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ON THE REGULARITY OF THE KOWALSKY COMPLETION

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Cauchy spaces were introduced by Kowalsky in 1954 [**9**]. In that paper a first completion method for these spaces was given. In 1968 Keller [**5**] has shown that the Cauchy space axioms characterize the collections of Cauchy filters of uniform convergence spaces in the sense of [**1**]. Moreover in the completion theory of uniform convergence spaces the associated Cauchy structures play an essential role [**12**]. This fact explains why in the past ten years in the theory of Cauchy spaces, much attention has been given to the study of completions.

In 1971 a very large and natural class of completions was constructed by E. Reed, unifying all completion methods known up until then, including the Kowalsky [**9**] and the Wyler completion [**15**]. Other completion methods were developed for certain subclasses of Cauchy spaces [**7**], [**4**].

In the study of completions, properties such as strictness and regularity play an important role. In fact in [**7**] Kent and Richardson have shown that a strict regular completion, if it exists, is unique up to an equivalence.

In [**13**] the same authors characterized Cauchy spaces having a regular completion and proved that not every space with a regular completion has a strict regular completion.

The subject of this paper is to find conditions on a space to ensure the existence of regular completions in the collection of E. Reed. The completions in this collection are all strict. We give an example of a Cauchy space having a strict regular completion but having no regular completion in the collection of E. Reed. However, if the Cauchy space is totally bounded and pseudotopological then the property of having a (strict) regular completion is equivalent to the existence of a regular completion in the collection of E. Reed. In fact, then both properties are equivalent to the uniformizability of the space.

In particular we investigate the regularity of the Kowalsky completion. We give an internal condition which characterizes the regularity of this

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completion. In the pretopological case this condition is equivalent to the space being relatively round. In general however, even in the totally bounded pseudotopological case, being relatively round is not a sufficient condition to ensure the regularity of the Kowalsky completion. A $C^{\wedge}$-embedded Cauchy space always has a regular completion, for instance the natural completion [**4**]. The position of the natural completion with respect to the collection of E. Reed is investigated.

The paper is subdivided in sections treating respectively: the general case, the pretopological, topological, totally bounded and $C^{\wedge}$-embedded case. At the top of each section a diagram summarizes implications holding in that particular case and in the corresponding section we prove those results necessary to deduce the non trivial implications. In each of the charts the only valid implications are those indicated by the arrows or obtained from them by transitivity. In the final section we give examples to show that all other possible implications are false.

I thank Professor D. Kent and Professor R. Frič for posing some interesting questions which initiated the investigations of this paper.

1. Preliminaries. Throughout the paper we use notations, constructions of completions and results from [**7**] and [**12**]. From these papers we also use basic definitions and notations on Cauchy spaces. A few changes in the notations of [**7**] will be made. The closure operator will be denoted by cl instead of Γ and the set of equivalence classes in a Cauchy space $(X, \mathscr{C})$ will be denoted by X^*. For a Cauchy filter $\mathscr{F}$ its equivalence class is $\langle \mathscr{F} \rangle$.

A Cauchy space is *uniformizable* if it is the collection of Cauchy filters in some uniform space, *totally bounded* if every ultrafilter is Cauchy, *pseudotopological* (full in [**12**]) if a filter $\mathscr{F}$ is Cauchy if and only if all finer ultrafilters are in the same equivalence class, *pretopological* (*topological*) if it is a Cauchy subspace of a pretopological (topological) convergence space, considered as a complete Cauchy space. Pretopological (topological) Cauchy spaces are spaces with minima (minima with an open base) in their equivalence classes [**11**]. A Cauchy space is *Hausdorff* if the induced convergence structure is Hausdorff, *regular* if cl $\mathscr{F}$ is Cauchy whenever $\mathscr{F}$ is Cauchy, *completely regular* if it is a subspace of a completely regular topological space.

Throughout the paper the original space $(X, \mathscr{C})$ is assumed to be regular and Hausdorff. By the term "completion" is meant a Hausdorff completion.

A subset X of Y is *strictly dense* in Y if cl $X = Y$ and whenever ψ is a filter on Y converging to y in Y, there is a filter $\mathscr{G}$ on Y converging to y

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such that $X \in \mathcal{G}$ and $\psi \supset \mathrm{cl}\, \mathcal{G}$. If $(Y, \mathcal{D})$ is a completion of $(X, \mathcal{C})$ and X is strictly dense in Y then $(Y, \mathcal{D})$ is a *strict completion* [**4**].

