Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1186–1191

RIGHT INVARIANT RIGHT HEREDITARY RINGS

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Let R be a right hereditary domain in which all right ideals are two-sided (i.e., R is right invariant). We show that R is the intersection of generalized discrete valuation rings and that every right ideal is the product of prime ideals. This class of rings seems comparable with (and contains) the class of commutative Dedekind domains, but the rings considered here are in general not maximal orders and not Dedekind rings in the terminology of Robson [**9**]. The left order of a right ideal of such a ring is a ring of the same kind and the class contains right principal ideal domains in which the maximal right ideals are two-sided [**6**]. Furthermore there is a one-to-one correspondence between fundamental sets of prime ideals and torsion theories (see section 4).

1. We assume in the following that R is a right hereditary domain in which all right ideals are two-sided. For any two nonzero elements r and s in R, there exists an element r' with $rs = sr'$. This implies that R can be embedded in a skew field of fractions $Q(R)$ and that for every maximal ideal M the set $S = R\backslash M$ is an Ore system. The ring $R_M = V$ of quotients rs^{-1}, $r \in R$, $s \in S$ exists and is again right hereditary. We define $B^{-1} = \{q \in Q(R) | qB \subset R\}$ for a nonzero right ideal B of R and BB^{-1} then contains the unit element 1 of R since R is right hereditary (Dual basis Lemma). It follows that both rings R and V are right noetherian. We obtain the first lemma:

LEMMA 1. *Let R be a right hereditary domain in which the right ideals are two-sided. Let $\{M_i\}$ be the set of maximal right ideals of R. Then $R = \bigcap V_i$, where $V_i = R_{M_i}$ are local principal right ideal rings in which the right ideals are inversely well-ordered.*

Proof. If $q \in Q(R)$ is contained in $\bigcap AV_i$, A any right ideal of R then $\{r \in R; qr \in A\} \not\subseteq M_i$ for every maximal right ideal M_i. Therefore $A = \bigcap A V_i$ and especially $R = \bigcap V_i$. The V_i's are local, right noetherian, right hereditary rings and therefore by Kaplansky's theorem [**7**] right principal ideal domains. It follows [**1**] that the right ideals are inversely well-ordered and two-sided. (Rings with this property were called generalized discrete valuation rings in [**1**].)

Let N_i be the maximal right ideal of V_i. Define $N_i^0 = V_i$, $N_i^{\alpha+1} = N_i^\alpha N_i$ and $N_i^\alpha = \bigcap_{\beta<\alpha} N_i^\beta$ for a limit ordinal α. The ideals N_i^α are called the transfinite (right) powers of N_i. Every ideal in V_i is of the form N_i^α for some α

Received March 30, 1973.

https://doi.org/10.4153/CJM-1974-111-3 Published online by Cambridge University Press

and we can define a mapping v_i from $V_i \backslash 0$ onto $\Gamma_i = \{\alpha; 0 \leqq \alpha < \omega^{j_i}\}$ for some ordinal j_i, by $v_i(a) = \alpha$ for $aV_i = N_i^\alpha$. It follows that $v_i(ab) = v_i(a) + v_i(b)$ with the usual addition of ordinals and aV_i is a prime ideal if and only if $v_i(a)$ is a power of ω. We set $v_i(aV_i) = v_i(a)$ as well and it follows that $v_i(aV_i) = v_i(bV_i)$ if and only if $aV_{i_e} = bV_i$. Every ordinal number α can be written as $\alpha = \omega^{e_1}n_1 + \ldots + \omega^{e_t}n_t$ with positive integers n_s and ordinals $e_1 > e_2 > \ldots > e_t$. Such a representation corresponds to the factorization

$$N_i^\alpha = P_{e_1}^{n_1} \ldots P_{e_t}^{n_t}$$

of N_i^α as a product of prime ideals $P_{e_j} = N_i^{\omega^{e_j}}$; $P_0 = N_i$. For details see [**1**] or [**3**].

