TOTALLY COMPLEX SUBMANIFOLDS OF THE CAYLEY PROJECTIVE PLANE

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Abstract. Let h be the second fundamental form of a compact submanifold M of the Cayley projective plane CaP^2. We determine all compact totally complex submanifolds of complex dimension n in CaP^2 satisfying $|h|^2 \leq n$.

1. Introduction. Let M be an n-dimensional compact Kaehler submanifold of the complex projective space $CP^m(1)$. Denote by h the second fundamental form of M and UM the unit tangent bundle over M. Ros showed in [**5**] that if $f(u) = |h(u, u)|^2 < \frac{1}{4}$ for any $u \in UM$, then M is totally geodesic. Moreover in [**6**], Ros gave a complete list of compact Kaehler submanifolds of $CP^m(1)$ satisfying the condition $\max_{u \in UM} f(u) = \frac{1}{4}$. The same type results for totally complex submanifolds of the quaternion projective space $HP^m(1)$ were obtained by Coulton and Gauchman [**3**]. In [**4**], Coulton and Glazebrook proved the analogous results in the case of totally complex submanifolds of the Cayley projective place CaP^2. In the present paper, we proved the following pinching theorem for the square of the norm of the second fundamental form.

THEOREM. *Let M be a compact complex submanifold of complex dimension n immersed in Cayley projective plane CaP^2. If the square of the norm of the second fundamental form of M satisfies $|h|^2 \leq n$, then either (i) or (ii) holds.*

(i) $|h|^2 = 0$, *M is totally geodesic in CaP^2, and M is $CP^1(1)$ or $CP^2(1)$.*

(ii) $|h|^2 = n$ *and M is $CP^1(\frac{1}{2})$.*

2. Cayley projective plane. In this section, we review the fundamental results about the Cayley projective plane; for details see [**4**].

Let us denote by Ca the set of Cayley numbers, It possesses a multiplicative identity 1 and a positive definite bilinear form $<,>$ with norm $\|a\| = <a, a>$ satisfying $\|ab\| = \|a\| \bullet \|b\|$, for $a, b \in Ca$. Every element $a \in Ca$ can be expressed in the form $a = a_0 1 + a_1$ with $a_0 \in R$ and $<a_1, 1> = 0$. The conjugation map $a \to a^* = a_0 1 - a_1$ is an anti-automorphism $(ab)^* = b^* a^*$.

A canonical basis for Ca is any basis of the form $\{1, e_0, e_1, ..., e_6\}$ satisfying: (i) $<e_1, 1> = 0$; (ii) $<e_i, e_j> = \{0$ for $i \neq j$, and 1 otherwise$\}$; (iii) $e_i^2 = -1$; $e_i e_j + e_j e_i = 0 (i \neq j)$; (iv) $e_i e_{i+1} = e_{i+3}$ for $i \in Z_7$.

Let V be a vector space of real dimension 16 with automorphism group $Spin(9)$. The splitting

$$V = Ca \oplus Ca$$

together with the above canonical basis on each summand, endows V with what we refer to as a Cayley structure. We know that the Cayley projective plane CaP^2 is a 16-dimensional Riemannian symmetric space whose tangent space admits the Cayley structure pointwise. In the following, Let $\{I_0, ..., I_6\}$ be the Cayley structure on CaP^2.

The curvature tensor $\overline{R}$ of CaP^2 is given in [2] as follows

$$\begin{aligned}\overline{R}((a,b),(c,d))(e,f) = \frac{1}{4}((&< c,e > a - 4 < a,e > c + (ed)b^* - (eb)d^* \\ &+ (ad-cb)f^*), (4 < d,f > b - 4 < b,f > d \\ &+ a^*(cf) - c^*(af) + e^*(ad-cb))) \end{aligned} \tag{1}$$

On $Ca \oplus Ca$ we have the positive definite bilinear form $<,>$ given by

$$< (a,b),(c,d) > \; = \; < a,c > + < b,d > \tag{2}$$

3. Totally complex submanifolds. Let $V \subset T_x CaP^2$ be a real vector subspace, we say that V is a totally complex subspace if there exists an I such that there is a basis with $I = I_0$ and (i) $I_0 \subset V$, and (ii) $I_k V$ is perpendicular to V for $1 \leq k \leq 6$. Clearly, if V is a maximal subspace of this kind, then $dim_R V = 4$.

Let M be a compact Riemannian manifold isometrically immersed in CaP^2 by $j : M \to CaP^2$. Denote by h and A the second fundamental form of j and the Weingarten endomorphism respectively. Then we have

$$< h(X,Y), N > \; = \; < X, A_N Y > \tag{3}$$

where $X, Y \in TM$, $N \in TM^{\perp}$. We take $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ to be respectively the Riemannian connections on CaP^2, M and the normal connection on M. The corresponding curvature tensors are denoted by $\overline{R}$, R, and $R^{\perp}$, respectively. The first and second covariant derivatives of h are given by

$$(\overline{\nabla} h)(X,Y,Z) = \nabla^{\perp}_Z (h(X,Y) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y), \tag{4}$$

$$\begin{aligned}(\overline{\nabla}^2 h)(X,Y,Z,W) = \nabla^{\perp}_W (\overline{\nabla} h)(X,Y,Z) &- (\overline{\nabla} h)(\nabla_W X, Y, Z) \\ - (\overline{\nabla} h)(X, \nabla_W Y, Z) &- (\overline{\nabla} h)(X, Y, \nabla_W Z),\end{aligned} \tag{5}$$

where $X, Y, Z, W \in TM$. The Codazzi equation takes the following form

$$(\overline{\nabla} h)(X_{r(1)}, X_{r(2)}, X_{r(3)}) = (\overline{\nabla} h)(X_1, X_2, X_3), \tag{6}$$

where $r \in S_3$, the permutation group, and the arguments are in the tangent space of M. Recalling that h and $\overline{\nabla} h$ are symmetric, we have the Ricci identity

$$\begin{aligned}(\overline{\nabla}^2 h)(X,Y,Z,W) - (\overline{\nabla}^2 h)(X,Y,W,Z) = &- R^{\perp}(Z,W)h(X,Y) \\ &+ h(R(Z,W)X,Y) + h(X, R(Z,W)Y).\end{aligned} \tag{7}$$

