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# THE RELATIVE PICARD GROUP OF A COMODULE ALGEBRA AND HARRISON COHOMOLOGY

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*Abstract* Let $A$ be a commutative comodule algebra over a commutative bialgebra $H$. The group of invertible relative Hopf modules maps to the Picard group of $A$, and the kernel is described as a quotient group of the group of invertible group-like elements of the coring $A \otimes H$, or as a Harrison cohomology group. Our methods are based on elementary $K$-theory. The Hilbert 90 theorem follows as a corollary. The part of the Picard group of the coinvariants that becomes trivial after base extension embeds in the Harrison cohomology group, and the image is contained in a well-defined subgroup $E$. It equals $E$ if $H$ is a cosemisimple Hopf algebra over a field.

*Keywords:* Picard group; coring; Harrison cohomology

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## 1. Introduction

Let $l$ be a cyclic Galois field extension of $k$. The Hilbert 90 theorem tells us that every cocycle in $Z^1(C_p, l^*)$ is a coboundary. There exist various generalizations of this result. For example, if we have a Galois extension $B \to A$ of commutative rings, with Galois group $G$, then the cohomology group $H^1(G, \mathbb{G}_m(A))$ is isomorphic to $\mathrm{Pic}(A/B)$, the kernel of the natural map from the Picard group of $B$ to the Picard group of $A$ (see, for example, [**9**]). Now we can ask the following question. Suppose that $G$ acts on $A$ as a group of isomorphisms. Can we still give an algebraic interpretation of $H^1(G, \mathbb{G}_m(A))$? A second problem is whether there is any relation between $H^1(G, \mathbb{G}_m(A))$ and the Picard group of the ring of invariants $B = A^G$.

In this paper we will discuss these two problems in a more general situation: we will assume that $A$ is a commutative $H$-comodule algebra, with $H$ an arbitrary commutative bialgebra over a commutative ring $k$. We then ask for an algebraic interpretation of the first Harrison cohomology group $H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$ (with notation as in [**6**]). If $H$ is finitely generated and projective, then this Harrison cohomology group is isomorphic to a Sweedler cohomology group $Z^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$, and if $H = \mathbb{Z}G$ with $G$ a finite group, then it reduces to the cohomology group $H^1(G, \mathbb{G}_m(A))$.

We proceed as follows: we introduce the relative Picard group $\mathrm{Pic}^H(A)$ as the Grothendieck group of the category of invertible relative Hopf modules. The forgetful functor to the category of invertible $A$-modules induces a $K$-theoretic exact sequence, linking the Picard group of $A$, the relative Picard group, and the groups of unit elements of $A$ and the coinvariants $B = A^{\mathrm{co}\,H}$; the middle term in the sequence can be computed, and it is the group of invertible group-like elements of the coring $A \otimes H$. We show also that these group-like elements are precisely the Harrison cocycles, and it follows from the exactness of the sequence that the first Harrison cohomology group is the kernel of the map $\mathrm{Pic}^H(A) \to \mathrm{Pic}(A)$, answering our first question.

Then we observe that there is a similar exact sequence associated with the induction functor $\underline{\underline{\mathrm{Pic}}}(B) \to \underline{\underline{\mathrm{Pic}}}(A)$, and that the two exact sequences fit into a commutative diagram. If $A$ is a faithfully flat Hopf Galois extension of $B$, then the categories of $B$-modules and relative Hopf modules are equivalent, hence $\mathrm{Pic}(B) \cong \mathrm{Pic}^H(A)$, and we recover Hilbert 90. In general, we have an injection $\mathrm{Pic}(A/B) \to H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$, and we can describe a subgroup of $H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$ that contains the image of $\mathrm{Pic}(A/B)$. The image is precisely this subgroup if $H$ is a cosemisimple Hopf algebra over a field $k$.

A special situation is the following: let $k$ be an algebraically closed field, $A$ a finitely generated commutative normal $k$-algebra, and $G$ a connected algebraic group acting rationally on $A$. Then $A$ is an $H$-comodule algebra, with $H$ the affine coordinate ring of $G$. In this case, our exact sequence was given by Magid in [**14**], but apparently Magid was not aware of the connection to Harrison cohomology, group-like elements of corings or the generalized Hilbert 90 theorem.

In § 5, we study the Harrison cocycles (or the group-like elements in $A \otimes H$) in some particular cases. First we look at the situation considered by Magid in [**14**], and then it turns out that the group-like elements of $G(A \otimes H)$ are induced by the group-like elements of $H$. In the situation where $A$ is a $\mathbb{Z}$-graded commutative $k$-algebra, the relative Picard group turns out to be the graded Picard group $\mathrm{Pic}_{\mathrm{g}}(A)$ studied by the first author in [**5**]. If $A$ is reduced, then the group-like elements of $A \otimes H$ can also be described using the group-like elements of $H$, according to a result in [**5**].

