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CLOSURE OPERATORS

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1. Introduction

A mapping $\kappa: \mathscr{P}(X) \to \mathscr{P}(X)$ is a *quasi-closure operator* (see Thron (1966) page 44) if (i) $\square\kappa = \square$, and for all $A, B \in \mathscr{P}(X)$ we have (ii) $A \subseteq A\kappa$, and (iii) $(A \cup B)\kappa = A\kappa \cup B\kappa$; one easily deduces that such operators have the further property: (iv) if $A \subseteq B \subseteq X$, then $A\kappa \subseteq B\kappa$; if κ also satisfies: (v) $A\kappa^2 \subseteq A\kappa$ for all $A \subseteq X$, then κ is called a *Kuratowski closure operator.*

Birkhoff (1967) and Morgado (1960) have taken statements analogous to (ii) (iv) and (v) as their axioms for an *algberaic closure operator* defined on an arbitrary complete lattice L. One reason for doing this is to ensure that the set $\phi(L)$ of all such operators is itself a lattice, in fact a complete lattice, under the partial order defined by:

$$(1) \qquad \phi \leqq \psi \text{ if and only if } a\phi \leqq a\psi \text{ for all } a \in L$$

Morgado (1960) discusses the existence of atoms and dual atoms in $\phi(L)$ and in Morgado (1961) describes all the lattice automorphisms of $\phi(L)$; in Morgado (1961b), he generalises the latter work and determines the conditions under which $\phi(L_1)$ is lattice-isomorphic to $\phi(L_2)$.

It is well known that not all algebraic closure operators satisfy axiom (iii) (see Example 1 below for a non-trivial instance) and that in general $K(X)$, the set of all Kuratowski closure operators, is not a lattice under the order defined in (1) (see Example 5). On the other hand, if $T(X)$ denotes the set of all topologies defined on X and $\Phi: K(X) \to T(X)$ is the 1-1 anti-order-preserving correspondence defined by $\eta\Phi = \{A \subseteq X: A'\eta = A'\}$ for all $\eta \in K(X)$ (see [17]), then $K(X)$ inherits a lattice-structure from that usually defined in $T(X)$; however this is not

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so of $Q(X)$, the set of all quasi-closure operators (see Example 6). In addition, the operations induced on $K(X)$ under Φ are particularly cumbersome and do not appear to simplify the analysis of $K(X)$ as a poset.

For these reasons we now commence a study of the posets $K(X)$ and $Q(X)$, modelling our approach on that of Morgado. In section 2, we determine all the atoms of $K(X)$ and show that the atoms of $Q(X)$ are precisely those of $K(X)$; in doing this we give an alternative proof of Frohlich's result characterising the maximal topologies definable on a set X (Frohlich (1964)). In section 3, we describe all the dual atoms of $K(X)$ and $Q(X)$ .These results are then used in section 4 to discuss the existence of order-isomorphisms between $K(X)$ and $K(Y)$ and between $Q(X)$ and $Q(Y)$; this extends Frohlich's work in characterising the lattice-automorphisms of $T(X)$ (Frohlich (1964); see also Steiner (1966)). Another feature of Morgado's work, the embedding of $\phi(L)$ in the Cartesian product $\phi(M) \times \phi(N)$ for some lattices M and N (see Morgado (1961a), (1963), (1966)), will be investigated for the case of $K(X)$ and $Q(X)$ in Sullivan (to appear). However, unlike Morgado, we shall restrict our attention throughout to operators defined on the lattice $\mathscr{P}(X)$ rather than on an arbitrary lattice L. For otherwise it seems that to achieve comparable results, we would have to first assume that L was complemented and distributive, and so was uniquely complemented (Birkhoff (1967), page 17), and that it was atomic, and so was in any case a sublattice of $\mathscr{P}(X)$ for some set X (Morgado, (1963), page 85).

2. Minimal closure operators

Notation will be that of Banach (1932) and Birkhoff (1967) unless otherwise specified. We write α for the identity mapping on $\mathscr{P}(X)$ and ω for the mapping defined on $\mathscr{P}(X)$ by setting $\square\omega = \square$ and $A\omega = X$ for all non-empty $A \subseteq X$. An order is defined on $Q(X)$ by the relation:

$$\kappa \leqq \eta \text{ if and only if } A\kappa \subseteq A\eta \text{ for all } A \subseteq X.$$

Under this order, $Q(X)$ is a poset with least element α and greatest element ω, and in general we have $K(X) \subset Q(X)$ and $K(X) \subset A(X)$, the set of all algebraic closure operators defined on $\mathscr{P}(X)$.

EXAMPLE 1. Let X be infinite and fix some infinite $C \subseteq X$. Define $\eta: \mathscr{P}(X) \to \mathscr{P}(X)$ as follows:

$$\begin{aligned} A\eta &= A \cup C \text{ if } C \setminus A \text{ is finite,} \\ &= A \text{ otherwise.} \end{aligned}$$

It can be readily checked that $\eta \in A(X)$. If however we fix x, $y \in C$, and write $C \setminus \{x, y\} = A \cup B$, where $A \cap B = \square$ and both A, B are infinite, then $(A \cup B)\eta \neq A\eta \cup B\eta$. Hence $K(X) \subset A(X)$.

EXAMPLE 2. Let X be any set, and fix some non-empty $A \subseteq X$ such that $|A'| \geqq 2$ and some $x \notin A$. Define $q: \mathscr{P}(X) \to \mathscr{P}(X)$ as follows:

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$$\begin{aligned} Cq &= A \cup x \text{ if } C \subseteq A,\ C \neq \square \\ &= X \quad \text{if } C \nsubseteq A \\ &= \square \quad \text{if } C = \square. \end{aligned}$$

It is easy to see that $q \in Q(X)$; however since $Aq^2 \supset Aq$, we find that $q \notin K(X)$, and so $K(X) \subset Q(X)$.

In fact, there are other more important, non-trivial examples to show that $K(X) \subset Q(X)$: for situations in analysis, see Thron (1966), page 46 and Banach (1932), page 208; the following example arises naturally in the theory of partial left translations of a semigroup (Sullivan (1969), Chapter 3).

EXAMPLE 3. Let S be a semigroup and fix some $x \in S$. Define $\eta: \mathscr{P}(S) \to \mathscr{P}(S)$ as follows:

$$C\eta = C \cup xC \text{ for all } C \subseteq S$$

It is easily checked that $\eta \in Q(S)$. But

$$C\eta = C \cup xC \subseteq C \cup xC \cup x^2C = C\eta^2;$$

if S is the positive integers under multiplication, and $C = \{1\}$, $x = 2$, the containment is proper, and so in this case, η will not belong to $K(S)$.

Although $A(X)$ is a lattice under the partial order defined in (1), this is not so of $K(X)$: we choose to show this by first defining certain elements of $K(X)$ which will have a role of some importance in our characterisation of the atoms of $K(X)$ for X finite.

