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A REMARK ON THE KRULL-SCHMIDT-AZUMAYA THEOREM

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It is well known that if a module M is expressible as a direct sum of modules with local endomorphism rings, then such a decomposition is essentially unique. That is, if $M=\bigoplus_{i\in I} M_i=\bigoplus_{j\in J} N_j$, then there is a bijection $f: I\to J$ such that M_i is isomorphic to $N_{f(i)}$ for all $i\in I$ (see [1]). On the other hand, a nonprincipal ideal in a Dedekind domain provides an example where such a theorem fails in the absence of the local hypothesis. Group algebras of certain groups over rings R of algebraic integers is another such example, where even the rank as R-modules of indecomposable summands of a module is not uniquely determined (see [2]). Both of these examples yield modules which are expressible as direct sums of two indecomposable modules in distinct ways. In this note we construct a family of rings which show that the number of summands in a representation of a module M as a direct sum of indecomposable modules is also not unique unless one has additional hypotheses. In these rings the identity may be expressed as a sum of sets of orthogonal primitive idempotents of differing cardinalities (finite of course), two decompositions may have the same cardinality but not isomorphic summands, and 1 may be a (finite) sum of orthogonal primitive idempotents in a ring with infinite sets of orthogonal primitive idempotents.

Let A be any set, and $\sim$ an equivalence relation on A. Let F be the polynomial ring over $\mathbf{Z}/2\mathbf{Z}$ in noncommuting indeterminants $\{e_\alpha \mid \alpha\in A\}$. Let I be the ideal of F generated by the set

$$\{e_\alpha^2-e_\alpha,\ e_\alpha e_\beta \mid \alpha,\beta\in A,\ \alpha\neq\beta,\ \alpha\sim\beta\}.$$

Let $R=F/I$. Denote the image of $x\in F$ in R by x'. By definition, each e'_α is idempotent, and for an equivalence class $\mathrm{cl}(\beta)$, $\{e'_\alpha \mid \alpha\in \mathrm{cl}(\beta)\}$ are orthogonal.

Monomials in F will be denoted by the letters u, v, w. The *degree* of a monomial is the sum of the exponents of the factors e_α. For convenience, 1 and 0 will both be considered monomials of degree 0.

A monomial w is called reduced if w is a constant or if

$$w = e_{\alpha(1)}e_{\alpha(2)}\cdots e_{\alpha(n)}$$

where $\alpha(i)\nsim\alpha(i+1)$ for $1\le i\le n-1$.

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A *chain in I* is a set $\{w_j(e^2_{\alpha(j)}-e_{\alpha(j)})u_j \mid \alpha(j)\in A,\ 1\le j\le m\}$, such that

$$\sum_{j=1}^{k} w_j(e^2_{\alpha(j)}-e_{\alpha(j)})u_j = v_k - w_1 e_{\alpha(1)}u_1$$

for all k between 1 and m, where $v_k = w_k e^n_{\alpha(k)} u_k$ for n either 1 or 2.

A monomial u is *linked* to a monomial v if $u-v$ is the sum of some chain in I.

LEMMA 1. *Let $x'\in R$. Then $x'=\sum_{i=1}^m w_i'$, where each w_i is reduced.*

Proof. Since the natural map: $F\to R$ is epic, $x'=\sum_{i=1}^n u_i'$. If u_i is not reduced, either u_i contains a product $e_\alpha e_\beta$ where $\alpha\sim\beta$, $\alpha\ne\beta$ in which case delete i from the indexing set since $u_i'=0$; or u_i contains some e_α's to a higher than first power. Modulo I, however, such a u_i is congruent to a word w_i where all higher powers are replaced by first powers. Such a w_i is reduced, and $x'=\sum w_i'$.