## 2. Preliminary results

### 2.1. The language of corings

Relative Hopf modules can be viewed as comodules over a coring. This will be used below, and this is why we briefly recall some properties of corings. Recall that an $A$-coring is a comonoid in the monoidal category ${}_A\mathcal{M}_A$ of $A$-bimodules. Thus an $A$-coring $\mathfrak{C}$ is an $A$-bimodule together with two $A$-bimodule maps,

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C} \quad \text{and} \quad \varepsilon_{\mathfrak{C}} : \mathfrak{C} \to A,$$

satisfying the usual coassociativity and counit properties. We refer to [**2**–**4**, **11**, **18**] for a detailed discussion of corings. The set of group-like elements of $\mathfrak{C}$ is given by

$$G(\mathfrak{C}) = \{X \in \mathfrak{C} \mid \Delta_{\mathfrak{C}}(X) = X \otimes_A X \text{ and } \varepsilon_{\mathfrak{C}}(X) = 1\}.$$

A right $\mathfrak{C}$-comodule $M$ is a right $A$-module together with a right $A$-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$ satisfying

$$(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M \quad \text{and} \quad (M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M.$$

A morphism of right $\mathfrak{C}$-comodules $f : M \to N$ is an $A$-linear map $f$ such that

$$\rho_N \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M.$$

$\mathcal{M}^{\mathfrak{C}}$ will be the category of right $\mathfrak{C}$-comodules and comodule morphisms. We have the following interpretation of the group-like elements of $\mathfrak{C}$.

**Lemma 2.1.** *Let $\mathfrak{C}$ be an $A$-coring. Then there is a bijective correspondence between $G(\mathfrak{C})$ and the set of maps $\rho : A \to A \otimes_A \mathfrak{C} = \mathfrak{C}$, making $A$ into a right $\mathfrak{C}$-comodule. The coaction $\rho_X$ corresponding to $X \in G(\mathfrak{C})$ is given by*

$$\rho_X(a) = Xa.$$

*With this notation, $A^X = (A, \rho_X)$ is isomorphic to $A^Y = (A, \rho_Y)$ as a right $\mathfrak{C}$-comodule if and only if there exists an invertible $b \in A$ such that $\rho_Y(b) = Yb = bX$.*

**Proof.** The first part is well known (and straightforward) (see, for example, [**3**]). Let $f : A^X \to A^Y$ be a right $\mathfrak{C}$-colinear isomorphism. Then $f(a) = ba$ for some $b \in A$, which is invertible since $f$ is an isomorphism. The fact that $f$ is $\mathfrak{C}$-colinear tells us that

$$Yb = \rho_Y(f(1)) = (f \otimes_A \mathfrak{C})(\rho_X(1)) = bX.$$

The converse property is obvious. □

### 2.2. Relative Hopf modules

Let $H$ be a bialgebra over a commutative ring $k$, and $A$ a right $H$-comodule algebra. Throughout this paper we will assume that $H$ and $A$ are commutative. Then $A$ is a commutative algebra and we have a right $H$-coaction $\rho$ on $A$ such that

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]},$$

for all $a, b \in A$. Here we use the Sweedler–Heyneman notation for the coaction $\rho$: $\rho(a) = a_{[0]} \otimes a_{[1]}$, with summation implicitly understood. For the comultiplication on $H$, we use the notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

A relative Hopf module $M$ is a $k$-module, together with a right $A$-action and a right $C$-coaction $\rho_M$ such that

$$\rho_M(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]},$$

for all $a \in A$ and $m \in M$. The category of relative Hopf modules and $A$-linear $H$-colinear maps will be denoted by $\mathcal{M}_A^H$. The coinvariant submodule $M^{\operatorname{co} H}$ of $M \in \mathcal{M}_A^H$ is defined by

$$M^{\operatorname{co} H} = \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

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$A^{\mathrm{co}\,H} = B$ is a $k$-subalgebra of $A$, and $M^{\mathrm{co}\,H}$ is a $B$-module. We obtain a functor $(\cdot)^{\mathrm{co}\,H} : \mathcal{M}_A^H \to \mathcal{M}_B$, which has a left adjoint $T = -\otimes_B A : \mathcal{M}_B \to \mathcal{M}_A^H$. The right $H$-coaction on $N \otimes_B A$ is $N \otimes_B \rho$. The unit $u$ and counit $c$ of the adjunction are given by the following formulae, for $N \in \mathcal{M}_B$ and $M \in \mathcal{M}_A^H$:

$$u_N : N \to (N \otimes_B A)^{\mathrm{co}\,H}, \quad u_N(n) = n \otimes 1,$$
$$c_M : M^{\mathrm{co}\,H} \otimes_B A \to M, \quad c_M(m \otimes a) = ma.$$

$A$ is called a Hopf algebra extension of $B = A^{\mathrm{co}\,H}$ if the canonical map

$$\mathrm{can} : A \otimes_B A \to A \otimes H, \quad \mathrm{can}(a \otimes_B b) = ab_{[0]} \otimes b_{[1]}$$

is an isomorphism. If $A$ is a faithfully flat Hopf Galois extension, then the adjunction $(-\otimes_B A, (\cdot)^{\mathrm{co}\,H})$ is a pair of inverse equivalences. We refer to [**10**, **15**, **17**] for a detailed discussion of Hopf algebras and relative Hopf modules.