Given a Cauchy space $(X, \mathcal{C})$ let $j:X \rightarrow X^*$ be the map $j(x) = \langle \dot{x} \rangle$ for $x \in X$. In [**12**] a family of completions is constructed on X^*, using collections of maps that pick out a filter for each equivalence class. $\Lambda(\mathcal{C})$ is the set of all maps λ from X^* to $F(X)$ (the set of all filters on X) such that $\lambda(p)$ is contained in some filter in p for $p \in X^*$ and $\lambda j(x) = \dot{x}$ for $x \in X$. For every $\Gamma \subset \Lambda(\mathcal{C})$ a completion $K_\Gamma(X, \mathcal{C})$ is constructed. The collection of completions thus obtained is called the collection of E. Reed and will be denoted by $\mathcal{R}(X, \mathcal{C})$ or if no confusion can arise simply $\mathcal{R}$.

In particular for

$$\Sigma = \{\sigma \in \Lambda(\mathcal{C})\, |\sigma(p) \in p \text{ for } p \in X^*\}$$

the associated completion $K_\Sigma(X, \mathcal{C})$ is the Kowalsky completion which we will shortly denote $K(X, \mathcal{C})$. If $\Gamma = \{\sigma\}$ then we write $K_\sigma(X, \mathcal{C})$ for the corresponding completion. $\mathcal{R}$ is a collection of strict completions. We call $\mathcal{R}$ *regular* if it contains a regular completion.

In [7] an extension of a Cauchy space $(X, \mathcal{C})$ is constructed which is not always a completion. The construction is based on a certain Σ-operation: In order to avoid confusion with the Σ defined above we will change the notation of [7]. For $A \subset X$ we have

$$\Delta A = \{p \in X^*|A \in \mathcal{G} \text{ for some } \mathcal{G} \in p\}.$$

For a filter $\mathcal{F}$ on X we have $\Delta\mathcal{F}$ the filter on X^* generated by $\{\Delta F|F \in \mathcal{F}\}$. The construction $(X^*, \mathcal{C}_p)$ in [7] will be denoted by $R(X, \mathcal{C})$. Every strict completion in standard form is finer than $R(X, \mathcal{C})$ and a strict regular completion in standard form, if it exists, is unique and equal to $R(X, \mathcal{C})$. These results from [7] will be used several times in this paper without explicit mentioning. For a Cauchy space we use the notation SC_3 from [**13**] to denote the property that it has a strict regular completion.

A convergence space (X, q) is *ω-regular* if $\mathrm{cl}_{\omega q}\mathcal{F}$ converges whenever $\mathcal{F}$ converges, where ωq is the finest completely regular topology on X coarser than q. (X, q) is *completely regular* if it is regular and has the same ultrafilter convergence as a completely regular topological space [**8**]. $C^\wedge$-embedded spaces are introduced in [**4**]. They can be characterized by the property of having an ω-regular pseudotopological completion. Such a completion can be obtained by embedding $(X, \mathcal{C})$ in a function algebra and this construction, called the natural completion, will be denoted by $N(X, \mathcal{C})$.

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2. General case. Let $(X, \mathscr{C})$ be a Cauchy space.

$\mathscr{R}$ regular $(X, \mathscr{C})$ relatively round	$\Leftrightarrow$	$K(X, \mathscr{C})$ regular	$\Rightarrow$	$(X, \mathscr{C})$ relatively round
		$\Vert$		
		$\mathscr{R}$ regular	$\Rightarrow$	SC_3

LEMMA 2.1. *$\mathscr{R}$ is regular if and only if there exists a $\sigma \in \sum$ such that $K_\sigma(X, \mathscr{C})$ is regular, where σ can be chosen to map on ultrafilters.*

Proof. Let $\Gamma \subset \Lambda(\mathscr{C})$ such that $K_\Gamma(X, \mathscr{C})$ is regular. Choose $\gamma \in \Gamma$ and for $y \in X^* \setminus j(X)$ an (ultra) filter $\mathscr{G}_y \in y$ such that $\gamma(y) \subset \mathscr{G}_y$. Let $\sigma: X^* \to F(X)$ mapping $\langle \dot{x} \rangle$ to $\dot{x}$ for $x \in X$ and $y \in X^* \setminus j(X)$ to $\mathscr{G}_y$. Then $\sigma \in \sum$ and

$$K_\Gamma(X, \mathscr{C}) \geqq K_\gamma(X, \mathscr{C}) \geqq K_\sigma(X, \mathscr{C}) \geqq R(X, \mathscr{C}).$$

Clearly the regularity of $K_\Gamma(X, \mathscr{C})$ implies the regularity of $K_\sigma(X, \mathscr{C})$.

For $\sigma \in \sum$ and F and A subsets of X and $\mathscr{F}$ a filter on X let $F <_\sigma A$ if and only if for every $p \in X^*$, A or $X \setminus F$ belongs to $\sigma(p)$ [**12**].

Let $F <^\sigma A$ if and only if for every $p \in X^*$, if $F \in \mathscr{H}$ for some $\mathscr{H} \in p$ then $A \in \sigma(p)$.