LEMMA 2. *Let $u=e^{i(1)}_{\alpha(1)}e^{i(2)}_{\alpha(2)}\dots e^{i(n)}_{\alpha(n)}$, where each $i(j)\ge 1$, and let u be linked to v. Then $v=e^{k(1)}_{\alpha(1)}e^{k(2)}_{\alpha(2)}\dots e^{k(n)}_{\alpha(n)}$ where each $k(j)\ge 1$.*

Proof. We use induction on the number of terms in a chain whose sum is $u-v$. If $u-v=w(e^2_\alpha-e_\alpha)\bar{w}$, the result is clear. If $u-v=\sum_{i=1}^m w_i(e^2_{\alpha(i)}-e_{\alpha(i)})\bar{w}_i$, then the sum of the first $m-1$ terms of the chain is, by induction, equal to $\bar{u}-v$, where $\bar{u}$ is of the required form. Then $u-\bar{u}=w_m(e^2_{\alpha(m)}-e_{\alpha(m)})\bar{w}_m$, so by the first case u is of the required form.

LEMMA 3. *Let $0\ne x=\sum_{i=1}^m w_i$, where each w_i is reduced. Then $x'\ne 0$.*

Proof. Assume not. Then

$$x = \sum_{j=1}^{k} u_j g_j v_j,$$

where each

$$g_j \in \{e^2_\alpha - e_\alpha,\ e_\alpha e_\beta \mid \alpha,\beta\in A,\ \alpha\sim\beta,\ \alpha\ne\beta\}.$$

Assume k and x have been selected so that any sum of less than k terms ugv, g a generator of I, has at least one nonreduced monomial in its unique expression as a sum of monomials in F, but x is a sum of reduced monomials. Since w_1 is reduced, it cannot contain a factor $e_\alpha e_\beta$, $\alpha\sim\beta$, $\alpha\ne\beta$. Hence some $u_jg_jv_j$ is of the form $\bar{u}(e^2_\alpha-e_\alpha)\bar{v}$, where $\bar{u}e_\alpha\bar{v}=w_1$. Let $\bar{u}_1=\bar{u}$, $\bar{v}_1=\bar{v}$, $\alpha(1)=\alpha$. Assume we have a chain

$$\{\bar{u}_i(e^2_{\alpha(i)}-e_{\alpha(i)})\bar{v}_i \mid 1\le i\le n\}\subseteq\{u_jg_jv_j \mid 1\le j\le k\},$$

such that $\sum_{i=1}^n \bar{u}_i(e^2_{\alpha(i)}-e_{\alpha(i)})\bar{v}_i=u-w_1$. If u is not reduced, u cannot contain a subproduct $e_\alpha e_\beta$, $\alpha\sim\beta$, $\alpha\ne\beta$, by Lemma 2, and since x is a sum of reduced monomials, some one of the remaining $u_jg_jv_j$ must be of the form $\bar{u}_{n+1}(e^2_{\alpha(n+1)}-e_{\alpha(n+1)})\bar{v}_{n+1}$, where u is one of the two monomials appearing in this element of F. (Here we have used the fact that the only nonzero coefficient is 1.) This element can then be used to increase the chain by one element. Continuing in this manner, we either find a monomial which is not reduced which actually appears in x (a contradiction), or we find a reduced monomial which is linked to w_1. By Lemma 2, such a reduced monomial must equal w_1, so the sum of the chain is 0. Deleting the chain from the set of $u_j g_j v_j$ gives a sum of fewer than k terms ugv which has no nonreduced monomials, contradicting the minimality of k.

Lemmas 1 and 3 together state that every element x' of R has a unique expression $x'=\sum_{i=1}^{m} w_i'$, where the w_i are reduced. Henceforth we will drop the prime in denoting elements of R, and all monomials will be reduced.

PROPOSITION. *$\{e_\alpha \mid \alpha \in A\}$ are primitive idempotents. Moreover, if* cl (β) *is finite, then $1-\sum_{\alpha\sim\beta} e_\alpha$ is also primitive.*

Proof. Let $0\neq ke_\alpha+e_\alpha\sum_{i=1}^{n} w_i e_\alpha$ be an idempotent of R, where $k\in \mathbf{Z}/2\mathbf{Z}$, each w_i has degree at least 1, and each $e_\alpha w_i e_\alpha$ is reduced. Then

$$x = e_\alpha\left(\sum_{i,j=1}^{n} w_i e_\alpha w_j - \sum_{m=1}^{n} w_m\right)e_\alpha$$

is in I, and when multiplied out, all monomials which appear in the expression for x are reduced. If the largest degree of a monomial w_i is d, then the monomial $e_\alpha w_i e_\alpha w_i e_\alpha$ of degree $2d+3$ occurs precisely once in the reduced expression for x, so $x'=0/S$ contradict Lemma 3. Thus no w_i can occur, and the idempotent is ke_α.