$\mathfrak{C} = A \otimes H$ is a coring, with structure maps

$$a'(b \otimes h)a = a'ba_{[0]} \otimes ha_{[1]},$$
$$\Delta_{\mathfrak{C}}(a \otimes h) = (a \otimes h_{(1)}) \otimes_A (1 \otimes h_{(2)}),$$
$$\varepsilon_{\mathfrak{C}}(a \otimes h) = a\varepsilon(h).$$

The category $\mathcal{M}^{A\otimes H}$ is isomorphic to the category $\mathcal{M}_A^H$ of relative Hopf modules; we refer to [**2**, **4**] for full details. Note that $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$ if and only if

$$\sum_i (a_i \otimes h_{i(1)} \otimes h_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes h_i a_{j[1]} \otimes h_j) \quad \text{and} \quad \sum a_i \varepsilon(h_i) = 1. \tag{2.1}$$

$A \otimes H$ is also a commutative algebra, with multiplication

$$(a \otimes h)(b \otimes k) = ab \otimes hk.$$

The product of two group-like elements is a group-like element, and $1_A \otimes 1_H$ is group-like. Hence $G^i(A \otimes H)$, the set of invertible group-like elements, is an abelian group. Also observe that an invertible group-like element is precisely a normalized Harrison 1-cocycle (see, for example, [**6**, § 9.2] for the definition of the Harrison complex).

Let $H$ be a finitely generated projective cocommutative Hopf algebra, and let $A$ be a commutative left $H$-module algebra. Then $H^*$ is a commutative Hopf algebra and $A$ is a right $H^*$-comodule algebra. If $\sum_i a_i \otimes f_i \in A \otimes H^*$ is an invertible group-like element (or a normalized Harrison cocycle), then

$$\phi : H \to A, \quad \phi(h) = \sum_i a_i f_i(h), \tag{2.2}$$

is a normalized Sweedler 1-cocycle. This means that $\phi(1_H) = 1_A$, and the cocycle condition

$$\phi(hh') = \sum_i (h_{(1)} \cdot (\phi(h')))\phi(h_{(2)}) \tag{2.3}$$

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is satisfied. This gives a bijective correspondence between Harrison and Sweedler cocycles, see [**6**, Proposition 9.2.3]. For the definition of the Sweedler complex, see [**16**] or [**6**, § 9.1]. In the case where $H = kG$, with $G$ a finite group, Sweedler cohomology reduces to group cohomology.

### 2.3. Elementary algebraic $K$-theory

Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, J)$ be skeletally small symmetric monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a cofinal, strong monoidal functor. Then we can consider the Grothendieck and Whitehead groups of $\mathcal{C}$ and $\mathcal{D}$, and we have an exact sequence connecting them (see, for example, [**1**, Chapter VII]):

$$K_1\mathcal{C} \xrightarrow{K_1F} K_1\mathcal{D} \xrightarrow{d} K_1\underline{\phi F} \xrightarrow{g} K_0\mathcal{C} \xrightarrow{K_0F} K_0\mathcal{D}. \tag{2.4}$$

$C \in \mathcal{C}$ is called invertible if there exists $C' \in \mathcal{C}$ such that $C \otimes C' \cong I$. If all elements of $\mathcal{C}$ and $\mathcal{D}$ are invertible, then the description of the five groups in (2.4) and the connecting maps simplifies (see [**6**, Appendix C]). $K_0\mathcal{C}$ is the group of isomorphism classes of objects in $\mathcal{C}$ and $K_1\mathcal{C} \cong \mathrm{Aut}_{\mathcal{C}}(I)$ (which is then an abelian group). Let $\underline{\Psi F}$ be the following category: objects are couples $(C, \alpha)$, with $C \in \mathcal{C}$ and $\alpha : F(C) \to J$ an isomorphism in $\mathcal{D}$. A morphism between $(C, \alpha)$ and $(C', \alpha')$ is an isomorphism $f : C \to C'$ in $\mathcal{C}$ such that $\alpha' = F(f) \circ \alpha$. $\underline{\Psi F}$ is monoidal, every object is invertible and

$$K_1\underline{\phi F} \xrightarrow{g} \cong K_0\underline{\Psi F}.$$

The maps $d$ and $g$ are given as follows: $d(\alpha) = [(I, \alpha)]$ and $g[(C, \alpha)] = [C]$.

A typical example is the following: for a commutative ring $A$, let $\underline{\underline{\mathrm{Pic}}}(A)$ be the category of invertible $A$-modules. If $i : B \to A$ is a morphism of commutative rings, then we have the cofinal strongly monoidal functor

$$G = - \otimes_B A : \underline{\underline{\mathrm{Pic}}}(B) \to \underline{\underline{\mathrm{Pic}}}(A),$$

and (2.4) takes the form

$$1 \to \mathbb{G}_m(B) \to \mathbb{G}_m(A) \xrightarrow{d'} K_1\underline{\phi G} \xrightarrow{g'} \mathrm{Pic}(B) \to \mathrm{Pic}(A). \tag{2.5}$$