DEFINITION 1. *For $A, B \subseteq X$, define $\kappa_{AB}: \mathscr{P}(X) \to \mathscr{P}(X)$ as follows*:

$$\begin{aligned} C\kappa_{AB} &= C \text{ if } A \cap C = \square \\ &= C \cup B \textit{ otherwise.} \end{aligned}$$

As an abbreviation we shall simply write κ_{aB} or κ_{Ab} when $A = \{a\}$ or $B = \{b\}$. It is then easy to see that in general we have

$$\alpha = \kappa_{A\square} = \kappa_{\square B} = \kappa_{aa} \text{ and } \omega = \kappa_{XX}$$

In fact these are the only κ_{AB} which equal α and ω when $|X| > 1$: the proof of this fact is straightforward and so shall be omitted.

LEMMA 1. *Suppose* $|X| > 1$.

(i) $\kappa_{AB} = \alpha$ *if and only if either* (*a*) $A = \square$ *or* $B = \square$, *or* (*b*) $A = B = \{a\}$ *for some* $a \in X$.

(ii) $\kappa_{AB} = \omega$ *if and only if* $A = X = B$.

LEMMA 2. $\kappa_{AB} \in K(X)$ *for all* $A, B \subseteq X$.

Unfortunately, even for X finite, not every $\eta \in K(X)$ is of the form κ_{AB} for some $A, B \subseteq X$. In showing this we shall use the additional abbreviation of writing $a\eta$ for $\{a\}\eta$.

EXAMPLE 4. Let $X = \{a,b,c,d\}$ and define $\eta: \mathscr{P}(X) \to \mathscr{P}(X)$ by setting $\square\eta = \square$ and $a\eta = a$, $d\eta = d$, $b\eta = b \cup d$, $c\eta = c \cup a$, and $Y\eta = \{y\eta : y \in Y\}$ for $Y \subseteq X$. It is then readily checked that $\eta \in K(X)$. But suppose $\eta = \kappa_{AB}$ for some $A, B \subseteq X$. If $b \notin A$, we have $b\eta = b\kappa_{AB} = b$, a contradiction. Hence $b \in A$, and $b \cup d = b\eta = b \cup B$, $c \cup a = c\eta = c \cup B$: this implies $d \in B \subseteq \{c, a\}$, a contradiction.

However this particular kind of closure operator can be used to show that $K(X)$ is not a lattice under $\leqq$ as defined in (1). Although the infimum under $\leqq$ of ϕ, $\psi \in A(X)$ is the operator $\phi \wedge \psi$ defined by:

$$B(\phi \wedge \psi) = B\phi \cap B\psi \text{ for all } B \subseteq X,$$

when $\wedge$ is restricted to $K(X)$ we find:

EXAMPLE 5. If $|X| \geqq 3$ and we choose $a,b,c \in X$ and put $\eta = \kappa_{ba} \wedge \kappa_{ca}$, then by (1) we have

$$\{b, c\}\eta = \{b, c\}\kappa_{ba} \cap \{b, c\}\kappa_{ca} = \{b,c,a\},$$

but $b\eta = \{b, a\} \cap b = b$ and likewise $c\eta = c$. Hence $(\{b\} \cup \{c\})\eta \neq \{b\}\eta \cup \{c\}\eta$, and so $\eta \notin K(X)$.

On the other hand, $K(X)$ inherits a lattice-structure from $T(X)$ by defining, for all η, $\xi \in K(X)$,

$$\eta \vee \xi = \lambda \text{ if and only if } T_\eta \wedge T_\xi = L \text{ and } \lambda\Phi = L$$

$$\eta \wedge \xi = \mu \text{ if and only if } T_\eta \wedge T_\xi = M \text{ and } \mu\Phi = M.$$

However under these operations $Q(X)$ is not a lattice. To see this, let π_a denote the quasi-closure operator defined in Example 2 for the case $A = \{a\}$; we then have

EXAMPLE 6. For all $a \in X$, $\pi_a\Phi = \{\square, X\} = T_\omega$, and so, if $\wedge$ is well defined on $Q(X)$, we obtain $\pi_a \wedge \pi_b = \omega$; clearly, $Q(X)$ cannot also be a semilattice under $\wedge$.

The elements κ_{ab} of $K(X)$ introduced in Example 5 and corresponding to distinct a, $b \in X$ prove to be extremely important. They are *minimal over* α in the sense that $\alpha < \kappa_{ab}$ and if $\xi \in K(X)$ and $\alpha \leqq \xi \leqq \kappa_{ab}$, then ξ equals α or κ_{ab}, and they provide a "factorization" theory for certain $\eta \in K(X)$ (see Collyer (to appear)). However before proving the former assertion we now determine the relationship in general between those κ_{AB} which do not equal α.

LEMMA 3. *Suppose* $\kappa_{AB} \neq \alpha$ *for* $A, B \subseteq X$. *Then* $\kappa_{AB} \leqq \kappa_{CD}$ *if and only if* *(i)* $A \subseteq C$ *and* $B \backslash a \subseteq D \backslash a$ $\forall a \in A$, *or (ii) there exists* $a \in A$ *such that* $B = \{a\} = A \backslash C \subseteq D$.

PROOF. Let $E \subseteq X$ and put $\xi = \kappa_{AB}$, $\eta = \kappa_{CD}$. If $E \cap A = \square$, then $E\xi = E \subseteq E\eta$. So now suppose (i) holds and $E \cap A \neq \square$. Then $A \neq \square$ and $E \cap C \neq \square$, and so for $a \in A \cap E$,

$$E\xi = E \cup B = E \cup a \cup B \backslash a \subseteq E \cup a \cup D \backslash a = E \cup D = E\eta.$$

And if (ii) holds and $E \cap A \neq \square$, then $E \cap C = \square$ will imply $E \cap A = \{a\}$ and so $E\xi = E \cup B = E \cup a = E = E\eta$. On the other hand, $E \cap C \neq \square$ will imply $E\xi = E \cup B = E \cup a \subseteq E \cup D = E\eta$. Hence in all cases, $E\xi \subseteq E\eta$ and so $\xi \leqq \eta$.

Conversely, suppose $\xi \leqq \eta$. Now either $A \subseteq C$ or $A \nsubseteq C$. In the first case, $\forall a \in A$, $a \cup B = a\xi \subseteq a\eta = a \cup D$, and so (i) holds. If there exists $a \in A \backslash C$, then $a \cup B = a\xi \subseteq a\eta = a$, and so $B = a = A \backslash C$ since $B \neq \square$ and fixed. Now if $A \cap C = \square$, then $A = B = a$ and $\xi = \alpha$, contrary to assumption. So, letting $x \in A \cap C$, we obtain $x\xi = \{a, x\} \subseteq \{x\} \cup D = x\eta$ and hence $a \in D$.

When we reduce to the special case where both A, B are singletons, we obtain:

LEMMA 4. *For all distinct* a, $b \in X$, κ_{ab} *is minimal over* α.

PROOF. Suppose $\xi \in K(X)$ and $\alpha \leqq \xi \leqq \kappa_{ab}$ and let $A \subseteq X$. If $a \notin A$, then $A \subseteq A\xi \subseteq A\kappa_{ab} = A$ and so $A\xi = A$ for all $A \subseteq X$ not containing a. Suppose $a \in A$. Now $a \subseteq a\xi \subseteq \{a, b\}$, and so $a\xi$ equals a or $\{a, b\}$. In the first case, we obtain $A\xi = (A \backslash a \cup a)\xi = A \backslash a \cup a = A$, and in the second case

$$A\xi = (A \backslash a)\xi \cup a\xi = A \cup b.$$

Hence ξ equals either α or κ_{ab}, and the result follows.