We can define a mapping w from the set of nonzero right ideals of R into the direct product Γ of the Γ_i by $w(A) = (v_i(AV_i))$ for a right ideal A of R. This mapping is one-to-one since $A = \bigcap AV_i$ and satisfies $w(AB) = w(A) + w(B)$ for right ideals A, B of R. Examples in section 5 show that this mapping is in general neither onto nor is the image contained in the direct sum of the Γ_i.

2. Turning to the factorization of right ideals in R we begin with the following remark: $A^{-1}A = R$ for every nonzero right ideal A of R. Otherwise $A \subset AA^{-1}A \subset AM$ for a maximal ideal M of R which implies $A = AM$, a contradiction since $w(A) \neq w(AM)$. We observed earlier that $AA^{-1} \supset R$, but there will not be equality in general.

Now let A be a right ideal in R maximal with the property that it can not be factored in prime ideals. $A \neq R$ implies that $AV_i \neq V_i$ for some i. Then

$$AV_i = P_{e_{1,i}}^{n_{1,i}} \ldots P_{e_{t,i}}^{n_{t,i}} V_i$$

with $P_{e_j,i} = N_i^{\omega^{e_j}}$ different prime ideals in V_i with $P_{e_j,i} \supsetneqq P_{e_{j-1},i}$. The $n_{j,i}$ are positive integers. The intersection $P_{e_1,i} \cap R = Q_{1,i}$ is a prime ideal in R with $Q_{1,i}^{-1}A = A_1 \supsetneqq A$ and $Q_{1,i}A_1 = A$, a contradiction. That $A_1 \supset A$ is obvious and the strict inequality follows from $A_1V_i \supsetneqq AV_i$. The equality $Q_{1,i}A_1 = A$ is correct since it is locally (i.e., if extended to each V_k) correct and is the content of the next lemma:

LEMMA 2. *Let* A, $P_{e_j,i}$, $Q_{1,i}$ *and* A_1 *be as above. Then* $Q_{1,i}A_1V_k = AV_k$ *for every* k.

For a proof consider two different cases: Let $AV_k = P_kB_kV_k$ where P_k is a minimal prime ideal of V_k containing AV_k. In the first case: $P_k \cap R = Q_{1,i}$ and it is obvious that $Q_{1,i}Q_{1,i}^{-1}P_kB_kV_k = AV_k$. In the second case assume $Q_k = P_k \cap R \neq Q_{1,i}$. It follows that $Q_k + Q_{1,i}$ cannot be contained in any maximal ideal of R and is therefore equal to R. We obtain $Q_{1,i}V_k = V_k$ and $R \subset Q_{1,i}^{-1} \subset V_k$. This implies $Q_{1,i}Q_{1,i}^{-1}AV_k = AV_k$ as desired.

https://doi.org/10.4153/CJM-1974-111-3 Published online by Cambridge University Press

With the notation as above and $P_{e_{j},i} \cap R = Q_{j,i}$ we obtain the following corollary:

COROLLARY 1. *There exists an ideal C in R such that*

$$A = Q_{1,i}^{n_{1,i}} \ldots Q_{t,i}^{n_{t,i}} C.$$

So far we proved the existence part of the following theorem:

THEOREM 1. *Let R be a right hereditary right invariant ring. Then every right ideal in R is the product of prime ideals. Prime ideals P_1, P_2 of R with $P_1 \not\subseteq P_2 \not\subseteq P_1$ commute, and $P_1P_2 = P_2$ if $P_1 \supsetneqq P_2$. Let $A = P_1 \ldots P_n = Q_1 \ldots Q_m$ be two factorizations of a right ideal A of R in prime ideals $\neq 0$, $\neq R$ such that for $i < j$ neither $P_i \supsetneqq P_j$ or $Q_i \supsetneqq Q_j$. Then $n = m$ and the two factorizations are equal up to the order of commuting factors.*