## 3. The relative Picard group

If $M, N \in \mathcal{M}_A^H$, then $M \otimes_A N \in \mathcal{M}_A^H$, with right $H$-coaction

$$\rho_{M\otimes_A N}(m \otimes_A n) = m_{[0]} \otimes_A n_{[0]} \otimes m_{[1]}n_{[1]}.$$

So we have a symmetric monoidal category $(\mathcal{M}_A^H, \otimes_A, A)$. Let $\underline{\underline{\mathrm{Pic}}}^H(A)$ be the full subcategory consisting of invertible objects. $\mathrm{Pic}^H(A) = K_0\underline{\underline{\mathrm{Pic}}}^H(A)$, the group of isomorphism classes of relative Hopf modules, will be called the relative Picard group of $A$ and $H$. The isomorphism class in $\mathrm{Pic}^H(A)$ represented by an invertible relative Hopf module $M$ will be denoted by $\{M\}$. This new invariant fits into an exact sequence.

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**Proposition 3.1.** *We have an exact sequence*

$$1 \to \mathbb{G}_m(B) \to \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H) \xrightarrow{g} \mathrm{Pic}^H(A) \to \mathrm{Pic}(A). \tag{3.1}$$

**Proof.** This result can be proved in two ways: a first possibility is to show that (3.1) is precisely the exact sequence (2.4), associated with the functor $\underline{\underline{\mathrm{Pic}}}^H(A) \to \underline{\underline{\mathrm{Pic}}}(A)$ forgetting the $H$-coaction. Let us present an easy direct proof.

The map $\mathbb{G}_m(B) \to \mathbb{G}_m(A)$ is the natural inclusion. Take $a \in A$ invertible, and let $d(a) = X = a^{-1}a_{[0]} \otimes a_{[1]}$. $X$ is group-like, since $a^{-1}a_{[0]}\varepsilon(a_{[1]}) = 1$, and

$$\begin{aligned} X \otimes_A X &= (a^{-1}a_{[0]} \otimes a_{[1]}) \otimes_A (b^{-1}b_{[0]} \otimes b_{[1]}) \\ &= a^{-1}a_{[0]}(b^{-1})_{[0]}b_{[0]} \otimes a_{[1]}(b^{-1})_{[1]}b_{[1]} \otimes b_{[2]} \\ &= a^{-1}b_{[0]} \otimes b_{[1]} \otimes b_{[2]} \\ &= (a^{-1}b_{[0]} \otimes b_{[1]}) \otimes_A (1 \otimes b_{[2]}) = \Delta(X), \end{aligned}$$

where we identified $(A \otimes H) \otimes_A (A \otimes H) = A \otimes H \otimes H$ and we wrote $a = b$. The inverse of $X$ is $X^{-1} = a(a^{-1})_{[0]} \otimes (a^{-1})_{[1]}$, so $X \in G^i(A \otimes H)$.

If $d(a) = a^{-1}a_{[0]} \otimes a_{[1]} = 1_A \otimes 1_H$, then $a_{[0]} \otimes a_{[1]} = a \otimes 1_H$, so $a \in B$, and the sequence is exact at $\mathbb{G}_m(A)$.

For $X \in G^i(A \otimes H)$, let $g(X) = A^X$, with notation as in Lemma 2.1. $g$ is multiplicative: take $X = \sum_i a_i \otimes h_i$ and $Y = \sum_j b_j \otimes k_j$ in $G^i(A \otimes H)$, then $A^X \otimes_A A^Y = A$ as an $A$-bimodule, with comultiplication given by

$$\rho_{A^X \otimes_A A^Y}(1) = \sum_{i,j} a_i \otimes_A b_j \otimes h_i k_j = XY,$$

as needed.

If $g(X) = \{A\}$ in $\mathrm{Pic}^H(A)$, then there exists an $H$-colinear $A$-linear isomorphism $f : A^X \to A$. Then $f(1) = a$ is invertible in $A$, and, since $f$ is $H$-colinear, $a_{[0]} \otimes a_{[1]} = \rho(a) = (f \otimes H)(X) = aX$, so $X = a^{-1}a_{[0]} \otimes a_{[1]} = d(a)$, and the sequence is also exact at $G^i(A \otimes H)$.

The exactness of the sequence at $\mathrm{Pic}^H(A)$ follows from Lemma 2.1. □

**Remark 3.2.** Let $H = k\mathbb{Z}$, and let $A$ be a commutative $\mathbb{Z}$-graded $k$-algebra. Then $\mathrm{Pic}^H(A) = \mathrm{Pic}_{\mathrm{g}}(A)$, the graded Picard group of $A$, as introduced in [**5**] (see also [**8**]). The exact sequence (3.1) reduces to the exact sequence in [**5**, Proposition 2.1].

The map $d : \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H)$ is precisely the map $\mathbb{G}_m(A) \to \mathbb{G}_m(A \otimes H)$ in the Harrison complex. From Proposition 3.1, we therefore immediately obtain the following corollary.

**Corollary 3.3.** *With $H$ and $A$ as in Proposition 3.1, we have an isomorphism of abelian groups*

$$\mathrm{Pic}^H(A) \cong H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m).$$

This is the promised algebraic interpretation of the first Harrison cohomology group. Note that there are no flatness or projectivity assumptions on $H$ or $A$. We have Hilbert 90 as an easy consequence.