We now proceed to determine all minimal Kuratowski closure operators on an arbitrary X. As a first step, we prove:

LEMMA 5. If $\eta \in K(X)$, $\eta \neq \alpha$ and $a \subset a\eta$ for some $a \in X$, then $\kappa_{ab} \leqq \eta$ for some $b \in X$.

PROOF. Since $a \subset a\eta$, we can choose $b \in a\eta \backslash a$. Then if $C \subseteq X$ and $a \notin C$, we have $C\kappa_{ab} = C \subseteq C\eta$, and if $a \in C$, then $b \in a\eta \subseteq C\eta$ and so $C\kappa_{ab} = C \cup b \subseteq C\eta$, and the result follows.

This result together with Lemma 4 suffice to characterise the atoms in $K(X)$ for X finite. For then, if η is minimal in $K(X)$ and $a = a\eta$ for all $a \in X$, then $\eta = \alpha$, a contradiction. Hence, $a \subset a\eta$ for some $a \in X$ and so by Lemma 5, we have $\alpha < \kappa_{ab} \leqq \eta$ for some $b \neq a$; the minimality of η then implies the result. The general result however is considerably more difficult: without the assumption a $\subset a\eta$ for some $a \in X$, it is not always true that for each $\eta \in K(X)$, there exist x, $y \in X$ with $\kappa_{xy} \leqq \eta$. As an example, let X be infinite and consider the closure operator η

defined by the conditions: $C\eta = C$ for every finite subset C of X and $C\eta = X$ otherwise. For this reason we now begin with

DEFINITION 2. *$\mathscr{L} \subseteq \mathscr{P}(X)$ is a prime sublattice if it is a lattice and if $C \cup D \in \mathscr{L}$ implies that $C \in \mathscr{L}$ or $D \in \mathscr{L}$.*

DEFINITION 3. *$\mathscr{M} \subset \mathscr{P}(X)$ is a maximal prime sublattice if it is a prime sublattice and if $\mathscr{P}(X)$ is the only prime sublattice properly containing $\mathscr{M}$.*

Our next result shows the existence of prime and maximal prime sublattices in $\mathscr{P}(X)$ for every X containing more than one element; in order to state this result, we put $\mathscr{B}_x = \{C \subseteq X : x \in C\}$.

LEMMA 6. *If $|X| > 1$ and $x \in X$, then $\mathscr{B}_x$ is a prime sublattice of $\mathscr{P}(X)$ and any proper prime sublattice containing $\mathscr{B}_x$ is contained in a maximal prime sublattice of $\mathscr{P}(X)$.*

PROOF. Since $X \neq x$, we have $\mathscr{B}_x \subset \mathscr{P}(X)$ and it is easy to see that $\mathscr{B}_x$ is a sublattice and prime.

Now suppose $\mathscr{B}_x \subseteq \mathscr{L} \subset \mathscr{P}(X)$ and $\mathscr{L}$ is prime. Let Σ be the set of all proper prime sublattices containing $\mathscr{L}$: this is non-empty since it contains $\mathscr{L}$. Suppose Γ is a chain in Σ and put $\mathscr{M} = \cup \Gamma$. If $C, D \in \mathscr{M}$, then $C \in \mathscr{C}$ and $D \in \mathscr{D}$ for some $\mathscr{C}, \mathscr{D} \in \Gamma$, and since Γ is a chain, we may suppose without loss of generality that $\mathscr{C} \subseteq \mathscr{D}$. Then both $C, D \in \mathscr{D}$ and so both $C \cap D$, $C \cup D \in \mathscr{D}$ since $\mathscr{D}$ is a sublattice. It follows that $\mathscr{M}$ is a sublattice. Now suppose $C \cup D \in \mathscr{M}$. Then $C \cup D \in \mathscr{F}$ for some $\mathscr{F} \in \Gamma$, and since $\mathscr{F}$ is prime, we obtain $C \in \mathscr{M}$ or $D \in \mathscr{M}$; that is, $\mathscr{M}$ is prime.

If $\mathscr{M} = \mathscr{P}(X)$, then in particular $X \setminus x \in \mathscr{G}$ for some $\mathscr{G} \in \Gamma$. Hence if $A \subseteq X$ and $x \notin A$, then $A \cup x \in \mathscr{B}_x \subseteq \mathscr{G}$ implies that $(X \setminus x) \cap (A \cup x) \in \mathscr{G}$ since $\mathscr{G}$ is closed under $\cap$. Hence $\mathscr{G}$ contains every $A \subseteq X$ not containing x, and so $\mathscr{G} = \mathscr{P}(X)$ a contradiction of the fact that $\mathscr{G}$ is a proper prime sublattice of $\mathscr{P}(X)$. Hence $\mathscr{M} \in \Sigma$ and so Zorn's Lemma implies that $\mathscr{L}$ is contained in a maximal prime sublattice of $\mathscr{P}(X)$ as required.

The relevance of the above is shown by the following result

LEMMA 7. *Suppose $x \in X$ and let $\mathscr{L}$ be a maximal prime sublattice of $\mathscr{P}(X)$ containing $\mathscr{B}_x$. Define $\lambda : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows:*

$$C\lambda = C \cup x \text{ if } C \notin \mathscr{L},\ C \neq \square$$
$$= C \textit{ otherwise.}$$

Then $\lambda \in K(X)$ and is minimal over α.

PROOF. From the definition, we have $\square\lambda = \square$ and $C \subseteq C\lambda$ for all $C \subseteq X$. If $C \subseteq X$, $C \notin \mathscr{L}$ and $C \neq \square$, then $C\lambda^2 = (C \cup x)\lambda = C \cup x$ since $C \cup x \in \mathscr{B}_x \subseteq \mathscr{L}$ in this case $C\lambda^2 = C\lambda$. In the other cases $C\lambda^2 = C\lambda = C$ trivially.

Now suppose $C \cup D \notin \mathscr{L}$ and $C \cup D \neq \square$. If $C = \square$, then $D \notin \mathscr{L}$ and $D \neq \square$, and so we have $(C \cup D)\lambda = C \cup D \cup x = C\lambda \cup D\lambda$. Since the same result follows if $D = \square$, we may therefore assume that both C, D are non-empty. Now since $\mathscr{L}$ is a sublattice and $C \cup D \notin \mathscr{L}$, we may also suppose without loss of generality that $C \notin \mathscr{L}$. We therefore have $C \notin \mathscr{L}$ and $C \neq \square$, and hence $(C \cup D)\lambda = C \cup D \cup x$ and $C\lambda = C \cup x$. But by definition $D\lambda$ equals D or $D \cup x$, and hence we obtain $C\lambda \cup D\lambda = (C \cup D)\lambda$.

If $C \cup D = \square$, then $C = D = \square$ and we have $(C \cup D)\lambda = C\lambda \cup D\lambda$.