Let

$$r_\sigma\mathscr{F} = \{A \subset X | F <_\sigma A \text{ for some } F \in \mathscr{F}\} \quad [\mathbf{12}]$$

and

$$s_\sigma\mathscr{F} = \{A \subset X | F <^\sigma A \text{ for some } F \in \mathscr{F}\}.$$

Then $r_\sigma\mathscr{F}$ and $s_\sigma\mathscr{F}$ are filters on X, $s_\sigma\mathscr{F} \subset \mathrm{cl}\,\mathscr{F}$ and $s_\sigma\mathscr{F} \subset r_\sigma\mathscr{F}$.

A Cauchy space is *relatively round* if for every $\sigma \in \sum$ the filter $r_\sigma\mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$ [**12**].

THEOREM 2.2. *$K(X, \mathscr{C})$ is regular if and only if for every $\sigma \in \sum$ the filter $s_\sigma\mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$. $\mathscr{R}$ is regular if and only if there exists a $\sigma \in \sum$ such that $s_\sigma\mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$.*

Proof. If $K(X, \mathscr{C})$ is regular and $\mathscr{F} \in \mathscr{C}$ then $\Delta\mathscr{F}$ converges in $K(X, \mathscr{C})$. For every $\sigma \in \sum$ there exists $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{G}^\sigma \subset \Delta\mathscr{F}$. Then we have $\mathscr{G} \subset s_\sigma\mathscr{F}$ and $s_\sigma\mathscr{F} \in \mathscr{C}$. For the converse suppose the condition is fulfilled, that Ψ is a filter on X^* converging to y in $K(X, \mathscr{C})$ and that $\sigma \in \sum$. Choose $\mathscr{F} \in y$ with $\mathscr{F}^\sigma \subset \Psi$. For $A \in s_\sigma\mathscr{F}$ let $F \in \mathscr{F}$ with $F <^\sigma A$. For $q \in \mathrm{cl}\,(F^\sigma)$

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let $\mathscr{H}$ be a filter in q such that $F^\sigma \vee \mathscr{H}^\sigma$ exists, then $F \vee \mathscr{H}$ exists and $A \in \sigma(q)$. Then

$$\text{cl } (F^\sigma) \subset A^\sigma \quad \text{and} \quad (s_\sigma \mathscr{F})^\sigma \subset \text{cl } \Psi.$$

So cl Ψ converges to y.

Using Lemma 2.1 the second part can be shown analogously.

THEOREM 2.3. *$K(X, \mathscr{C})$ is regular if and only if $(X, \mathscr{C})$ is relatively round and $\mathscr{R}$ is regular.*

Proof. If $K(X, \mathscr{C})$ is regular then $(X, \mathscr{C})$ is relatively round since $s_\sigma \mathscr{F} \subset r_\sigma \mathscr{F}$ for every $\sigma \in \sum$. The other implication follows from the fact that in a relatively round space all the completions $K_\Gamma(X, \mathscr{C})$ for $\Gamma \subset \sum$ coincide [**12**].

THEOREM 2.4. *If $K(X, \mathscr{C})$ is regular then the structure induced on $X^* \setminus j(X)$ is diagonal (in the sense of* [**9**] *).*

Proof. If

$$\mu: X^* \setminus j(X) \rightarrow F(X^* \setminus j(X))$$

and $\mu(p)$ converges to p for every $p \in X^* \setminus j(X)$ and Ψ is a filter on $X^* \setminus j(X)$ converging to q in $X^* \setminus j(X)$ let $[\mu(p)]$, $p \in X^* \setminus j(X)$ and $[\Psi]$ be the filters generated on X^*. For $p \in X^* \setminus j(X)$ choose $\eta(p) \in p$ with $\Delta\eta(p) \subset [\mu(p)]$. For $x \in X$ let $\eta\langle \dot{x} \rangle = \dot{x}$. Choose $\mathscr{F} \in q$ with $\Delta\mathscr{F} \subset [\Psi]$. Then

$$s_\eta(\mathscr{F}) \in q \quad \text{and} \quad \Delta s_\eta \mathscr{F} \subset [\bigcup_{\mathscr{A} \in \Psi} \bigcap_{p \in \mathscr{A}} \mu(p)].$$

It follows that $\bigcup_{\mathscr{A} \in \Psi} \bigcap_{p \in \mathscr{A}} \mu(p)$ converges to q in $X^* \setminus j(X)$.