**Corollary 3.4 (Hilbert 90).** *Let $H$, $A$, $B$ be as in Proposition 3.1. If $A$ is a faithfully flat $H$-Galois extension of $B$, then we have an isomorphism of abelian groups:*

$$\mathrm{Pic}(A/B) \cong H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m).$$

**Proof.** From the fact that the monoidal categories $\mathcal{M}_B$ and $\mathcal{M}_A^H$ are equivalent, it follows that $\mathrm{Pic}(B) \cong \mathrm{Pic}^H(A)$. □

Take the exact sequences (2.5) and (3.1), and observe that they fit into a commutative diagram:

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d'} & K_1\underline{\phi G} & \xrightarrow{g'} & \mathrm{Pic}(B) & \longrightarrow & \mathrm{Pic}(A) \\
 & & \downarrow{=} & & \downarrow{=} & & & & \downarrow{j} & & \downarrow{=} \\
1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d} & G^i(A \otimes H) & \xrightarrow{g} & \mathrm{Pic}^H(A) & \longrightarrow & \mathrm{Pic}(A)
\end{array}$$

The map $j$ maps $[N] \in \mathrm{Pic}(B)$ to $\{N \otimes_B A\} \in \mathrm{Pic}^H(A)$. Using the 'five lemma', we find a map $i : K_1\underline{\phi G} \to G^i(A \otimes H)$.

**Lemma 3.5.** *With notation as above, the maps $i$ and $j$ are injective.*

**Proof.** From the fact that $u$ is a natural transformation between additive endofunctors of the category of $B$-modules, and since $u_B$ is an isomorphism, it follows that $u_N : N \to (N \otimes_B A)^{\mathrm{co}\,H}$ is an isomorphism if $N$ is finitely generated and projective as a $B$-module. So if $N \otimes_B A \cong A$, then $N \cong (N \otimes_B A)^{\mathrm{co}\,H} \cong A^{\mathrm{co}\,H} = B$, and $j$ is injective. The injectivity of $i$ then follows from an easy diagram-chasing argument. □

Our next aim is to characterize the image of $i$. This will be the topic of § 4; it will turn out that we obtain nice results in the case where $H$ is cosemisimple.

## 4. Coinvariantly generated relative Hopf modules

Some of our results will be more specific if we assume that $H$ is a cosemisimple Hopf algebra over a field $k$. Recall that $H$ is cosemisimple if there exists a left integral $\phi$ on $H^*$ such that $\phi(1) = 1$ (see, for example, [**18**]). In this case, the coinvariants functor $(\cdot)^{\mathrm{co}\,H} : \mathcal{M}_A^H \to \mathcal{M}_B$ is exact (see [**15**, Lemma 2.4.3]).

A relative Hopf module $M$ is called *coinvariantly generated* if $c_M$ is surjective, or, equivalently, if $M = M^{\mathrm{co}\,H}A$. If $M$ is coinvariantly generated, and finitely generated as an $A$-module, then we can find a finite set $\{m_1, \dots, m_n\} \in M^{\mathrm{co}\,H}$ that generates $M$.

It follows immediately from the properties of adjoint functors that $N \otimes_B A$ is coinvariantly generated, for every $N \in \mathcal{M}_B$; in particular, $A$ is coinvariantly generated. We also have the following lemma.

**Lemma 4.1.** *Let $M \in \mathcal{M}_A^H$ and $N \in \mathcal{M}_B$. If $M$ is an epimorphic image of $N \otimes_B A$ in $\mathcal{M}_A^H$, then $M^{\mathrm{co}\,H} = 0$ implies that $M = 0$.*

**Proof.** If $M^{\mathrm{co}\,H}=0$, then

$$\mathrm{Hom}_A^H(N\otimes_B A, M)=\mathrm{Hom}_B(N, M^{\mathrm{co}\,H})=0.$$

But $\mathrm{Hom}_A^H(N\otimes_B A, M)$ contains the epimorphism of relative Hopf modules $N\otimes_B A\to M$, so $M=0$. □

If $N$ is an epimorphic image of $M$ in $\mathcal{M}_A^H$, and if $M$ is coinvariantly generated, then $N$ is also coinvariantly generated.

**Lemma 4.2.** *Assume that $H$ is a cosemisimple Hopf algebra over a field $k$. If $N\in\mathcal{M}_B$ is projective, then $N\otimes_B A$ is projective in $\mathcal{M}_A^H$.*

**Proof.** See [**7**, Proposition 2.5]. □

**Lemma 4.3.** *Let $k$ be a field.*

(1) *The forgetful functor $\mathcal{M}_A^H\to\mathcal{M}_A$ preserves projectives.*

(2) *If $H$ is cosemisimple, then the forgetful functor also reflects projectivity of finitely generated modules.*

**Proof.** (1) Take $M\in\mathcal{M}_A^H$ projective, and consider the epimorphism $p: M\otimes A\to M$, $p(m\otimes a)=ma$ in $\mathcal{M}_A^H$. The exact sequence

$$0\to \mathrm{Ker}\,p\to M\otimes A\xrightarrow{p} M\to 0$$

splits in $\mathcal{M}_A^H$, since $M$ is a projective object, and *a fortiori* in $\mathcal{M}_A$. Hence $M$ is a direct factor of $M\otimes A$, which is a projective $A$-module, so $M$ is also a projective $A$-module.