So, suppose $C \cup D \in \mathscr{L}$ and $C \cup D \neq \square$, and since $\mathscr{L}$ is prime we may therefore assume without loss of generality that $C \in \mathscr{L}$. Hence we have $(C \cup D)\lambda = C \cup D$ and $C\lambda = C$. Now if $D \in \mathscr{L}$, we also have $D\lambda = D$ from the definition, and so in this case $(C \cup D)\lambda = C\lambda \cup D\lambda$. So suppose $D \notin \mathscr{L}$. Then $D \notin \mathscr{B}_x$ since $\mathscr{B}_x \subseteq \mathscr{L}$, and so $x \notin D$. Suppose we also have $x \notin C$. Then $x \notin C \cup D$, $C \cup D \neq \square$ $C \cup D \in \mathscr{L}$, a sublattice and $D \cup x \in \mathscr{B}_x \subseteq \mathscr{L}$ together imply that

$$D = (C \cup D) \cap (D \cup x) \in \mathscr{L},$$

a contradiction. Hence if $D \notin \mathscr{L}$ we must have $x \in C$ and so

$$C\lambda \cup D\lambda = C \cup D \cup x = C \cup D = (C \cup D)\lambda.$$

This gives $\lambda \in K(X)$.

Finally suppose that $\alpha \leqq \xi \leqq \lambda$ for some $\xi \in K(X)$ and put

$$\mathscr{L}_\xi = \{C \subseteq X : C\xi = C\}.$$

Since $\xi \in K(X)$, $\mathscr{L}_\xi$ is a sublattice of $\mathscr{P}(X)$, and if $C \cup D \in \mathscr{L}_\xi$ with $C \notin \mathscr{L}_\xi$, then $C \subset C\xi \subseteq C\lambda$. But $C\lambda$ equals $C \cup x$ or C, and hence in this case $C\xi = C \cup x$ with $x \notin C$. Then $C \cup D = (C \cup D)\xi = C\xi \cup D\xi = C \cup x \cup D\xi$, and so $x \in D$; that is, $D \in \mathscr{B}_x \subseteq \mathscr{L}$ and so $D \subseteq D\xi \subseteq D\lambda = D$ implies that $D \in \mathscr{L}_\xi$, and hence $\mathscr{L}_\xi$ is also prime. In fact, if $E \subseteq X$ and $E \in \mathscr{L}$, then $E \subseteq E\xi \subseteq \lambda = E$ and so $\mathscr{L}_\xi$ is a prime sublattice of $\mathscr{P}(X)$ such that $\mathscr{B}_x \subseteq \mathscr{L} \subseteq \mathscr{L}_\xi \subseteq \mathscr{P}(X)$. By the maximality of $\mathscr{L}$, we therefore know that $\mathscr{L}_\xi$ equals $\mathscr{L}$ or $\mathscr{P}(X)$; that is ξ equals λ or α, and so λ is minimal in $K(X)$.

We shall now prove that conversely every atom of $K(X)$ is of the form described in Lemma 7. In order to do this however, we must first show that for any $\eta \in K(X)$, there exists a $\xi \in K(X)$ just a little "smaller" than η; we state this result in more general terms so that it can be used in Theorem 3 to show that the atoms of $Q(X)$ are precisely those of $K(X)$.

LEMMA 8. *Let $\eta \in Q(X)$ and $x \in X$. Define $\sigma_x : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows:*

$$C\sigma_x = C\eta \backslash x \text{ if } x \notin C$$

$$= C\eta \text{ if } x \in C.$$

Then $\sigma_x \in Q(X)$ *and* $\sigma_x \leqq \eta$. *Moreover, if* $\eta \in K(X)$ *then* $\sigma_x \in K(X)$.

PROOF. From the definition we have $\square\sigma_x = \square\eta \setminus x = \square$, and if $C \subseteq X$ with $x \notin C$, then $C \subseteq C\eta$ and so $C \subseteq C\eta \setminus x$. Hence $C \subseteq C\sigma_x$ for all $C \subseteq X$.

Now if $x \notin C \cup D$, then $x \notin C$ and $x \notin D$, and so

$$(C \cup D)\sigma_x = (C \cup D)\eta \setminus x = (C\eta \cup D\eta) \setminus x = C\sigma_x \cup D\sigma_x.$$

If $x \in C \cup D$, then without loss of generality, we may suppose $x \in C$. Then $(C \cup D)\sigma_x = (C \cup D)\eta = C\eta \cup D\eta$ and $x \in C\eta$. If $x \in D$, we have $C\sigma_x \cup D\sigma_x = C\eta \cup D\eta$, and if $x \notin D$, we have $C\sigma_x \cup D\sigma_x = C\eta \cup D\eta \setminus x = C\eta \cup D\eta$ since $x \in C\eta$. Hence $\sigma_x \in Q(X)$.

Now let $C \subseteq X$. If $x \in C$, we have $C\sigma_x = C\eta$, and if $x \notin C$, we have $C\sigma_x = C\eta \setminus x \subseteq C\eta$, and so $\sigma_x \leqq \eta$.

Finally suppose $\eta \in K(X)$ and let $C \subseteq X$ with $x \notin C$. Then $C\sigma_x^2 = (C\eta \setminus x)\sigma_x = (C\eta \setminus x)\eta \setminus x$ since $x \notin C\eta \setminus x$. But $C\eta \setminus x \subseteq C\eta$ implies $(C\eta \setminus x)\eta \subseteq C\eta^2 = C\eta$ and so $(C\eta \setminus x)\eta \setminus x \subseteq C\eta \setminus x \subseteq (C\eta \setminus x)\eta$. But $x \notin C\eta \setminus x$ implies that $C\eta \setminus x \subseteq (C\eta \setminus x)\eta \setminus x$, and so in case $x \notin C$ we have $C\sigma_x^2 = C\eta \setminus x = C\sigma_x$. If $x \in C$, then $x \in C\eta$ and so $C\sigma_x^2 = C\eta\sigma_x = C\eta^2 = C\eta = C\sigma_x$.

LEMMA 9. *If* $\eta \in K(X)$ *is minimal over* α *and if as before* $\mathscr{L}_\eta$ *denotes the set* $\{C \subseteq X: C\eta = C\}$, *then there exists* $x \in X$ *such that* $\mathscr{B}_x \subseteq \mathscr{L}_\eta$ *and* $\mathscr{L}_\eta$ *is a maximal prime sublattice of* $\mathscr{P}(X)$.

PROOF. Since $\eta \neq \alpha$, there exists $A \subset X$ such that $A \subset A\eta$. Let $x \in A\eta \setminus A$. Then Lemma 7 implies that $\alpha \leqq \sigma_x \leqq \eta$, and since $A\sigma_x = A\eta \setminus x \subset A\eta$, then minimality of η implies $\sigma_x = \alpha$. We therefore have $A = A\sigma_x = A\eta \setminus x$, and so using $x \in A\eta$, we obtain $A\eta = A \cup x$.