Proof. If P_1 and P_2 are prime ideals with $P_1 \not\subseteq P_2 \not\subseteq P_1$, the sum $P_1 + P_2$ is not contained in any maximal ideal of R and $w(P_1P_2) = w(P_2P_1)$ follows. This means $P_1P_2 = P_2P_1$. We obtain $P_1P_2V_k = P_2V_k$ for all k in the case $P_1 \supsetneqq P_2$, which implies $P_1P_2 = P_2$. Every factorization of A in prime ideals can therefore be brought in the standard form as described in the theorem. Assume A is a counterexample to the uniqueness statement with n minimal. Let $P_1 \subset M_i$ for a maximal ideal M_i and $V_i = R_{M_i}$. Then $AV_i = P_1P_{i_2} \ldots P_{i_s}V_i$ with $P_1 \subset P_{i_2} \subset \ldots \subset P_{i_s} \subset M_i$ for certain of the P_j's and $AV_i = Q_{k_1} \ldots Q_{k_t}V_i$ with $Q_{k_1} \subset \ldots \subset Q_{k_t} \subset M_i$, $k_1 < k_2 < \ldots < k_t$. It follows that $t = s$, $P_1V_i = Q_{k_1}V_i$ and therefore $P_1 = Q_{k_1}$. If $k_1 = 1$ consider $P_1^{-1}A$ and we are finished by induction. Since $j < k_1 \neq 1$ implies $Q_j \not\supseteq Q_{k_1}$ and $Q_{k_1} \not\supseteq Q_j$ we have $Q_jQ_{k_1} = Q_{k_1}Q_j$ and induction applies again.

COROLLARY 2. $|\{AV_i \cap R\}| < \aleph_0$ *for every right ideal A of R.*

Proof. Let $AV_i = P_{i_1} \ldots P_{i_t}V_i$ for prime ideals $P_{i_1} \subset \ldots \subset P_{i_t} \subset M_i$ of R. We set $AV_i \cap R = B$. Then $AV_i = BV_i$ and there exists an ideal C in R with $P_{i_1} \ldots P_{i_t} \subset B = P_{i_1} \ldots P_{i_t}C$ (Corollary 1). We obtain $AV_i \cap R = B = P_{i_1} \ldots P_{i_t}$ where the P_{i_j} are exactly those prime ideals in a factorization of A (in standard form) which are contained in M_i. The corollary follows now immediately from the theorem.

The property proved in the corollary for the family of the V_i's is a generalization of the property of being 'of finite character' which plays an important role in the commutative case. (See for example [**4**, § 35].)

3. The above results enable us to determine the left orders of all right ideals of R. Let I be any right ideal $\neq 0$ of R. The left order $O_l(I)$ is defined as $O_l(\mathrm{I}) = \{q \in Q(R);\ qI \subset I\}$. We will show that $O_l(\mathrm{I}) = \cap\, a_iV_ia_i^{-1}$ if $IV_i = a_iV_i$ for $a_i \in I$. It is clear that $qa_iV_i \subset a_iV_i$ implies $q \in a_iV_ia_i^{-1}$. But $I = \cap\, a_iV_i$ and $qI \subset I$ if and only if $qa_iV_i \subset a_iV_i$ for all i.

https://doi.org/10.4153/CJM-1974-111-3 Published online by Cambridge University Press

LEMMA 3. *Let R be a right hereditary right invariant ring and let I be any nonzero right ideal in R. Then $O_l(I)$ is again a right hereditary right invariant ring. $O_l(I) = II^{-1} = \bigcap a_iV_ia_i^{-1}$ for $IV_i = a_iV_i$.*

Proof. Every right ideal in $O_l(I)$ is two-sided since this is true for the rings $a_iV_ia_i^{-1}$ and hence for their intersection $O_l(I)$. Let $J \neq 0$ be any right ideal in $O_l(I)$ and put $JI = J_0$, which is a right ideal in R. We want to show that JJ^{-1} contains the unit element of $O_l(I)$. But $J = JII^{-1} = J_0I^{-1}$ and $J^{-1} \supset IJ_0^{-1}$, since $II^{-1} = O_l(I) \supset R$. Therefore $JJ^{-1} \supset J_0I^{-1}IJ_0^{-1} \supset R$ contains 1 and this means that J is a projective right $O_l(I)$ ideal.