Now assume cl (β) is finite, and let $f=1-\sum_{\alpha\sim\beta} e_\alpha$. If $0\neq kf+f\sum w_i f$ is an idempotent of R, we may assume that no w_i starts or ends with e_α where $\alpha\sim\beta$. Proceeding exactly as above, replacing e_α by f in the expression for x and multiplying out, we get a reduced monomial $e_\beta w_i e_\beta w_i e_\beta$ which cannot be cancelled out in the reduced expression for x, so again there are no w_i and $k=1$.

The idempotents in the above proposition are not the only idempotents in the ring R. For example, if $\alpha\nsim\beta$, $e_\alpha+e_\alpha e_\beta+e_\alpha e_\beta e_\alpha$ is idempotent. However, we can say the following: Let $f=f^2\in R, f=l+\sum w_i$, where l is a sum of products containing at most one element and every word w_i has degree at least 2. Let w_1 have smallest degree $d\geq 2$ among the w_i, and assume $l=0$. Then every reduced monomial in f^2 has degree $\geq 2d-1>d$ so $f^2\neq f$. Now assume l contains a subsum $e_\alpha+e_\beta$ where $\alpha\nsim\beta$. Let $P(n)$ be the property that every reduced product of k e_α's and e_β's is one of the w_i for $1\leq k\leq n$. We are assuming $P(1)$. If $P(n)$ holds,

$$f = e_\alpha+e_\beta+\sum_{k=1}^{n}\underbrace{e_\alpha e_\beta\ldots}_{k \text{ factors}}+\sum_{k=1}^{n}\underbrace{e_\beta e_\alpha\ldots}_{k \text{ factors}}+\sum w_i'.$$

If $P(n+1)$ fails, then one product of $n+1$ elements, say $u=e_\alpha e_\beta\ldots$ ($n+1$ factors) is not among the w_i. In f^2, the monomial u is obtained by multiplying e_α by $e_\beta\ldots$ (n factors), by multiplying each $e_\alpha e_\beta\ldots e_\beta$ ($2k$ factors) by both $e_\beta\ldots$ ($n-2k+2$ factors) and $e_\alpha\ldots$ ($n-2k+1$ factors), and by multiplying $e_\alpha\ldots e_\alpha$ ($2k+1$ factors) by both $e_\alpha e_\beta\ldots$ ($n-2k+1$ factors) and $e_\beta\ldots$ ($n-2k$ factors). In particular, u is obtained an odd number of times and so actually appears in f^2 but not in f. Since

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some $P(n)$ must fail, we conclude that l contains summands from at most one equivalence class and in particular, $l^2 = l$. Moreover, $l \neq 1$ since $1-f$ is idempotent, so one equivalence class must occur.

LEMMA 4. *Let $f^2 = f = l + \sum w_i$, $g^2 = g = l' + \sum v_i$, $fg = gf = 0$, where l and l' contain all monomials of f and g of degree ≤ 1. Then the equivalence class determined by l' must have a corresponding idempotent appearing in l or some w_i.*

Proof. Assume not. Then $l = e_\alpha + \cdots$, $l' = e_\beta + \cdots$, $\alpha \nsim \beta$. Let $P'(n)$ be the property that every reduced product of k e_α's and e_β's appears among the v_i for $2 \leq k \leq n$. We will use induction. $P'(1)$ vacuously holds. To show $P'(n) \Rightarrow P'(n+1)$ we first distinguish two cases.