3. Pretopological case. Let $(X, \mathscr{C})$ be a pretopological Cauchy space.

$(X, \mathscr{C})$ relatively round	$\Leftrightarrow$	$K(X, \mathscr{C})$ regular		
		$\Downarrow$		
		$\mathscr{R}$ regular $\Rightarrow$	SC_3 $\Leftrightarrow$	$(X, \mathscr{C})$ a strict subspace of a regular pretopological space

THEOREM 3.1. *$K(X, \mathscr{C})$ is regular if and only if $(X, \mathscr{C})$ is relatively round.*

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Proof. In view of Theorem 2.3 we only need to prove the "if" part. Let $\mu: X^* \to F(X)$, $\mu\langle\dot{x}\rangle = \dot{x}$ for $x \in X$ and $\mu(p) = \mathscr{M}_p$, the minimum element in p, for $p \in X^* \setminus j(X)$. Let $\mathscr{M}$ be a minimal Cauchy filter in $\mathscr{C}$. Since $(X, \mathscr{C})$ is regular and relatively round we have $r_\mu(\text{cl } \mathscr{M}) = \mathscr{M}$. By straightforward verification we obtain that

$$s_\mu \mathscr{M} \supset r_\mu (\text{cl } \mathscr{M}).$$

Using Theorem 2.3 and the fact that $(X, \mathscr{C})$ is pretopological it follows that $K(X, \mathscr{C})$ is regular.

THEOREM 3.2. *$(X, \mathscr{C})$ is SC_3 if and only if it is a strict subspace of a regular pretopological space.*

Proof. If $(X, \mathscr{C})$ is a strict subspace of a regular pretopological space then the closure of X in this space is a strict regular completion.

If $(X, \mathscr{C})$ is SC_3 then $R(X, \mathscr{C})$ is a pretopological strict regular completion with neighborhood filter $\Delta\mathscr{M}_p$ in p, where $\mathscr{M}_p$ is the minimum element in p.

THEOREM 3.3. *If $K(X, \mathscr{C})$ is regular then the induced structure on $X^* \setminus j(X)$ is topological.*

Proof. If $(X, \mathscr{C})$ is pretopological then so is $K(X, \mathscr{C})$. If $K(X, \mathscr{C})$ is regular then $X^* \setminus j(X)$ is diagonal (Theorem 2.4). It follows that $X^* \setminus j(X)$ is topological [**9**].

4. Topological Cauchy spaces. Let $(X, \mathscr{C})$ be a topological Cauchy space.

$(X, \mathscr{C})$ relatively round	$\Leftrightarrow$	$K(X, \mathscr{C})$ regular	$\Leftrightarrow$	$(X, \mathscr{C})$ a subspace of a regular topological space
		$\Downarrow$		
		$\mathscr{R}$ regular		
		$\Downarrow$		
		SC_3		

THEOREM 4.1. *$K(X, \mathscr{C})$ is regular if and only if $(X, \mathscr{C})$ is a subspace of a regular topological space.*

Proof. If $(X, \mathscr{C})$ is topological then so is $K(X, \mathscr{C})$ [**11**]. If $(X, \mathscr{C})$ is a subspace of a regular topological space Y then the closure cl X of X in Y is

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a topological regular completion of $(X, \mathscr{C})$. Let φ:cl $X \to X^*$ be the map which to $y \in$ cl X assigns $\langle \mathscr{M}_y \rangle$, where $\mathscr{M}_y$ is the trace on X of the neighborhoodfilter $\mathscr{V}(y)$ of y. Since $\mathscr{V}(y)$ has an open base, we have $\varphi(\mathscr{V}(y)) \supset \mathscr{M}_y^\mu$, where $\mu:X^* \to F(X)$ maps $p \in X^* \setminus j(X)$ to the minimum element in p and $\langle \dot{x} \rangle$ to $\dot{x}$ for $x \in X$. Hence

$$\varphi\colon \text{cl } X \to K(X, \mathscr{C})$$

is continuous. Since $K(X, \mathscr{C}) \geqq R(X, \mathscr{C})$ the map φ^{-1} also is continuous. It follows that $K(X, \mathscr{C})$ is regular.

5. Totally bounded pseudotopological Cauchy spaces. Let $(X, \mathscr{C})$ be totally bounded and pseudotopological.

$K(X, \mathscr{C})$ regular	$\Leftrightarrow$	$\mathscr{R}$ regular	$\Leftrightarrow$	SC_3	$\Leftrightarrow$	$(X, \mathscr{C})$ uniformizable
		$\Downarrow$				
		$(X, \mathscr{C})$ relatively round				

THEOREM 5.1. *The following properties are equivalent*:
(1) *$K(X, \mathscr{C})$ is regular*
(2) *$\mathscr{R}$ is regular*
(3) *$(X, \mathscr{C})$ is SC_3*
(4) *$(X, \mathscr{C})$ has a regular completion*
(5) *$(X, \mathscr{C})$ is uniformizable.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial implications. (4) $\Rightarrow$ (5): Suppose $(X, \mathscr{C})$ has a regular completion. Then from Lemma 2.3 in [**2**] and Theorem 2.2 in [7] it follows that $R(X, \mathscr{C})$ is an almost topological regular completion in the sense of [7]. If $\omega \in \Sigma$ maps every $p \in X^*$ to an ultrafilter $\omega(p)$ then $K_\omega(X, \mathscr{C})$ is compact Hausdorff and pseudotopological [**12**] and therefore it is minimal Hausdorff [**6**]. It follows that $K_\omega(X, \mathscr{C})$ and $R(X, \mathscr{C})$ coincide and that $R(X, \mathscr{C})$ is topological. From the proof of Theorem 4.1 we have that $K(X, \mathscr{C})$ is equal to $R(X, \mathscr{C})$ and that $K(X, \mathscr{C})$ is a compact Hausdorff topological space. Hence $(X, \mathscr{C})$ is uniformizable [**10**].

