2} = y_{k-3}$. By Lemma 5.5, either $x_{k-3} < x_{k-2}$ or $x_{k-2}\lambda x_{k-3}$. Since $a_* < y_{k-3}$ and $\langle p_{k-3}, J_{k-3}\rangle$ is a minimal pair, $x_{k-3} \vee a_* \ngeqq p_{k-3}$. If $x_{k-3} < x_{k-2}$ or $x_{k-2}\lambda x_{k-3}\lambda y_{k-3}$, then $p_{k-3} < z_{k-3} < x_{k-3} \vee a$; therefore from Lemma 5.3 there exists $x' \leqq x_{k-3}$ such that $\langle p_{k-3}, \{x', a\}\rangle$ is a minimal pair. This contradicts Lemma 6.7, and we conclude that $y_{k-3}\lambda x_{k-3}$. It follows that $v\lambda x_{k-3}$, and hence $x_{k-2} \vee a_* \ngeqq p_{k-3}$. Now, if $x_{k-3} < x_{k-2} \vee a$ then $p_{k-3} < z_{k-3} \leqq x_{k-2} \vee a$; from Lemma 5.3 there exists $x' \leqq x_{k-2}$ such that $\langle p_{k-3}, \{x', a\}\rangle$ is a minimal pair, contradicting Lemma 6.7 once more. Thus it must be that $x_{k-2} \vee a\lambda x_{k-3}$, and so $x_{k-3} > a^*$. Finally, if $x_{k-2} \vee y_{k-1} = x_{k-2} \vee a$, then by $(\mathbf{SD}_\vee)$ we have $y_{k-1} < x_{k-2} \vee y_{k-1} = x_{k-2} \vee (y_{k-1} \wedge a) = x_{k-2} \vee a_* \leqq v$, a contradiction; thus $x_{k-2} \vee y_{k-1} \ngeqq a$. Now if $x_{k-3} > y_{k-2}$ then

$$\{y_{k-3}, y_{k-3} \vee y_{k-1}, z_{k-1}, y_{k-2} \vee a, a, x_{k-2} \vee y_{k-1}, x_{k-3}, v\} \cong M1,$$

while if $x_{k-3} \ngeqq y_{k-2}$ then

$$\{y_{k-2}, z_{k-1}, y_{k-1}, a^*, a, x_{k-2} \vee y_{k-1}, x_{k-3}, z_{k-2}\} \cong M1,$$

showing that Case (ii) is impossible.

Case (iii). Let $x_{k-2}\lambda y_{k-2}\lambda w$.

Suppose that $x_{k-2} < z_{k-1}$. Then $z_{k-1} = x_{k-1} \vee y_{k-1} = x_{k-2} \vee y_{k-1}$, and $(\mathbf{SD}_\vee)$ implies that $z_{k-1} = (x_{k-1} \wedge x_{k-2}) \vee y_{k-1}$. Since $\langle p_{k-1}, J_{k-1}\rangle$ is a minimal pair, $x_{k-2} > x_{k-1}$. Now, $x_{k-1} < p_{k-2} < z_{k-2} \leqq z_{k-1}$, and $p_{k-2}\lambda p_{k-1}$ or $p_{k-2} > p_{k-1}$; hence from Lemma 5.4 and Lemma 5.1(v) $\langle p_{k-2}, J_{k-1}\rangle$ is a minimal pair of type (a), which is impossible. Therefore $x_{k-2}\lambda z_{k-1}$. If $z_{k-2} > w$ then $z_{k-2} = x_{k-2} \vee y_{k-2} = x_{k-2} \vee w$, and by $(\mathbf{SD}_\vee)$ $z_{k-2} = x_{k-2} \vee (y_{k-2} \wedge w)$. Since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair, this is a contradiction; thus $z_{k-2}\lambda w$. Since $z_{k-2} > x_{k-1}$ and $w = x_{k-1} \vee a_*$, it follows that $z_{k-2}\lambda a_*$. Finally we have that $x_{k-2} \vee x_{k-1}\lambda y_{k-2}$ since $\langle p_{k-2}, J_{k-2}\rangle$ is a minimal pair. Therefore

$$\{z_{k-2}, y_{k-2}, z_{k-1}, y_{k-1}, a^*, x_{k-1}, a_*, x_{k-2} \vee x_{k-1}, a\} \cong (M2)^d.$$

With this contradiction we have succeeded in showing that $p_{k-1} > x_{k-1}$ is impossible.

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B. Finally we assume $p_{k-1} > y_{k-1}$. If $p_{k-1} \leqq a^*$ then $\langle p_{k-1}, J_k \rangle$ is a minimal pair of type (a), which is impossible. Hence $p_{k-1} \lambda a^*$.

Recall that $p_{k-1} = y_{k-2}$, and first assume that $x_{k-2} \lambda y_{k-2}$. By Lemma 5.5, $x_{k-2} < x_{k-1}$ or $x_{k-1} \lambda x_{k-2}$, and so $z_{k-2} \leqq x_{k-1} \vee y_{k-2} = z_{k-1}$. If $p_{k-2} > y_{k-2}$, then $\langle p_{k-2}, J_{k-1} \rangle$ is a minimal pair of type (a), which is impossible; therefore $p_{k-2} > x_{k-2}$. Also, $x_{k-2} \lambda y_{k-2} \lambda a$ implies $p_{k-2} < x_{k-2} \vee a$, but since $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair, $p_{k-2} \nleqq x_{k-2} \vee a_*$. Thus by Lemma 5.3 there exists $x' \leqq x_{k-2}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), contradicting Lemma 6.6. We conclude that $y_{k-2} \lambda x_{k-2}$, and hence $x_{k-2} > a_*$.