Case (i). $n = 2m$ is even, $m \geq 1$. Let $u = e_\alpha e_\beta \ldots e_\alpha$ ($2m+1$ factors). Then

$$g = e_\beta + \sum_{k=2}^{2m} \underbrace{e_\alpha \cdots}_{k \text{ factors}} + \sum_{k=2}^{2m} \underbrace{e_\beta \cdots}_{k \text{ factors}} + \sum v_i'$$

and in g^2, u is obtained by multiplying $e_\alpha \ldots e_\beta$ ($2k$ factors) by $e_\beta \ldots e_\alpha$ ($2m-2k+2$ factors) or by $e_\alpha \ldots e_\alpha$ ($2m-2k+1$ factors) if $k \neq m$, by multiplying $e_\alpha \ldots e_\alpha$ ($2k-1$ factors, $k \geq 2$) by $e_\alpha \ldots e_\alpha$ ($2m-2k+3$ factors) or by $e_\beta \cdots e_\alpha$ ($2m-2k+2$ factors), and by multiplying $e_\alpha \ldots e_\beta$ ($2m$ factors) by $e_\beta e_\alpha$. In particular, it occurs an odd number of times so that u appears in g^2 and hence in g. If $u' = e_\beta \ldots e_\beta$ ($2m+1$ factors), u' arises in g^2 by multiplying e_β by $e_\alpha \ldots e_\beta$ ($2m$ factors) and all other initial segments by two distinct monomials of g. If u' appears in g, $e_\beta u'$ and $u' e_\beta$ cancel out in g^2, so u' must appear in g^2 and hence in g.

Case (ii). Assume $P'(2m+1)$. Then

$$g = e_\beta + \sum_{k=2}^{2m+1} \underbrace{e_\alpha \cdots}_{k \text{ factors}} + \sum_{k=2}^{2m+1} \underbrace{e_\beta \cdots}_{k \text{ factors}} + \sum v_i'$$

fg contains the term $e_\alpha e_\beta \ldots e_\beta$ ($2m+2$ factors) but no w_i is a product $e_\alpha e_\beta \ldots$. Hence $fg = 0$ implies $e_\alpha v_i' = e_\alpha e_\beta \ldots e_\beta$ ($2m+2$ factors) for some v_i', and that v_i' must equal $e_\alpha e_\beta \ldots e_\beta$ ($2m+2$ factors). Moreover, gf contains the term $e_\beta \ldots e_\beta e_\alpha$ ($2m+2$ factors) so $e_\beta \ldots e_\beta e_\alpha$ ($2m+2$ factors) likewise occurs among the v_i'.

By induction, $P'(n)$ holds for all n, a contradiction. These computations may be summarized by:

THEOREM. *For each of the following properties, there exists a ring R having the required property.*

(i) *R can be expressed as a finite direct sum of indecomposable left ideals but R contains an infinite set of primitive orthogonal idempotents.*

(ii) *R can be expressed as a direct sum of n indecomposable left ideals for each integer $n \geq 2$ but R contains no infinite set of orthogonal idempotents.*

Proof. (i) Let $A=\omega$, $n\sim m \Leftrightarrow mn\neq 0$. Then $R=Re_0\oplus R(1-e_0)$ with e_0 and $(1-e_0)$ primitive idempotents, but $\{e_n \mid n\in\omega-\{0\}\}$ is an infinite set of orthogonal primitive idempotents.

(ii) Let A be the disjoint union of sets of cardinality $n+1$ for $n\in\omega$, $\alpha\sim\beta$ if and only if α and β belong to the same set in this disjoint union. Then the corresponding ring is a direct sum of $n+2$ indecomposable left ideals for each n. By Lemma 4 if E is a set of orthogonal idempotents, $f=\sum w_i\in E$, then f determines a finite number of equivalence classes which may serve for the linear parts of other idempotents in E. Hence there are at most a finite number of linear parts appearing in elements of E, and since $g^2=g=l+\sum w_i$, $h^2=h=l+\sum v_i$ implies $gh=l+\sum w_i\sum v_j+\sum lv_i+\sum w_il$ has linear part $l\neq 0$, E must be finite. We remark that, in all of these rings, $\alpha\neq\beta$ implies Re_α is not isomorphic to Re_β since $e_\alpha=e_\alpha fe_\beta ge_\alpha$ has no solutions f and g. Moreover, for all α and β, Re_α is not isomorphic to $R(1-e_\beta)$ for the same reason.

Remark. It has been brought to the author's attention that E. C. Dade also has an example of an entirely different ring satisfying the property (i) of the theorem.

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