(5) $\Rightarrow$ (1) is evident since $K(X, \mathscr{C})$ preserves uniformizability [**12**].

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6. $C^{\wedge}$-embedded Cauchy spaces. Let $(X, \mathscr{C})$ be a $C^{\wedge}$-embedded space.

$$\begin{array}{ccccc}
\begin{array}{c}(X, \mathscr{C}) \text{ completely} \\ \text{regular}\end{array} & & & & \\
\Updownarrow & & & & \\
\begin{array}{c}K(X, \mathscr{C}) \text{ completely} \\ \text{regular}\end{array} & & & & \\
\Downarrow & & & & \\
K(X, \mathscr{C})\ \omega\text{-regular} & \overset{\cdot}{\Rightarrow} & \begin{array}{c}\text{there is an } \omega\text{-regular} \\ \text{completion in } \mathscr{R}\end{array} & & \\
\Updownarrow & & \Downarrow & & \\
K(X, \mathscr{C}) \approx N(X, \mathscr{C}) & \Rightarrow & N(X, \mathscr{C}) \text{ ``}\in\text{'' } \mathscr{R} & \Rightarrow & N(X, \mathscr{C}) \text{ strict} \\
\Downarrow & & \Downarrow & & \Downarrow \\
K(X, \mathscr{C}) \text{ regular} & \Rightarrow & \mathscr{R} \text{ regular} & \Rightarrow & SC_3
\end{array}$$

THEOREM 6.1. *$K(X, \mathscr{C})$ is a completely regular convergence space if and only if $(X, \mathscr{C})$ is a completely regular Cauchy space.*

Proof. $K(X, \mathscr{C})$ is pseudotopological so if it is completely regular then it is topological and then $(X, \mathscr{C})$ is a completely regular Cauchy space.

The other implication follows from the proof of Theorem 4.1.

We use the following notations.

If $\sigma \in \sum$, F and A are subsets of X and $\mathscr{F}$ is a filter on X then we put

$$F \ll^{\sigma} A$$

if and only if for every $p \in X^{x}$, we have $A \in \sigma(p)$ whenever for every Cauchy continuous real valued map f on $(X, \mathscr{C})$,

$$\lim f(\mathscr{H}) \in \overline{f(F)} \text{ for some } \mathscr{H} \in p$$

and

$$t_{\sigma}\mathscr{F} = \{A \subset X | F \ll^{\sigma} A \text{ for some } F \in \mathscr{F}\}.$$

Then $t_{\sigma}\mathscr{F}$ is a filter on X and $t_{\sigma}\mathscr{F} \subset s_{\sigma}\mathscr{F}$.

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THEOREM 6.2. *The following properties are equivalent.*

(1) *$K(X, \mathscr{C})$ is ω-regular.*

(2) *For every $\sigma \in \sum$ the filter $t_\sigma \mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$.*

(3) *$K(X, \mathscr{C})$ is equivalent to $N(X, \mathscr{C})$.*

(4) *$K(X, \mathscr{C})$ is regular and $N(X, \mathscr{C})$ is strict.*

Proof. (1) $\Rightarrow$ (2). If $K(X, \mathscr{C})$ is ω-regular, $\sigma \in \sum$ and $\mathscr{F} \in \mathscr{C}$ then $[\mathscr{F}]$ converges in $K(X, \mathscr{C})$ and so does $\mathrm{cl}_{\omega K}[\mathscr{F}]$. Let $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{G}^\sigma \subset \mathrm{cl}_{\omega K}[\mathscr{F}]$ and let $G \in \mathscr{G}$. Choose $F \in \mathscr{F}$ satisfying $\mathrm{cl}_{\omega K} F \subset G^\sigma$. If $p \in X^*$ is such that for every real valued Cauchy continuous map f on $(X, \mathscr{C})$, $\lim f(\mathscr{H}) \in \overline{f(F)}$ for some $\mathscr{H} \in p$, then $p \in \mathrm{cl}_{\omega K} F$. If not, then there would exist a continuous function $h{:}K(X, \mathscr{C}) \to \mathbf{R}$ such that

$$h(\mathrm{cl}_{\omega K} F) = \{0\} \quad \text{and} \quad h(p) = 1$$

and for the restriction f to X we would have

$$f(F) = h(F) \subset h(\mathrm{cl}_{\omega K} F) = \{0\} \quad \text{and}$$

$$\lim f(\mathscr{H}) \notin \overline{f(F)} \quad \text{for every } \mathscr{H} \in p,$$

which is impossible. Since $p \in \mathrm{cl}_{\omega K} F$ we have $G \in \sigma(p)$. Finally $F \ll^\sigma G$ and $G \in t^\sigma \mathscr{F}$. Since $\mathscr{G} \subset t_\sigma \mathscr{F}$ we have that $t_\sigma \mathscr{F} \in \mathscr{C}$.