Now suppose $B \subseteq X$ and $B \subset B\eta$. As above we will have $B\eta = B \cup y$ for some $y \notin B$ and $\sigma_y = \alpha$. If $x \in B$, then from the definition of σ_x and the fact that $\sigma_x = \alpha$, we obtain $B = B\sigma_x = B\eta = B \cup y$, and so $y \in B$, a contradiction. Hence $x \notin B$ and similarly $y \notin A$. So if $D = A \cup B$ (which is non-empty) we have $D \cap \{x, y\} = \square$ and hence again using the definition of σ_x and σ_y, we have $D = D\sigma_x = D\eta \setminus x$ and $D = D\sigma_y = D\eta \setminus y$. Therefore if $x \neq y$,

$$D\eta = (D\eta \setminus x) \cup (D\eta \setminus y) = D.$$

But $D\eta = A\eta \cup B\eta = A \cup x \cup B \cup y$, and so $x \in A \cup B$, a contradiction. Hence we may fix some $A \subseteq X$ such that $A \subset A\eta$, and then if $x \in A\eta \setminus A$, we obtain $B\eta = B \cup x$ for all $B \subseteq X$ such that $B \subset B\eta$. Moreover for such $x \in X$, $\sigma_x = \alpha$ and so $C\eta = C$ for all C containing x. Hence, we have $\mathscr{B}_x \subseteq \mathscr{L}_\eta$ and the fact that $\eta \in K(X)$ implies $\mathscr{L}_\eta$ is a sublattice.

Since $\eta \neq \alpha$, $\mathscr{L}_\eta \subset \mathscr{P}(X)$. To show $\mathscr{L}_\eta$ is prime, suppose $C \cup D \in \mathscr{L}_\eta$ and $C \notin \mathscr{L}_\eta$. Then $C \cup D = (C \cup D)\eta = C\eta \cup D\eta = C \cup x \cup D\eta$ since $C \subset C\eta$, and so $x \in C \cup D$. But $x \notin C$ (otherwise $C\eta = C$ and so $C \in \mathscr{L}_\eta$, a contradiction), and so $x \in D$. Then $D = D\eta$ and so $D \in \mathscr{L}_\eta$.

By Lemma 6 we may choose a maximal prime $\mathscr{L}$ such that

$$\mathscr{B}_x \subseteq \mathscr{L}_\eta \subseteq \mathscr{L} \subset \mathscr{P}(X).$$

Let $\lambda \in K(X)$ be defined as in Lemma 7. Then if $C \subseteq X$ and $C \in \mathscr{L}$, we have $C\lambda = C \subseteq C\eta$, and if $C \notin \mathscr{L}$, then $C \notin \mathscr{L}_\eta$ and so $C\eta = C \cup x = C\lambda$. Hence $\alpha < \lambda \leqq \eta$, and so $\lambda = \eta$ by minimality of η. Therefore $\mathscr{L}_\eta = \mathscr{L}$ and is a maximal prime sublattice of $\mathscr{P}(X)$.

If we write $\lambda(\mathscr{L}, x)$ for the operator defined in Lemma 7, we can summarise our work thus far in the following:

THEOREM 1. *If $|X| > 1$, then $\eta \in K(X)$ is minimal over α if and only if $\eta = \lambda(\mathscr{L}, x)$ for some maximal prime sublattice $\mathscr{L}$ containing $\mathscr{B}_x$.*

REMARK 1. We note that, under the lattice-anti-isomorphism Φ, this result also provides a characterisation of the maximal topologies on X (cf. Frohlich (1964)).

For completeness, we now determine which $\lambda(\mathscr{L}, x)$ equal κ_{ab} for some $a, b \in X$ (note Lemma 4).

THEOREM 2. *If $|X| \geqq 3$ and $a, b \in X$, $a \neq b$, then $\kappa_{ab} = \lambda(\mathscr{L}, x)$ for some $x \in X$ and some maximal prime sublattice $\mathscr{L}$ containing $\mathscr{B}_x$ iff $x = b$ and $\mathscr{L} = \mathscr{B}'_a \cup \mathscr{B}_b$.*

PROOF. Put $\lambda = \lambda(\mathscr{L}, x)$ and suppose $\lambda = \kappa_{ab}$ for $a \neq b$. Now $a\kappa_{ab} = a \cup b$, and so if $\{a\} \in \mathscr{L}$, $a\lambda = a$, a contradiction. Hence $\{a\} \notin \mathscr{L}$, and so $a\lambda = a \cup x$, which implies $x = b$ and $\mathscr{B}_b \subseteq \mathscr{L}$. If now $C \in \mathscr{B}'_a$, then $C\kappa_{ab} = C$, and if in addition $C \notin \mathscr{L}$ and $C \neq \square$, then $C\lambda = C \cup b$; that is, $b \in C$ and $C \in \mathscr{B}_b \subseteq \mathscr{L}$, a contradiction. Hence $C \in \mathscr{L} \cup \square$ and we have $\mathscr{B}'_a \cup \mathscr{B}_b \subseteq \mathscr{L} \cup \square$. But $\square \in \mathscr{L}$ since $\mathscr{L}$ is a maximal prime sublattice, and so we have $\mathscr{B}'_a \cup \mathscr{B}_b \subseteq \mathscr{L}$. On the other hand, if $C \in \mathscr{L}$ and $b \notin C$, then $C\lambda = C$, and so if $a \in C$, then $C\kappa_{ab} = C \cup b$, a contradiction. Hence, $a \notin C$ and we have $C \in \mathscr{B}'_a$. We have therefore shown that $\mathscr{L} = \mathscr{B}'_a \cup \mathscr{B}_b$.

Conversely, suppose that $\mathscr{L} = \mathscr{B}'_a \cup \mathscr{B}_b$, which is clearly a prime sublattice of $\mathscr{P}(X)$. If $C \subseteq X$ and $a \notin C$, then $C \in \mathscr{B}'_a \subseteq \mathscr{L}$ and $C\lambda = C = C\kappa_{ab}$. Suppose $a \in C$. Now either $C \in \mathscr{B}_b$ or $C \notin \mathscr{B}_b$, and in the first case, $C \in \mathscr{L}$ and $C\lambda = C = C \cup b = C\kappa_{ab}$, and in the second case, $C \notin \mathscr{L}$, $C \neq \square$, and $C\lambda = C \cup b = C\kappa_{ab}$. Hence we have $\lambda(\mathscr{L}, b) = \kappa_{ab}$ where $\mathscr{L} = \mathscr{B}'_a \cup \mathscr{B}_b$.

We are now in a position to prove

THEOREM 3. *The atoms of $Q(X)$ are precisely those of $K(X)$.*

PROOF. We first show that any atom of $K(X)$ is an atom of $Q(X)$. To do this, let $\lambda = \lambda(\mathscr{L}, x)$ be an atom of $K(X)$ (see Theorem 1), and suppose there exists $q \in Q(X)$ such that $\alpha \leqq q \leqq \lambda$. Then if $C \subseteq X$ and $C \in \mathscr{L}$, we have $C \subseteq Cq \subseteq C\lambda$

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$= C$ and so $C = Cq = Cq^2$. On the other hand, if $C \notin \mathscr{L}$, we have $C \subseteq Cq \subseteq C \cup x$. Now if $Cq \subset Cq^2$, then $C \subset Cq$ and so $Cq = C \cup x$; that is, $x \in Cq$, $Cq \in \mathscr{L}$, and $(Cq)q = Cq$, a contradiction. Hence for $C \notin \mathscr{L}$, we also have $Cq = Cq^2$; that is, $q \in K(X)$, and so since λ is an atom of $K(X)$, we find that q equals α or λ.