Every $O_l(I)$ is an order equivalent to R since $R \subset O_l(I)$ and $O_l(I)a \subset R$ for a right ideal $I \neq 0$ in R and $a \neq 0$ in I. But in general infinite ascending chains of equivalent orders will appear.

As an example consider a generalized discrete vaulation domain V of type $\omega^2 + 1$ (see [**1**] or [**3**]). Let $V \supset xV \supset yV \supset 0$ be its prime ideals. Then $xy = y\epsilon$ for a unit ϵ in V. Every element $\neq 0$ in V can be written uniquely in the form y^mx^nu for non negative integers n and m and a unit u. We will show that $bV \subsetneq aV$ implies $O_l(bV) \supsetneq O_l(aV)$. It was observed earlier that $O_l(aV) = aVa^{-1}$ and it is clear that $O_l(aV) \subseteq O_l(bV)$. To show the strict inequality consider two cases: First let $b = y^nx^mu$ with $m \geqq 1$, u a unit in V. Then $y^nx^myx^{-m}y^{-n}$ is an element in $O_l(bV)$ but not in $O_l(y^nx^{m-1}V)$. Otherwise $y^nx^myx^{-m}y^{-n} = y^nx^{m-1}rx^{-(m-1)}y^{-n}$ for some r in V. This leads to

$$y^nx^myx^{-m}y^{-n}y^nx^{m-1} = y^nx^{m-1}r = y^kx^l\alpha$$

for a unit α and nonnegative integers k and l. The left hand side is equal to $y^{n+1}\beta x^{-1}$ and $y^{n+1}\beta = y^kx^{l+1}\alpha'$ with units β and α' follows. This is a contradiction to the uniqueness of such a representation. One shows in a similar fashion that y^nxy^{-n} is an element in $O_l(y^nV)$ which is not contained in $O_l(y^mx^kV)$ for $m < n$ and arbitrary $k \geqq 0$. This proves that we get a strictly ascending chain

$$V \subsetneq O_l(xV) = xVx^{-1} \subsetneq x^2Vx^{-2} \subsetneq \ldots \subsetneq x^nVx^{-n} \subsetneq \ldots yVy^{-1} \subsetneq yxVx^{-1}y^{-1} \subsetneq \ldots$$

of orders equivalent to V.

We might remark that the right orders $O_r(I)$ of right ideals $I \neq O$ of the rings considered in this paper are equal to $R = I^{-1}I$.

4. We now consider idempotent filters of right ideals (see [**8**] and [**11**]). Again let R be a domain satisfying our general conditions: R is right hereditary and the right ideals are two-sided. Say $\{P_j\}_{j\in\Lambda} = \pi$ is a fundamental set of prime ideals if every P_j is a prime ideal in R and $P_j \in \pi$, $P_j \subseteq P$ implies $P \in \pi$ for a prime ideal P of R.

LEMMA 4. *There is a one-to-one correspondence between fundamental sets of prime ideals of R and idempotent filters of right ideals of R.*

Let $\pi = \{P_j\}$ be a fundamental set of prime ideals of R. We show that the set

$$\mathscr{F}_\pi = \{A; A = P_1, \ldots, P_k; P_j \in \pi\}$$

is an idempotent filter of right ideals. (The P_j's are prime ideals).

If $A \in \mathscr{F}_\pi$, $r \in R$ then $r^{-1}A \supset A$ and $r^{-1}A$ is contained in $\mathscr{F}_\pi$. If $A \in \mathscr{F}_\pi$ and $a^{-1}B \in \mathscr{F}_\pi$ for every $a \in A$ and a right ideal B of R, then we like to conclude that $B \in \mathscr{F}_\pi$.

