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REMARKS ON A PROBLEM OF MOSER

BY

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In memory of Leo Moser

Let $M(n)$ be the set of all the points $(x_1, x_2, \ldots, x_n) \in E^n$ such that $x_i \in \{0, 1, 2\}$ for each $i=1, 2, \ldots, n$ and let $f(n)$ be the cardinality of a largest subset of $M(n)$ containing no three distinct collinear points. L. Moser [4] asked for a proof of the inequality $f(n) \geq c3^n/\sqrt{n}$.

Let us consider the set S_n of those points $(x_1, x_2, \ldots, x_n) \in M(n)$ which satisfy $|\{i : x_i = 1\}| = [(n+1)/3]$. As S_n is a subset of the sphere with center at $(1, 1, \ldots, 1)$ and radius $(n-[(n+1)/3])^{1/2}$, no three distinct points of S_n are collinear. Thus we have

$$(1) \qquad f(n) \geq |S_n| = \binom{n}{[(n+1)/3]} 2^{n-[(n+1)/3]}.$$

This is the desired result as Stirling's formula implies

$$\binom{n}{[(n+1)/3]} 2^{n-[(n+1)/3]} \sim \left(\frac{9}{4\pi}\right)^{1/2} \cdot 3^n/\sqrt{n}.$$

Now we are going to improve (1). Let k, n be integers such that $0 \leq k < n$. A family F of sets will be called an (n, k) family if:

(i) all the members of F are subsets of the same set with n elements,

(ii) $|X \triangle Y| > k$ whenever X, Y are distinct members of F ($X \triangle Y$ denotes the symmetric difference $(X-Y) \cup (Y-X)$).

We denote by $G(n, k)$ the maximum cardinality of an (n, k)-family. It is easy to show that $G(n, k) \leq 2^{n-k}$; the determination of $G(n, k)$ is essentially a problem from coding theory. Given any $x = (x_1, x_2, \ldots, x_n) \in M(n)$ we set $P(x) = \{i : x_i = 0\}$ and $Q(x) = \{i : x_i = 1\}$. Given any set $X \subset \{1, 2, \ldots, n\}$ such that $|X| = j$, take an $(n-j, k-j)$-family $F(X)$ of subsets of $\{1, 2, \ldots, n\} - X$ such that $|F(X)| = G(n-j, k-j)$ and put

$$R(X) = \{u \in M(n) : Q(u) = X, P(u) \in F(X)\}.$$

Set $R_{n,k} = \bigcup R(X)$ where X ranges over all subsets of $\{1, 2, \ldots, n\}$ with at most k elements. Assume that $R_{n,k}$ contains three distinct collinear points x, y, z with y between x and z. Then $2y = x+z$ and so

$$(2) \qquad Q(x) = Q(z), \quad Q(y) = Q(x) \cup (P(x) \triangle P(z)).$$

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In particular, we have $x, z \in R(Q(x))$. But then $P(x)$ and $P(z)$ are distinct members of $F(Q(x))$ and so $|P(x) \triangle P(z)| > k-|Q(x)|$. By (2) we then have $|Q(y)| = |Q(x)| + |P(x) \triangle P(z)| > k$ which is a contradiction, as $|Q(y)| \le k$ whenever $y \in R_{n,k}$. As k was arbitrary, we have

$$f(n) \ge \max_{0 \le k < n} |R_{n,k}| = \max_{0 \le k < n} \sum_{j=0}^{k} \binom{n}{j} G(n-j, k-j). \tag{3}$$

Asymptotically (3) is not much of an improvement over (1), for one has

$$\max_{0 \le k < n} \sum_{j=0}^{k} \binom{n}{j} G(n-j, k-j) \le \max_{0 \le k < n} 2^{n-k} \sum_{j=0}^{k} \binom{n}{j}$$

and, as Professor J. G. Kalbfleisch pointed out to me,

$$\max_{0 \le k < n} 2^{n-k} \sum_{j=0}^{k} \binom{n}{j} \sim 2\left(\frac{9}{4\pi}\right)^{1/2} 3^n/\sqrt{n}.$$

However, (3) gives better lower bounds for $f(n)$ than (1) whenever $n \ge 2$. In particular, it gives exact values of $f(n)$ for $n=1, 2, 3$—one has $f(1)=2$, $f(2)=6$, $f(3)=16$. Nevertheless, (3) only yields $f(4) \ge 42$ whereas $f(4) \ge 43$. Indeed, the set $A \cup B \cup C$ where

$$A = \{x \in M(4) : |Q(x)| = 2\},$$

$$B = \{x \in M(4) : |Q(x)| = 1 \text{ and } |P(x)| \text{ is even}\},$$

$$C = \{(0, 0, 0, 0), (0, 0, 0, 2), (2, 2, 2, 2)\}$$

contains no three distinct collinear points. I do not know the exact value of $f(4)$.

We conclude with a few remarks setting the present problem in a more general context. Firstly, for integers k and n such that $3 \le k \le n$ we denote by $r(k, n)$ the cardinality of a largest subset of $\{1, 2, \ldots, n\}$ containing no k distinct integers in an arithmetic progression. It has been conjectured for a long time that

$$r(k, n) = o(n) \tag{4}$$

for all k. The relation (4) would imply the existence of $g(k, p)$ such that whenever $n \ge g(k, p)$ and the set $\{1, 2, \ldots, n\}$ is partitioned into p parts, one of the parts contains k distinct integers in an arithmetic progression. The existence of $g(k, p)$ was first proved by Van der Waerden [7]; some small values of $g(k, p)$ can be found in [1]. Roth [5] proved (4) for $k=3$; in fact, he proved $r(3, n) < cn/\log\log n$. Recently Szemerédi [6] proved (4) for $k=4$. The relation $f(n) = o(3^n)$ would imply $r(3, n) = o(n)$. More generally, one could define $M(k, n)$ as the set of all the points $(x_1, x_2, \ldots, x_n) \in E^n$ such that $x_i \in \{0, 1, \ldots, k-1\}$ for each $i = 1, 2, \ldots, n$, and $f(k, n)$ as the cardinality of a largest subset of $M(k, n)$ containing no k distinct collinear points. Then the relation

$$f(k, n) = o(k^n) \quad \text{for all } k \tag{5}$$

would imply (4)—indeed, one has $r(k, k^n) \leq f(k, n)$. This has been already remarked by Moser [3]. The relation (5) would also imply the existence of $h(k, p)$ such that whenever $n \geq h(k, p)$ and $M(n, k)$ is partitioned into p parts, one of the parts contains k distinct collinear points. Actually, $h(k, p)$ exists for any k and p; this follows from a more general theorem of Hales and Jewett [2]. It is easy to see that the existence of $h(k, p)$ implies the existence of $g(k, p)$ as one has $g(k, p) \leq k^{h(k, p)}$.

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