(2) Let $M$ and $N$ be relative Hopf modules, and assume that $M$ is finitely generated and projective in $\mathcal{M}_A$. According to [**7**, Proposition 4.2], $\mathrm{Hom}_A(M,N)\in\mathcal{M}_A^H$, and it is easy to show that $\mathrm{Hom}_A(M,N)^{\mathrm{co}\,H}=\mathrm{Hom}_A^H(M,N)$. It follows that the functor $\mathrm{Hom}_A^H(M,-):\mathcal{M}_A^H\to\mathcal{M}$ is exact, since it is the composition of the exact functors $\mathrm{Hom}_A(M,-):\mathcal{M}_A^H\to\mathcal{M}^H$ ($M\in\mathcal{M}_A$ is projective) and $(\cdot)^{\mathrm{co}\,H}:\mathcal{M}^H\to\mathcal{M}$ ($H$ is cosemisimple). □

**Lemma 4.4.** *Let $H$ be a cosemisimple Hopf algebra over a field $k$, and take $P,Q\in\mathcal{M}_A^H$ finitely generated as $A$-modules. Assume that $Q$ is a projective object of $\mathcal{M}_A^H$. Then every epimorphism $f:P\to Q$ in $\mathcal{M}_A^H$ has a right inverse in $\mathcal{M}_A^H$.*

**Proof.** It is clear that $\mathrm{Hom}_A(Q,P)$ and $\mathrm{Hom}_A(Q,Q)$ are right $H$-comodules, and the map

$$f^*=\mathrm{Hom}_A(Q,f):\mathrm{Hom}_A(Q,P)\to\mathrm{Hom}_A(Q,Q)$$

is right $H$-colinear. It follows from Lemma 4.3 that $Q$ is projective as an $A$-module, so $f^*$ is surjective. Since $f^*$ is $H$-colinear, $f^*$ restricts to a surjection

$$\mathrm{Hom}_A^H(Q,P)=\mathrm{Hom}_A(Q,P)^{\mathrm{co}\,H}\to\mathrm{Hom}_A^H(Q,Q)=\mathrm{Hom}_A(Q,Q)^{\mathrm{co}\,H}.$$

Take a preimage $g\in\mathrm{Hom}_A^H(Q,P)$ of the identity map $\mathrm{id}_Q$ on $Q$. Then $f\circ g=\mathrm{id}_Q$, and the result follows. □

For $M \in \mathcal{M}_A$, we will denote the dual module by $M^* = \mathrm{Hom}_A(M, A)$.

**Proposition 4.5.** *Let $H$ be cosemisimple, and assume that $P \in \mathcal{M}_A^H$ is coinvariantly generated and finitely generated projective as an $A$-module. Then*

(1) *$P^{\mathrm{co}\,H}$ is a finitely generated projective $B$-module;*

(2) *$P^*$ is coinvariantly generated;*

(3) *the map $c_P$ is an isomorphism in $\mathcal{M}_A^H$.*

**Proof.** (1) As we have seen, there exist $p_1, p_2, \dots, p_n \in P^{\mathrm{co}\,H}$ such that $P = \sum_i p_i A$. Set $F = A^n$ and let $f : F \to P$ be the $A$-linear map given by $f(a_1, a_2, \dots, a_n) = \sum_i p_i a_i$. Then $F \in \mathcal{M}_A^H$ and $f$ is an epimorphism in $\mathcal{M}_A^H$. By Lemma 4.4, there exists a monomorphism $g \in \mathrm{Hom}_A(P, F)$ such that $f \circ g = \mathrm{id}_P$. The restriction of $g$ to $P^{\mathrm{co}\,H}$ is then a $B$-linear right inverse of the restriction of $f$ to $F^{\mathrm{co}\,H}$, and $F^{\mathrm{co}\,H} = B^n$, and we obtain (1).

(2) The map $g^* = \mathrm{Hom}_A(g, A) : F^* \to P^*$ is surjective and $H$-colinear. The fact that $F^*$ is coinvariantly generated then implies that $P^*$ is also coinvariantly generated.