Let $a_1, \ldots, a_n$ be a finite generating system of A as a right ideal. It follows that $B_i = a_i^{-1}B \in \mathscr{F}_\pi$ for every i and B is in $\mathscr{F}_\pi$, since $AB_1 \ldots B_n$ is contained in B.

Conversely, let $\mathscr{F}$ be any idempotent filter of right ideals of R. Consider $\pi = \{P_i\}$ the set of prime ideals in $\mathscr{F}$. Obviously $\mathscr{F} \subset \mathscr{F}_\pi$ since every ideal in $\mathscr{F}$ is a product of primes contained in π; and $\mathscr{F}_\pi \subset \mathscr{F}$ since with $A, B \in \mathscr{F}$ the product $A \cdot B$ is in $\mathscr{F}$ and every right ideal C with $A \subset C$.

If $\mathscr{F}$ is any idempotent filter of right ideals of R, set $\mathscr{F}_{V_i} = \{A V_i, A \in \mathscr{F}\}$. It can be shown that $\mathscr{F}_{V_i}$ is an idempotent filter of right ideals of V_i determined by a prime ideal P_i of V_i.

The ring of quotients of V_i with respect to $\mathscr{F}_{V_i}$ is nothing else but the localization $V_i S_i^{-1}$ of V_i with respect to the Ore system $S_i = V_i \backslash P_i$. The ring of quotients of R with respect to $\mathscr{F}$ is $R_{\mathscr{F}} = \bigcap V_i S_i^{-1}$ which in general will not be a localization of R with respect to some Ore system (see [**2**]).

5. Jategaonkar proved a result similar to Theorem 1 for principal right ideal domains in which every maximal right ideal is two-sided. These rings are of course right hereditary and we will prove that all right ideals are two-sided. We will show that for a maximal right ideal M of a principal right ideal domain whose maximal right ideals are two-sided the set $R \backslash M = S$ is an Ore system. It then follows as in the proof of Lemma 1 that $R_M = V$ is a generalized discrete valuation ring, all right ideals of V are two-sided and R as the intersection of such rings has the same property.

Let $s \in S$, $r \in R$ and we may assume that $sR + rR = R$. There exist elements a, b in R with $sa + rb = 1$. If b is in M consider $sar + rbr = r$ and $sar = r(1 - br)$ follows. This is a multiple of s and r of the desired form since $1 - br \notin M$. If $b \notin M$ consider $sas + rbs = s$; $rbs = s(1 - as)$ follows and our proof is completed since $bs \notin M$.

We do not know if in a right hereditary domain whose maximal right ideals are two-sided all right ideals have to be two-sided. Finally we show by an example that the mapping w (defined in section 1) is neither onto nor is it sufficient to consider the direct sum of the Γ_i. Let k be a commutative field with a monomorphism σ from $k[x]$ into k, and form the twisted power series ring $R = k[x][[y, \sigma]]$ whose multiplication is defined by $f(x)y = yf^\sigma(x)$. The maximal right ideals M_i of R are generated by the monic irreducible polynomials $f_i(x)$ of $k[x]$. Each Γ_i is of the form $\{\alpha; 0 \leqq \alpha < \omega^2\}$ and the value $w(yR) = (\omega, \omega, \omega, \ldots \omega, \ldots)$ is not contained in the direct sum of the Γ_i.

On the other hand $(\omega, 0, 0, \ldots, 0, \ldots)$ is not contained in the image of w.

It is clear from Jategaonkar's examples [**5**] that for every power of ω, say ω^{e_j}, e_j some ordinal, there is a ring of the kind considered in this paper such that the image of the mapping w is exactly $\Gamma_j = \{\alpha; 0 \leqq \alpha < \omega^{e_j}\}$. We do not know if there are rings such that the image of w is equal to the direct sum of arbitrary Γ_i's; and similarly we do not know which subsemigroups of the direct product of arbitrary Γ_i's can appear as images of w. One restriction is given by: COROLLARY 2. The direct product of infinitely many Γ_i's for example can never appear as the image of w.

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