Conversely, suppose η is an atom of $Q(X)$ and that $\eta \notin K(X)$. Then there exists $A \subseteq X$ such that $A\eta \subset A\eta^2$. Since this implies that $A \subset A\eta$, we can choose $x \in A\eta \setminus A$ and $y \in A\eta^2 \setminus A\eta$. Then $\alpha \leqq \sigma_x \leqq \eta$ for $\sigma_x \in Q(X)$ (see Lemma 8) and we have $A \subseteq A\sigma_x = A\eta \setminus x \subset A\eta$. Hence $\sigma_x < \eta$ and so minimality of η implies that $\sigma_x = \alpha$; we therefore have $A = A\sigma_x = A\eta \setminus x$ and so $A\eta = A \cup x$ since $x \in A\eta$. Hence $\alpha \leqq \sigma_y \leqq \eta$ and $y \notin A$ imply that

$$A\sigma_y = A\eta \setminus y = A\eta = A \cup x \supset A$$

since $x \notin A$, and so $\alpha < \sigma_y$; the minimality of η therefore implies that $\eta = \sigma_y$. But $y \notin \{x\}$, and so $x\eta = x\sigma_y = x\eta \setminus y$; that is, $y \notin x\eta$. Hence we have $A\eta = A \cup x$ and $A\eta^2 = A\eta \cup x\eta$ with $y \in A\eta^2$ but $y \notin A\eta$ and $y \notin x\eta$, a contradiction. Therefore $\eta \in K(X)$ and if $\alpha \leqq \xi \leqq \eta$ for some $\xi \in K(X)$, then $\xi \in Q(X)$ and so the minimality of η in $Q(X)$ implies that ξ equals α or η; that is, η is an atom of $K(X)$.

3. Maximal closure operators

The notion of a *dual atom* was investigated by Morgado (1960): we shall on occasion refer to them as *maximal elements under* ω.

DEFINITION 4. For $A \subseteq X$, define $\gamma_A : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\begin{aligned} C\gamma_A &= A \quad \text{if } C \subseteq A,\ C \neq \square \\ &= X \quad \text{if } C \nsubseteq A \\ &= \square \ \text{ if } C = \square. \end{aligned}$$

It is readily seen that $\gamma_A = \omega$ if and only if A equals $\square$ or X. But more generally we have:

LEMMA 10. *$\gamma_A \in K(X)$ for all $A \subseteq X$.*

PROOF. It is clear from the definition that $\square\gamma_A = \square$ and $C \subseteq C\gamma_A$ for all $C \subseteq X$. From the remark above we may suppose that $\square \subset A \subset X$. Then if $C \subseteq A$ and $C \neq \square$, we have $C\gamma_A^2 = A\gamma_A = A = C\gamma_A$, and if $C \nsubseteq A$, then $C\gamma_A^2 = X\gamma_A = X = C\gamma_A$ since $X \nsubseteq A$. Now suppose $C \cup D \subseteq A$ and $C \cup D \neq \square$. Then both $C, D \subseteq A$ and without loss of generality we may suppose $C \neq \square$. In this case therefore $(C \cup D)\gamma_A = A = C\gamma_A \cup D\gamma_A$. If $C \cup D \nsubseteq A$, then either $C \nsubseteq A$ or $D \nsubseteq A$, and so $(C \cup D)\gamma_A = X = C\gamma_A \cup D\gamma_A$. Finally if $C \cup D = \square$, then $C = D = \square$, and again $(C \cup D)\gamma_A = C\gamma_A \cup D\gamma_A$.

Closure operators of the form γ_A are of particular importance as the following result indicates:

THEOREM 4. *$\eta \in K(X)$ is maximal under ω if and only if $\eta = \gamma_A$ for some $A \subseteq X$ such that $\square \subset A \subset X$.*

PROOF. Suppose η is maximal under ω in $K(X)$. Since $\eta \neq \omega$, there exists $\square \subset B \subset X$ such that $B\eta \subset X$. Put $B\eta = A$: we assert that $\eta \leqq \gamma_A$ (we note that $A \neq \square$ since $B \subseteq B\eta$). If $C \subseteq X$ and $C \subseteq A$, $C \neq \square$ then $C\eta \subseteq A\eta = B\eta^2 = A = C\gamma_A$, and if $C \nsubseteq A$, then $C\eta \subseteq X = C\gamma_A$. Clearly, $\square\eta = \square = \square\gamma_A$, and so the assertion follows. But from a remark above, $\gamma_A \neq \omega$ and so maximality of η implies the result.

Now suppose $\square \subset A \subset X$ and $\gamma_A \leqq \xi \leqq \omega$. Then if $C \subseteq X$ and $C \nsubseteq A$, we haev $X \subseteq C\xi$; if $C \subseteq A$ and $C \neq \square$, then $C\xi \subseteq A\xi$ and $A \subseteq C\xi$, and so $C\xi = A\xi$ fo all such C. Now $A \subseteq A\xi$: if $A = A\xi$, then $\xi = \gamma_A$; if there exists $a \in A\xi \setminus A$, then $a \notin A$ implies that $X = a\xi \subseteq A\xi$, and so $\xi = \omega$.

REMARK 2. As before, we note that under Φ this result determines all minimal topologies on an arbitrary set X.

The next result reveals the relationship between the operators γ_A and the κ_{CD} introduced in section 2.

THEOREM 5. *Suppose $\gamma_A \neq \omega$. Then $\gamma_A = \kappa_{CD}$ if and only if one of the following occurs*

(i) $C = X$ *and* $A = D = X \setminus a$,
(ii) $C = X \setminus a$, $A = a$, $|X| = 2$ *and* $D = a$,
(iii) $C = X \setminus a$, $A = a$, $|X| > 2$ *and* $D = X$.

PROOF. Suppose $\gamma_A = \kappa_{CD}$ and $\square \subset A \subset X$. Then $\forall x \in A'$, $x\gamma_A = X = x\kappa_{CD}$; if $x \notin C$, then $X = x$ implies $A = x$, a contradiction. Hence $x \in C$ and we have $A' \subseteq C$ and $X = x \cup D$ $\forall x \in A'$. This gives $A = A \cap X = A \cap D$ and so $A \subseteq D$.

Now suppose $A \setminus C = \square$. Then $A \subseteq C$ and so $C = X$ since $A' \subseteq C$ from above. If there exists $d \in D \setminus A$, we obtain $X = d\gamma_A = d\kappa_{XD} = d \cup D = D$, and so Lemma 1 (ii) implies $\gamma_A = \omega$, a contradiction. Hence $D = A$. Let $a \in X \setminus A$. Then $X = a\gamma_A = a\kappa_{XA} = a \cup A$, and so $A = X \setminus a$.