(3) Consider the natural transformation $t : (\cdot)^{\mathrm{co}\,H} \otimes_B A \to (\cdot)$ given by

$$t_P : P^{\mathrm{co}\,H} \otimes_B A \to P, \quad t_P(p \otimes a) = pa.$$

The map $t_A$ is an isomorphism, so $t_F$ is an isomorphism by additivity. It follows that $t_P$ is an isomorphism, since $F = P \oplus \mathrm{Ker}\, f$ as $H$-comodules. □

Let $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$, and write

$$A_X = \left\{ a \in A \;\middle|\; \rho(a) = aX = \sum_i a a_i \otimes h_i \right\}$$

and

$$A_X^i = \{a \in A_X \mid a \text{ is invertible}\}.$$

Observe that

$$\mathrm{Im}(d) = \{X \in G^i(A \otimes H) \mid A_X^i \neq \emptyset\}$$

and

$$A_{1 \otimes 1} = A^{\mathrm{co}\,H}.$$

Furthermore, $A_X A_Y \subset A_{XY}$: take $a \in A_X$ and $b \in A_Y$, then $\rho(a) = aX = \sum_i a a_i \otimes h_i$, $\rho(b) = bY = \sum_j b b_j \otimes k_j$ and

$$\rho(ab) = a_{[0]} b_{[0]} \otimes a_{[1]} b_{[1]} = \sum_{i,j} a a_i b b_j \otimes h_i k_j = abXY.$$

Also $A_X^i \cap A_Y^i = \emptyset$ if $X \neq Y$.

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**Lemma 4.6.** *The set*

$$E = \{X \in G^i(A \otimes H) \mid AA_X = A \text{ and } AA_{X^{-1}} = A\}$$

*is a subgroup of* $G^i(A \otimes H)$ *containing* $\mathrm{Im}(d)$.

**Proof.** If $X \in \mathrm{Im}(d)$, then there exists an invertible $a \in A_X$, and then $AA_X = A$. Since $X^{-1} \in \mathrm{Im}(d)$, we also have $AA_{X^{-1}} = A$, hence $X \in E$. It is clear that $1 \otimes 1 \in E$. If $X, Y \in E$, then $AA_{XY} \supset AA_X A_Y = AA_Y = A$, and, in a similar way, $AA_{(XY)^{-1}} = A$, hence $XY \in E$. Finally, if $X \in E$, then obviously $X^{-1} \in E$. □

**Proposition 4.7.** *Consider the injective map* $j : \mathrm{Pic}(B) \to \mathrm{Pic}^H(A)$. *If* $H$ *is a cosemisimple Hopf algebra over a field* $k$, *then*

$$\mathrm{Im}(j) = \{\{M\} \in \mathrm{Pic}^H(A) \mid M \text{ is coinvariantly generated}\}.$$

**Proof.** $M \otimes_B A$ is coinvariantly generated, so $\mathrm{Im}(j)$ is contained in the desired set. If $H$ is cosemisimple, and $\{N\} \in \mathrm{Pic}^H(A)$, with $N$ coinvariantly generated, then $N = (N^{\mathrm{co}\,H}) \otimes_B A \in \mathrm{Im}(j)$, by Proposition 4.5 (3). □

**Lemma 4.8.** *Take* $X \in G^i(A \otimes H)$. *Then* $A^X$ *is coinvariantly generated if and only if* $AA_{X^{-1}} = A$. *If* $H$ *is cosemisimple, then this is also equivalent to* $X \in E$.

**Proof.** The first statement follows from the fact that $(A^X)^{\mathrm{co}\,H} = A_{X^{-1}}$. Indeed, $a \in (A^X)^{\mathrm{co}\,H}$ if and only if $\rho_X(a) = Xa = a \otimes 1$, if and only if $\rho(a) = (1 \otimes 1)a = X^{-1}(a \otimes 1) = aX^{-1}$, which means that $a \in A_{X^{-1}}$.

Let $H$ be cosemisimple. Note that $(A^X)^* \cong A^{X^{-1}}$ as relative Hopf modules. If $A^X$ is coinvariantly generated, then so is $A^{X^{-1}}$, by Proposition 4.5, and then $X \in E$. □

Now we are able to prove the main result of this section.

**Theorem 4.9.** *Consider the monomorphism* $i : K_1 \underline{\phi G} \to G^i(A \otimes H)$ *introduced in Lemma 3.5.*

*Then* $\mathrm{Im}(i) \subset E$ *and* $\mathrm{Im}(i) = E$ *if* $H$ *is a cosemisimple Hopf algebra over a field* $k$. *In this situation,* $\mathrm{Pic}(A/B) \cong E$.

**Proof.** Take $[(M, \alpha)] \in K_0 \underline{\psi G}$, and let $i[(M, \alpha)] = X \in G^i(A \otimes H)$. Then

$$\{A^X\} = j(g'[(M, \alpha)]) = j([M]) = \{M \otimes_B A\},$$

hence $A^X$ is coinvariantly generated and $AA_{X^{-1}} = A$, by Lemma 4.8. In a similar way, $i([(M, \alpha)]^{-1}) = X^{-1}$, and $A^{X^{-1}} \cong M^* \otimes_B A$ is coinvariantly generated, so $AA_X = A$, again by Lemma 4.8. This proves that $X \in E$.

Assume now that $H$ is cosemisimple, and take $X \in E$. It follows from Lemma 4.8 that $A^X$ is coinvariantly generated, and from Proposition 4.7 that $A^X = M \otimes_B A$ for some $M \in \underline{\underline{\mathrm{Pic}}}(B)$. Since the image of $M$ in $\mathrm{Pic}(A)$ is trivial, $[M] = g'[(M, \alpha)]$ for some $(M, \alpha) \in \overline{\mathcal{C}}$. Write $i[(M, \alpha)] = Y$. Then $X = Y d(a)$, for some $a \in \mathbb{G}_m(A)$. Consider the map $\alpha' : M \otimes_B A \to A$, $\alpha'(m \otimes b) = a^{-1}\alpha(m \otimes b)$. Then $i[(M, \alpha')] = X$. □

## 5. On the group-like elements

We have an injective map $i : G(H) \to G(A \otimes H)$, $i(g) = 1_A \otimes g$. Everything simplifies if $i$ is an isomorphism. We discuss two situations in which this is (almost) the case.