(2) $\Rightarrow$ (1). If (2) is satisfied and Ψ is a filter on X^* converging to p in $K(X, \mathscr{C})$ then we choose $\mathscr{F} \in p$ with $\Delta \mathscr{F} \subset \Psi$. For $\sigma \in \sum$ and $A \in t_\sigma \mathscr{F}$ let $F \in \mathscr{F}$ such that $F \ll^\sigma A$. For $p \in \mathrm{cl}_{\omega K} F$ and a real valued Cauchy continuous map f the set $(f^*)^{-1}(\overline{f(F)})$ is ωK-closed if f^* is the extension of f to $K(X, \mathscr{C})$. Hence

$$(f^*)^{-1}(\overline{f(F)}) \supset \mathrm{cl}_{\omega K} F.$$

So there exists an $\mathscr{H} \in p$ with $\lim f(\mathscr{H}) \in \overline{f(F)}$. It follows that $p \in A^\sigma$ and $\mathrm{cl}_{\omega K} F \subset A^\sigma$. Finally $K(X, \mathscr{C})$ is ω-regular since

$$(t_\sigma \mathscr{F})^\sigma \subset \mathrm{cl}_{\omega K} \mathscr{F} \subset \mathrm{cl}_{\omega K} \Psi.$$

(1) $\Rightarrow$ (3). If $K(X, \mathscr{C})$ is ω-regular then it is C-embedded and strict. So it is equivalent to $N(X, \mathscr{C})$ [**4**].

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) are trivial implications.

With an analogous proof we also obtain the following result.

THEOREM 6.3. *The following properties are equivalent.*

(1) *$N(X, \mathscr{C})$ is equivalent to some completion in $\mathscr{R}$.*

(2) *There exists a $\sigma \in \sum$ such that $N(X, \mathscr{C})$ is equivalent to $K_\sigma(X, \mathscr{C})$.*

(3) *There exists a $\sigma \in \sum$ such that $t_\sigma \mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$.*

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7. Examples.

Example 7.1. A $C^{\wedge}$-embedded topological Cauchy space, not relatively round, with a strict natural completion and for which $\mathscr{R}$ is not regular.

On the real line we choose an irrational number α and a descending sequence $(\alpha_j)_{j \in \mathbf{N}_0}$ of irrationals converging to α. For $j > 1$ we choose two disjoint sequences of rationals $(q^j_n)_{n \in \mathbf{N}_0}$ and $(r^j_n)_{n \in \mathbf{N}_0}$ in $]\alpha_j, \alpha_{j-1}[$ converging to α_j. For $j = 1$ we choose two disjoint sequences of rationals $(q^1_n)_{n \in \mathbf{N}_0}$ and $(r^1_n)_{n \in \mathbf{N}_0}$ in $]\alpha_1, +\infty[$ converging to α_1.

Let $X = ([\alpha, +\infty[\cap \mathbf{Q}) \cup \{\alpha\} \cup \{\alpha_j | j \in \mathbf{N}_0\}$ and $Y = [0, \Omega]$, where $\mathbf{Q}$ is the set of rationals and is Ω the first uncountable ordinal. $Z = X \times Y$ is equipped with the pretopological structure p defined by its neighborhood filters in the following way.

$$\mathscr{B}((q, y)) = (q; y) \text{ if } q \in \mathbf{Q}$$

$$\mathscr{B}((\alpha, \Omega)) = [\{(\left[\alpha, \alpha + \frac{1}{n}\right[\cap X) \times \{\Omega\} | n \in \mathbf{N}_0\}]$$

$$\mathscr{B}((\alpha, y)) = [\{F_n(y) \cup \{(\alpha, y)\} \cup \{(\alpha_j, y) | j \geqq n\} | n \in \mathbf{N}_0\}]$$

$$\text{for } y \in [0, \Omega[, \text{ with } F_n(y) = \bigcup_{j=n}^{\infty} \{(q^j_k, y) | k \in \mathbf{N}_0\}$$

$$\mathscr{B}((\alpha_j, \Omega)) = [\{(G^j_n \cup \{\alpha_j\}) \times]\gamma, \Omega] | n \in \mathbf{N}_0, \gamma < \Omega\}]$$

$$\text{for } j \in \mathbf{N}_0, \text{ with } G^j_n = \{r^j_m | m \geqq n\}$$

$$\mathscr{B}((\alpha_j, y)) = [\{]\alpha_j - \frac{1}{n}, \alpha_j + \frac{1}{n}[\cap X) \times \{y\} | n \in \mathbf{N}_0\}]$$

for $j \in \mathbf{N}_0$ and $y \in [0, \Omega[$.