Now suppose there exists $a \in A \setminus C$. Then $A = a\gamma_A = a\kappa_{CD} = a$ implies $A = a$ and $A \setminus C = a$. Hence $C \neq X$ and so $C = X \setminus A$ since $X \setminus a \subseteq C$ from above. Now we already have $X = x \cup D$ $\forall x \in C$, and $a \in D$. Hence if $|X| = 2$, we obtain $D = a$. But if $|X| > 2$, then $|D| \geqq 2$ and so there exists $d \in D \setminus a$, and once again we have $X = d\gamma_A = d\kappa_{CD} = d \cup D$ since $C = X \setminus a$ and $d \neq a$. It follows that in this case $D = X$, and hence (ii) and (iii) hold.

Conversely suppose (i) holds and $E \subseteq X$. If $E \subseteq A$ and $E \neq \square$, then $E\kappa X_A = E \cup A = A = E\gamma_A$ and if $E \nsubseteq A$, then $E = X$ and the result follows. If (ii) holds and we put $C = b$, it is readily checked that $\gamma_a = \kappa_{ba}$. Finally if (iii) holds, then first $a\kappa_{CX} = a = a\gamma_a$, and if $E \nsubseteq A$, then $E \cap C \neq \square$ and so $E\kappa_{CX} = E \cup X = X = E\gamma_a$, and again the result follows.

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Before proceeding to determine the dual atoms of $Q(X)$, we wish to point out that for any $\eta \in Q(X)$, there exists a $\xi \in Q(X)$ just a little "bigger" than η.

REMARK 3. If $\eta \in Q(X)$ and $x \in X$, and if we define $\beta_x: \mathscr{P}(X) \to \mathscr{P}(X)$ as follows:

$$C\beta_x = C\eta \cup x \text{ if } x \in C\eta^2$$
$$= C\eta \text{ otherwise,}$$

then $\beta_x \in Q(X)$ and $\eta \leqq \beta_x$. For, since $x \notin \square\eta^2 = \square$, we have $\square\beta_x = \square\eta = \square$, and clearly, $C \subseteq C\beta_x$ for all $C \subseteq X$. Moreover, if $C, D \subseteq X$ and $x \in (C \cup D)\eta^2$, then $(C \cup D)\beta_x = (C \cup D)\eta \cup x$ and without loss of generality we may suppose $x \in C\eta^2$. Then $C\beta_x \cup D\beta_x = C\eta \cup x \cup D\beta_x$ where $D\beta_x$ equals either $D\eta$ or $D\eta \cup x$: in either case we will obtain $(C \cup D)\beta_x = C\beta_x \cup D\beta_x$. If on the other hand, $x \notin (C \cup D)\eta^2$, then $x \notin C\eta^2$ and $x \notin D\eta^2$, and so

$$(C \cup D)\beta_x = (C \cup D)\eta = C\eta \cup D\eta = C\beta_x \cup D\beta_x.$$

Hence, $\beta_x \in Q(X)$ and it is obvious that $\eta \leqq \beta_x$.

The result indicated in Remark 3 can be regarded as a "dual" of Lemma 8. Yet another instance of some correlation between our earlier work on showing that the atoms of $Q(X)$ are precisely those of $K(X)$ and our aim to now determine the dual atoms of $Q(X)$ and relate them to those of $K(X)$ is indicated by the following

EXAMPLE 7. Suppose $\eta \in K(X)$ and choose $\mathscr{E} \subseteq \mathscr{P}(X)$ with the property

$$\square \in \mathscr{E} \text{ and } C \cup D \in \mathscr{E} \text{ if and only if both } C, D \in \mathscr{E}$$

Note that both $\mathscr{P}(X)$ and any set of the form $\{\{a\}, \square\}$, $a \in X$, has this property. Now define $\xi: \mathscr{P}(X) \to \mathscr{P}(X)$ by putting $C\xi = C\eta$ if $C \in \mathscr{E}$ and equal to X otherwise: it is then easily checked that $\xi \in Q(X)$ and that $\eta \leqq \xi \leqq \omega$.

However the analogy between the two situations is illusory. To show this we first state

DEFINITION 5. *For $a, b \in X$, $a \neq b$, define*

$$\phi_{ab}: \mathscr{P}(X) \to \mathscr{P}(X) \text{ by}$$
$$C\phi_{ab} = \square \quad \text{if } C = \square$$
$$= X \setminus b \quad \text{if } C = a$$
$$= X \quad \text{otherwise.}$$

Since $a \neq b$, ϕ_{ab} is only defined for $|X| \geqq 2$.

LEMMA 11. *$\phi_{ab} \in Q(X)$ for all $a, b \in X$, $a \neq b$.*

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PROOF. $\Box\phi_{ab} = \Box$ from the definition. Since $a \neq b$, $\{a\} \subseteq X \setminus b = a\phi_{ab}$ and for $C \neq a$, we see from the definition $C \subseteq C\phi_{ab}$. Now suppose $C \cup D \neq a$. If $C \cup D = \Box$, then C and D are both empty and $(C \cup D)\phi_{ab} = C\phi_{ab} \cup D\phi_{ab}$. If $C \cup D \neq \Box$, then at least one of C or D is both non-empty and not equal to $\{a\}$. Without loss of generality, let us suppose it is C. Then

$$C\phi_{ab} \cup D\phi_{ab} = X \cup D\phi_{ab} = X = (C \cup D)\phi_{ab}.$$

If $C \cup D = a$, then $C \subseteq a$ and $D \subseteq a$ and either C or D equals a. Without loss of generality, let us assume $C = a$. Now $D \subseteq a$ implies $D\phi_{ab}$ equals the empty set or $X \setminus b$. In either case we have $C\phi_{ab} \cup D\phi_{ab} = X \setminus b \cup D\phi_{ab} = X \setminus b = (C \cup D)\phi_{ab}$.

Just as the γ_A defined earlier were maximal under ω in $K(X)$, we now have a similar result for $Q(X)$ when $|X| \geqq 2$.

THEOREM 6: *$\eta \in Q(X)$ is maximal under ω if and only if $\eta = \phi_{ab}$ for some $a, b \in X$, $a \neq b$.*

PROOF. Suppose η is maximal under ω in $Q(X)$. Since $\eta \neq \omega$, there exists a non-empty $C \subset X$ such that $C\eta \neq X$. Choose $a \in C$, and $b \in X \setminus C\eta$. We assert that $\eta \leqq \phi_{ab}$. Clearly $\Box\eta = \Box = \Box\phi_{ab}$. If $D \neq a$, $D \neq \Box$, then $D\phi_{ab} = X \supseteq D\eta$. If $D = a$, then $D \subseteq C$, and

$$D\eta \subseteq C\eta \subseteq X \setminus b = D\phi_{ab},$$

and so the assertion follows. But $\phi_{ab} \neq \omega$ since $a\phi_{ab} = X \setminus b \neq X$, and so maximality of η implies $\eta = \phi_{ab}$.

Now suppose, there exists $\xi \in Q(X)$ such that $\phi_{ab} \leqq \xi \leqq \omega$. Then if $D \subseteq X$ is non-empty and $D \neq a$, then $X = D\phi_{ab} \subseteq D\xi \subseteq X$ gives $D\xi = X$ for all $D \neq a$. If $D = a$, then $D\phi_{ab} = X \setminus b \subseteq D\xi \subseteq X$. So $D\xi$ either equals $X \setminus b$ or X. If $D\xi = X | b$, *then* $\xi = \phi_{ab}$, *and if* $D\xi = X$, $\xi = \omega$.