We turn our attention briefly to J_{k+1}. Without loss of generality, let $x_{k+1} \leqq a_* \prec p_{k+1}$; then x_{k+1} is on the right boundary, so $y_{k+1} \lambda x_{k+1}$. If either $x_{k-1} \geqq y_{k+1}$ or $x_{k-1} \lambda y_{k+1}$, then $a = p_{k+1} < z_{k+1} \leqq x_{k+1} \vee x_{k-1} \leqq z_{k-1}$, a contradiction. Hence $y_{k+1} > x_{k-1}$ or $y_{k+1} \lambda x_{k-1}$, and it follows that $z_{k+1} = y_{k+1} \vee p_{k+1} \geqq x_{k-1} \vee a > p_{k-1}$. If $\langle p_{k-1}, J_{k+1} \rangle$ is a minimal pair, then we can construct a cycle $C \in \mathscr{C}$ such that $\alpha(C) < \alpha_0$, which is impossible; hence, letting $x_{k+1*} \prec x_{k+1}$ and $y_{k+1*} \prec y_{k+1}$, we conclude that either $y_{k+1} \vee x_{k+1*} > p_{k-1}$ or $y_{k+1*} \vee x_{k+1} > p_{k-1}$. Suppose the former; then, since $\langle p_{k+1}, J_{k+1} \rangle$ is a minimal pair, $z_{k+1} > y_{k+1} \vee x_{k+1*} = y_{k+1} \vee x_{k+1*} \vee p_{k-1} \geqq z_{k+1}$, a contradiction. Therefore $y_{k+1*} \vee x_{k+1} > p_{k-1}$, and so $y_{k+1*} \vee x_{k+1} \geqq x_{k-1} \vee p_{k-1} = z_{k-1}$. Clearly $y_{k+1*} \lambda w$, and we now have $y_{k+1*} \vee x_{k+1} \leqq y_{k+1*} \vee w \leqq y_{k+1*} \vee p_{k-1} = y_{k+1*} \vee x_{k+1}$, implying $y_{k+1*} \vee w = y_{k+1*} \vee p_{k-1}$. From the dual of Lemma 2.7 (ii) we get that $z_{k-1} = w \vee p_{k-1} < y_{k+1*} \vee x_{k+1}$, whence $y_{k+1*} \lambda z_{k-1}$. Since $\langle p_{k+1}, J_{k+1} \rangle$ is a minimal pair, we also have $y_{k+1*} \vee x_{k+1} \ngeqq p_{k+1}$.

Suppose that $x_{k-2} > a^*$; then

$$\{x_{k-1}, z_{k-1}, y_{k-1}, a^*, a, y_{k+1*} \vee x_{k+1}, x_{k-2}, y_{k+1*}\} \cong M1.$$

Hence $x_{k-2} \ngeqq a^*$, and it follows by Lemma 4.3 that $x_{k-2} < y_{k-2} \vee a \leqq x_{k-1} \vee a$. Furthermore $z_{k-2} < y_{k-2} \vee a$, for otherwise $z_{k-2} = x_{k-2} \vee y_{k-2} = y_{k-2} \vee a$, and by $(\mathbf{SD}_\vee)$ $z_{k-2} = y_{k-2} \vee (x_{k-2} \wedge a)$, contradicting the fact that $\langle p_{k-2}, J_{k-2} \rangle$ is a minimal pair; hence $z_{k-2} \lambda a$. If $p_{k-2} > y_{k-2}$ then by Lemma 5.3 there exists $x' \leqq y_{k-2}$ such that $\langle p_{k-2}, \{x', a\} \rangle$ is a minimal pair of type (a), which is a contradiction to Lemma 6.6. Hence $p_{k-2} > x_{k-2}$.

Let $p_{k-2} = y_{k-3}$. If $x_{k-3} \lambda y_{k-3}$ then by Lemma 4.3 $p_{k-3} < z_{k-3} \leqq x_{k-3} \vee a$, while $p_{k-3} \nleqq x_{k-3} \vee a_*$ since $\langle p_{k-3}, J_{k-3} \rangle$ is a minimal pair; by Lemma 5.3 there exists $x' \leqq x_{k-3}$ such that $\langle p_{k-3}, \{x', a\} \rangle$ is a minimal pair, contradicting Lemma 6.7. Thus $y_{k-3} \lambda x_{k-3}$. If $x_{k-3} > a^*$ then

$$\{x_{k-1}, z_{k-1}, y_{k-1}, a^*, a, y_{k+1*} \vee x_{k+1}, x_{k-3}, y_{k+1*}\} \cong M1;$$

hence $x_{k-3} \ngeqq a^*$, and so $x_{k-3} < x_{k-1} \vee a$. Now $p_{k-3} < x_{k-1} \vee a$ and $w \lambda p_{k-3}$, and by Lemma 5.3 there exists $x'' \leqq x_{k-1}$ such that $\langle p_{k-3}, \{x'', a\} \rangle$ is a minimal pair. This contradicts Lemma 6.7, and the proof of Theorem 6.3 is complete.

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