Let

$$D = ([\alpha, +\infty[\cap \mathbf{Q}) \times [0, \Omega].$$

On D we take the Cauchy subspace $\mathscr{C}$ induced by (Z, p). $(D, \mathscr{C})$ is a topological Cauchy subspace with minimal Cauchy filters

$$\mathscr{M}_{(x,y)} = \mathscr{B}((x, y))|_D \quad \text{for } (x, y) \in Z$$

and (Z, p) is a strict completion of $(D, \mathscr{C})$ since

$$\overline{\mathscr{M}_{(x,y)}} \subset \mathscr{B}(x, y) \quad \text{for every } (x, y) \in Z.$$

The structure p on Z is finer than the product topology τ on $X \times Y$ of X, with the trace of the discrete rational extension topology on $\mathbf{R}$ [**14**] and the topology on Y, which is discrete in every $\gamma < \Omega$ and has the neighborhood

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filter $[\{\,]\gamma, \Omega]\,|\gamma < \Omega\}\,]$ in Ω. The topology τ is completely regular and $\mathscr{B}(x, y)$ has a τ-closed base for every $(x, y) \in Z$. Hence (Z, p) is ω-regular and C-embedded. $(D, \mathscr{C})$ is $C^{\wedge}$-embedded and $N(X, \mathscr{C})$ is equivalent to (Z, p).

Next we show that $(D, \mathscr{C})$ has no regular completion in $\mathscr{R}$. Suppose on the contrary that $\mathscr{R}$ is regular. Using Lemma 2.1 we can choose $\sigma \in \Sigma$ such that $K_\sigma(D, \mathscr{C})$ is regular. Then

$$\varphi:(Z, p) \rightarrow K_\sigma(D, \mathscr{C})$$

mapping z to $\langle \mathscr{M}_z \rangle$ is an isomorphism. For $j \in \mathbf{N}_0$ we have

$$\mathscr{M}^{\sigma}_{(\alpha_j, \Omega)} \subset \varphi(\mathscr{B}(\,(\alpha_j, \Omega)\,)).$$

There exist $n_j \in \mathbf{N}_0$ and $\gamma_j < \Omega$ such that

$$\varphi(\,(G^j_{n_j} \cup \{\alpha_j\}\,) \times\,]\gamma_j, \Omega]\,) \subset (G^j_1 \times [0, \Omega]\,)^{\sigma}.$$

It follows that for every $y \in\,]\gamma_j, \Omega]$,

$$G^j_1 \times [0, \Omega] \in \sigma\, \langle \mathscr{M}_{(\alpha_j, y)} \rangle.$$

Since $\sigma\, \langle \mathscr{M}_{(\alpha_j, y)} \rangle \supset \mathscr{M}_{(\alpha_j, y)}$, we have

$$G^j_1 \times \{\,y\} \in \sigma\, \langle \mathscr{M}_{(\alpha_j, y)} \rangle.$$

Take

$$\gamma_0 = \sup_{j \in \mathbf{N}_0} \gamma_j$$

then we obtain that

$$(1) \quad G^j_1 \times \{\gamma_0\} \in \sigma(\,\langle \mathscr{M}_{\alpha_j, \gamma_0)} \rangle\,)$$

for every $j \in \mathbf{N}_0$.

On the other hand we also have

$$\mathscr{M}^{\sigma}_{(\alpha, \gamma_0)} \subset \varphi(\mathscr{B}(\,(\alpha, \gamma_0)\,)\,).$$

There exists an $m \in \mathbf{N}_0$ such that

$$\varphi(F_m(\gamma_0) \cup \{\,(\alpha, \gamma_0)\,\} \cup \{\,(\alpha_j, \gamma_0)\,|\,j \geqq m\} \subset (F_1(\gamma_0)\,)^{\sigma}.$$

Then

$$F_1(\gamma_0) \in \sigma\, \langle \mathscr{M}_{(\alpha_m, \gamma_0)} \rangle$$

and using (1) we have

$$F_1(\gamma_0) \cap (G^m_1 \times \{\gamma_0\}\,) \in \sigma\, \langle \mathscr{M}_{(\alpha_m, \gamma_0)} \rangle$$

which in view of the disjointness of the chosen sequences is a

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contradiction. In particular $K(D, \mathscr{C})$ is not regular and using Theorem 3.1, we obtain that $(D, \mathscr{C})$ is not relatively round.