REMARK 4. We note that $\phi_{ab} \in K(X)$ if and only if $|X| = 2$. For clearly we always have $\Box\phi_{ab}^2 = \Box\phi_{ab}$, and if $D \neq a$, $D \neq \Box$, then $D\phi_{ab}^2 = X\phi_{ab} = X = D\phi_{ab}$. If $D = a$, then $D\phi_{ab}^2 = (X \setminus b)\phi_{ab}$ and $D\phi_{ab} = X \setminus b$. If $|X| = 2$, then $X \setminus b = a$ and so $D\phi_{ab}^2 = a\phi_{ab} = D\phi_{ab}$ implying $\phi_{ab} \in K(X)$. If $|X| \neq 2$, then $X \setminus b \neq a$ and so $D\phi_{ab}^2 = (X \setminus b)\phi_{ab} = X \neq X \setminus b = D\phi_{ab}$, implying $\phi_{ab} \notin K(X)$.

4. Order-isomorphisms

We now use the results of section 3 to investigate the existence of bijections $\theta: K(X) \to K(Y)$ such that $\eta \leqq \xi$ in $K(X)$ if and only if $\eta\theta \leqq \xi\theta$ in $K(Y)$; that is, we shall attempt to determine the condition under which $K(X)$ is *order-isomorphic* to $K(Y)$, and in like manner the conditions under which $Q(X)$ is order-isomorphic to $Q(Y)$, where X and Y are arbitrary (cf. Birkhoff (1967), page 3). Before stating our first result in this direction, we note that if $|X| = 1$ and $K(X)$ is order-isomorphic to $K(Y)$, then $|Y| \leqq 1$: for otherwise there would exist $a, b \in Y$, $a \neq b$ such that $\kappa_{ab} \in K(Y)$ and $\kappa_{ab} \neq \alpha_Y$.

LEMMA 12: *If* $|X| > 1$ *and* $K(X)$ *is order-isomorphic to* $K(Y)$, *then* $2^{|X|} = 2^{|Y|}$.

PROOF. Suppose $\theta: K(X) \to K(Y)$ is an order-isomorphism and let $\square \subset A \subset X$. Then Theorem 5 states that γ_A is maximal in $K(X)$. Hence $\gamma_A\theta$ is maximal in $K(Y)$ since θ is an order-isomorphism, and so again by Theorem 5, $\gamma_A\theta = \gamma_B$ for some $\square \subset B \subset Y$. Define $\theta': \mathscr{P}(X) \to \mathscr{P}(Y)$ as follows:

$$\square\theta' = \square, \; X\theta' = Y$$

$$\text{and } \forall A \subset X, \; A\theta' = B \text{ if and only if } \gamma_A\theta = \gamma_B.$$

Then θ' is a bijection since θ is an order-isomorphism, and the result follows.

Before proceeding we note that if $|X| = |Y|$ and $\theta: X \to Y$ is a bijection, then $\tau: \mathscr{P}(X) \to \mathscr{P}(Y)$ defined by putting $C\tau = C\theta$ for all $C \subseteq X$ is an order-isomorphism of $\mathscr{P}(X)$ into $\mathscr{P}(Y)$, and that every order-isomorphism is obtained in this way. We can therefore formulate a partial converse of the above Lemma:

LEMMA 13: *If* $|X| = |Y|$, *then* $K(X)$ *is order-isomorphic to* $K(Y)$.

PROOF. Suppose $\theta: X \to Y$ is bijective and for each $\eta \in K(X)$, define $\bar{\eta}: \mathscr{P}(Y) \to \mathscr{P}(Y)$ by putting $C\bar{\eta} = ((C\theta^{-1})\eta)\theta$ for all $C \subseteq Y$. It is then easily checked that $\bar{\eta} \in K(Y)$. Now define the mapping $\theta^*: K(X) \to K(Y)$ by putting $\eta\theta^* = \bar{\eta}$ for all $\eta \in K(X)$; it can also be easily checked that θ^* is the required order-isomorphism.

Combining Lemmas 12 and 13 in the finite case, we obtain:

THEOREM 7: *If* X *is finite and* $|X| > 1$, *then* $K(X)$ *is order-isomorphic to* $K(Y)$ *if and only if* $|X| = |Y|$.

REMARK 3. For infinite X, a complete converse to Lemma 12 does not seem possible. By assuming the Generalised Continuum Hypothesis, we can deduce $|X| = |Y|$ when $2^{|X|} = 2^{|Y|}$, but without this assumptio) it still does not appear to be known whether the deduction is valid (see Morgafo (1966), page 155). An alternative approach to the problem would be to use the results of section 2. Clearly minimality will be preserved under order-isomorphisms, and so a characterisation of the maximal prime sublattices of $\mathscr{P}(X)$ containing $\mathscr{B}_x$, $x \in X$, may enable a "better" cardinality condition to be obtained when $K(X)$ is order-isomorphic to $K(Y)$ and X is infinite.

When we turn our attention to order-isomorphisms of $Q(X)$ we obtain a much better result. However before stating this we first note that if $|X| = 1$ and $Q(X)$ is order-isomorphic to $Q(Y)$, then $|Y| \leqq 1$; for otherwise we would have $\phi_{ab} \in Q(Y)$, $\phi_{ab} \neq \omega_Y$.

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THEOREM 8. *For* $|X| > 1$, $Q(X)$ *is order-isomorphic to* $Q(Y)$ *if and only if* $|X| = |Y|$.

PROOF. Suppose $\theta: Q(X) \to Q(Y)$ is an order-isomorphism and let $a, b \in X$, $a \neq b$. Then Theorem 6 states that $\phi_{ab}\theta$ is maximal in $Q(Y)$ since θ is an order-isomorphism, and so again by Theorem 6, $\phi_{ab}\theta = \phi_{cd}$ for some $c, d \in Y$, $c \neq d$. Define

$$\theta': (X \times X) \setminus \{(a, a): a \in X\} \to (Y \times Y) \setminus \{(c, c): c \in Y\}$$

as follows:

$$((a, b))\theta' = (c, d) \text{ iff } \phi_{ab}\theta = \phi_{cd}.$$

Then θ' is a bijection since θ is an order-isomorphism.

For $|X| > 1$ and $|X|$ finite, this implies $|X| = |Y|$.

For $|X|$ infinite, let $\iota_X = \{(a,a):\ a \in X\}$. Then if $a \in X$, we define $\mu: X \times (X \setminus a) \to (X \times X) \setminus \iota_X$ by

$$(c, d)\mu = (c, d) \text{ if } c \neq d,$$
$$= (c, a) \text{ if } c = d.$$

Then it is easily shown that μ is well defined, 1-1 and onto. Now for X infinite, $|X \setminus a| = |X|$ and so $|X \times (X \setminus a)| = |X \times X|$ and so we have

$$|(X \times X) \setminus \iota_X| = |X \times (X \setminus a)| = |X \times X| = |X|.$$

This gives us $|X| = |(X \times X) \setminus \iota_X| = |(Y \times Y) \setminus \iota_Y| = |Y|$.

A proof similar to that of Lemma 13 establishes the converse.

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