Recall that a commutative algebra which is an integral domain is called normal if it is integrally closed in its field of fractions.

**Proposition 5.1.** *Let $k$ be an algebraically closed field, let $A$ be a finitely generated commutative normal $k$-algebra and let $G$ be a connected algebraic group acting rationally on $A$. Let $H$ be the affine coordinate ring of $G$, and $\chi(G)$ be the group of characters of $G$. Then*

$$G(A \otimes H) = \{1 \otimes \phi \mid \phi \in G(H) = \chi(G)\}.$$

**Proof.** Let $x = \sum_i a_i \otimes f_i \in G(A \otimes H)$. Then we have

$$\sum_i (a_i \otimes f_{i(1)} \otimes f_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes (f_i * a_{j[1]}) \otimes f_j) \tag{5.1}$$

and $\sum a_i \varepsilon(f_i) = 1$. The map

$$\alpha : A \otimes H \to \mathrm{Hom}(kG, A), \quad \alpha(a \otimes f)(g) = af(g)$$

is injective. Let $\phi = \alpha(x)$. Using (5.1), we compute for all $g, g' \in G$ that

$$\begin{aligned} \phi(gg') &= \sum_i a_i f_i(gg') = \sum_i a_i f_{i(1)}(g) f_{i(2)}(g') \\ &= \sum_{i,j} a_i a_{j[0]}((f_i * a_{j[1]})(g)) f_j(g') \\ &= \sum_{i,j} a_i a_{j[0]} f_i(g) a_{j[1]}(g) f_j(g') \\ &= \sum_{i,j} (g \cdot a_j) f_j(g') a_i f_i(g) \\ &= \sum_{i,j} g \cdot (a_j f_j(g')) a_i f_i(g) \\ &= (g \cdot (\phi(g')))\phi(g). \end{aligned}$$

From the second equality, we have $1 = \sum_i a_i f_i(1_G) = \phi(1_G)$. For every $g \in G$, $\phi(g)$ is invertible in $A$, with inverse $g \cdot (\phi(g^{-1}))$. By the proof of [**13**, Proposition 1b, p. 46], $\phi(g) \in k$ for every $g \in G$, so $\phi \in \chi(G)$. Now $\chi(G) = G(H) \subset H$ (see [**12**, p. 25]), so it follows in particular that $\phi \in H$. For all $g \in G$ we now have that

$$\alpha(1 \otimes \phi)(g) = \phi(g) = \sum_i a_i f_i(g) = \alpha(x)(g),$$

hence $x = 1 \otimes \phi$, by the injectivity of $\alpha$. □

Now consider the situation from Remark 3.2: $H = k\mathbb{Z} \cong k[X, X^{-1}]$, and $A$ is a commutative $\mathbb{Z}$-graded algebra. In this situation $A \otimes H = A \otimes k[X, X^{-1}]$. Group-like elements in $A \otimes H$ can be constructed as follows. Let $1 = e_1 + \cdots + e_n$ with the $e_i$ orthogonal idempotents, and take $d_1, \dots, d_n \in \mathbb{Z}$. Then $\sum_{i=1}^n e_i \otimes X^{d_i}$ is a group-like element in $A \otimes k[X, X^{-1}]$. In this way, we have an embedding of $\mathcal{C}(\mathrm{Spec}(A), \mathbb{Z})$, the continuous functions from $\mathrm{Spec}(A)$ (with the Zariski topology) to $\mathbb{Z}$ (with the discrete topology), into $G(A \otimes k[X, X^{-1}])$. The first author was amazed to see that one of his first results, [**5**, Theorem 2.3], can be restated in such a way that it becomes a result about corings. Recall that a commutative ring is called reduced if it has no non-trivial nilpotents.

**Proposition 5.2.** *Let $A$ be a reduced $\mathbb{Z}$-graded commutative $k$-algebra. Then the map $\mathcal{C}(\mathrm{Spec}(A), \mathbb{Z}) \to G(A \otimes k[X, X^{-1}])$ is a bijection.*

**Example 5.3 (cf. Example 2.6 in [5]).** Proposition 5.2 does not hold if $A$ contains nilpotent elements; this is related to the fact that there exist non-homogeneous units in this situation. Let $A = k[x]$, with $x^2 = 0$, and put a $\mathbb{Z}$-grading on $A$ by taking $\deg(x) = 1$. Then $1 + ax \in \mathbb{G}_m(A)$, and $d(1 + ax) = (1 - ax) \otimes 1 + ax \otimes X$ is a group-like element in $G(A \otimes k[X, X^{-1}])$ which is not in the image of $\mathcal{C}(\mathrm{Spec}(A), \mathbb{Z})$.

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