Example 7.2. A $C^{\wedge}$-embedded topological Cauchy space, with a regular completion in $\mathscr{R}$, equivalent to the natural completion, which has a non-regular Kowalsky completion and is not relatively round.

On the real line we choose α, $(\alpha_j)_{j \in \mathbf{N}_0}$, $(q^j_n)_{n \in \mathbf{N}_0}$ for $j \in \mathbf{N}_0$, and X as in the previous example. X is equipped with the pretopological structure p with neighborhood filters defined in the following way.

$$\mathscr{B}(q) = \dot{q} \text{ for } q \in \mathbf{Q}$$

$$\mathscr{B}(\alpha) = [\{F_n \cup \{\alpha\} \cup \{\alpha_j \mid j \geqq n\} \mid n \in \mathbf{N}_0\}] \quad \text{where}$$

$$F_n = \bigcup_{j=n}^{\infty} \{q^j_k \mid k \in \mathbf{N}_0\}$$

$$\mathscr{B}(\alpha_j) = [\{]\alpha_j - \frac{1}{n}, \alpha_j + \frac{1}{n}[\cap X \mid n \in \mathbf{N}_0\}].$$

Let $E = [\alpha, +\infty[\cap \mathbf{Q}$ be equipped with the Cauchy structure $\mathscr{C}$ induced by (X, p). Then $(E, \mathscr{C})$ is a topological Cauchy space and $\mathscr{M}_x = \mathscr{B}(x)|_E$ is the minimal Cauchy filter corresponding to $x \in X$. (X, p) is a strict completion, and p is finer than the trace of the discrete rational extension topology τ on X. $\mathscr{B}(x)$ has a τ-closed base for $x \in X$. It follows that (X, p) is C-embedded and (E, C) is $C^{\wedge}$-embedded.

For $\sigma: E^* \to F(E)$, $\sigma(\langle \dot{x} \rangle) = \dot{x}$ if $x \in E$, $\sigma\langle \mathscr{M}_\alpha \rangle = \mathscr{M}_\alpha$ and

$$\sigma\langle \mathscr{M}_{\alpha_j} \rangle = [\{\{q^j_k \mid k \geqq m\} \mid m \in \mathbf{N}_0\}] \quad \text{for } j \in \mathbf{N}_0,$$

we have $s_\sigma(\dot{x}) = \dot{x}$, $s_\sigma \mathscr{M}_\alpha = \mathscr{M}_\alpha$ and $s_\sigma \mathscr{M}_{\alpha_j} = \mathscr{M}_{\alpha_j}$. Hence $K_\sigma(E, \mathscr{C})$ is regular.

For $\lambda: E^* \to F(E)$, $\lambda(\langle \dot{x} \rangle) = \dot{x}$ if $x \in E$, $\lambda\langle \mathscr{M}_\alpha \rangle = \mathscr{M}_\alpha$ and $\lambda\langle \mathscr{M}_{\alpha_j} \rangle = \mathscr{M}_{\alpha_j}$ for $j \in \mathbf{N}_0$ we have $s_\lambda \mathscr{M}_\alpha \neq \mathscr{M}_\alpha$. It follows that $K(E, \mathscr{C})$ is not regular and that $(E, \mathscr{C})$ is not relatively round. The completions (X, p), $N(E, \mathscr{C})$, $R(E, \mathscr{C})$ and $K_\sigma(E, \mathscr{C})$ are all equivalent.

Example 7.3. A totally bounded pseudotopological relatively round Cauchy space with no regular completion.

Let X be an infinite set. Fix a nonprincipal ultrafilter $\mathscr{U}_0$ and let $\mathscr{C}$ be the following Cauchy structure on X: A filter $\mathscr{F}$ belongs to $\mathscr{C}$ if and only if $\mathscr{F} = \dot{x}$ for some $x \in X$ or $\mathscr{F} = \mathscr{U}_0$ or every ultrafilter finer than $\mathscr{F}$ is nonprincipal and different from $\mathscr{U}_0$. $(X, \mathscr{C})$ clearly is totally bounded, pseudo-topological and regular. $(X, \mathscr{C})$ is not pretopological since

$$\mathscr{U}_0 \supset \cap \{\mathscr{U} \mid \mathscr{U} \text{ nonprincipal}, \mathscr{U} \neq \mathscr{U}_0\}$$

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and in view of Theorem 5.1, $(X, \mathscr{C})$ has no regular completion. However, $(X, \mathscr{C})$ is relatively round.

Example 7.4. A $C^{\wedge}$-embedded Cauchy space which is not completely regular, for which the Kowalsky and natural completion are equivalent.

Let X be any C-embedded non completely regular topological space and $\mathscr{C}$ the complete compatible Cauchy structure.

Example 7.5. [**3**] A $C^{\wedge}$-embedded Cauchy space with a strict regular completion and with a non strict natural completion.

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