We say that $j: M \to CaP^2$ is a totally complex immersion if $W = j_*(TM)$ is a totally complex subspace for each point of M. Observe that every totally complex submanifold of CaP^2 has a Kaehler structure. We set $I = I_0$, and consequently we have

$$\begin{aligned} &(a)\ \overline{\nabla}_X I = 0, \\ &(b)\ h(IX, Y) = Ih(X, Y), \\ &(c)\ A_{IN} = IA_N = -A_N I, \\ &(d)\ IR(X, IX)X = R(X, IX)IX, \end{aligned} \tag{8}$$

where $X, Y \in T_xM$ and $N \in T_xM^\perp$.

Define $f(u) = |h(u, u)|^2$, where $u \in UM$, the unit tangent bundle over M. Assume f attains its maximum at some vector $v \in UM_p$, then by [5] we have

$$A_{h(v,v)}v = |h(v, v)|^2 v. \tag{9}$$

Lemma 3.1. *Let M_n be a compact totally complex submanifold in CaP^2. Assume f attains its maximum at $v \in UM_p$, then*

$$3|h(v, v)|^2(1 - 4|h(v, v)|^2 + \sum_{i=1}^{6} < h(v, v), I_i v >^2 + 4|\nabla h)v, v, v)|^2 \leq 0. \tag{10}$$

Proof. Fix v in UM_p. For any $u \in UM_p$, let $r_u(t)$ be the geodesic in M satisfying the initial conditions $r_u(0) = p$, $r'_u(0) = u$. Parallel translating along $r_u(t)$ gives rise to a vector field $V_u(t)$. Put $f_u(t) = f(V_u(t))$, then

$$\frac{d^2}{dt^2} f_u(0) = 2 < (\overline{\nabla}^2 h)(u, u, v, v), h(v, v) > + 2|(\overline{\nabla} h)(u, v, v)|^2. \tag{11}$$

Using (6), (7) and (8), we have

$$\begin{aligned} < (\overline{\nabla}^2 h)(Iv, Iv, v, v), h(v, v) > &= < (\overline{\nabla}^2 h)(Iv, v, Iv, v), h(v, v) > \\ &= - < (\overline{\nabla}^2 h)(v, v, v, v), h(v, v) > + < R^\perp(Iv, v)h(Iv, v), h(v, v) > \\ &\quad - 2 < R(Iv, v)Iv, A_{h(v,v)}v > . \end{aligned} \tag{12}$$

From the Ricci equation, (1), (2) and (8), we obtain

$$\begin{aligned} < R^\perp(Iv, v)h(Iv, v), h(v, v) >&= < \overline{R}(Iv, v)h(Iv, v), h(v, v) > + < [A_{h(Iv,v)}, A_{h(v,v)}]Iv, v > \\ &= -\frac{1}{2}|h(v, v)|^2 - 2_{h(v,v)}v|^2 + \frac{1}{2}\sum_{i=1}^{6} < h(v, v), I_i v >^2 . \end{aligned} \tag{13}$$

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Now, by the Gauss equation and using (1), (2) and (8), we have

$$< R(Iv, v)Iv, A_{h(v,v)}v > = -|h(v, v)|^2 + 2|A_{h(v,v)}v|^2. \tag{14}$$

Since f attains its maximum at v, we have

$$\frac{d^2}{dt^2} f_v(0) + \frac{d^2}{dt^2} f_{Iv}(0) \leq 0. \tag{15}$$

Combining (11)–(15) and noticing (9), we get (10).

LEMMA 3.2. *Let M be a compact totally complex submanifold in CaP^2. Assume f attains its maximum at $v \in UM_p$, then for any $u \in UM_p$ with $< u, v > = < u, Iv > = 0$, we have*

$$|h(v, v)|^2(1 - 8|h(u, v)|^2) - |A_{h(v,v)}u|^2 + \sum_{i=1}^{6} < h(v, v), I_i u >^2 + 4|(\nabla h)(u, v, v)|^2 \leq 0. \tag{16}$$

Proof. Suppose $u \in UM_p$ such that $< u, v > = < u, Iv > = 0$. From (7), (8), (11) and the fact that f attains its maximum at v, we have

$$\begin{aligned} 0 \geq \frac{1}{2}\left(\frac{d^2}{dt^2} f_u(0) + \frac{d^2}{dt^2} f_{Iu}(0)\right) &= (\nabla^2 h)(u, u, v, v), h(v, v) > \\ &\quad + < (\nabla^2 h)(Iu, Iu, v, v), h(v, v) > + 2|(\nabla h)(u, v, v)|^2 \\ &= < R^{\perp}(Iu, u)h(Iv, v), h(v, v) > - 2 < R(Iu, u)Iv, A_{h(v,v)}v > \\ &\quad + 2|(\nabla h)(u, v, v)|^2. \end{aligned}$$

Using the Ricci equation, (1), (2), (8) and (9), we get

$$< R^{\perp}(Iu, u)h(Iv, v), h(v, v) > = -\frac{1}{2}|h(v, v)|^2 - |A_{h(v,v)}u|^2 + \sum_{i=1}^{6} < h(v, v), I_i u >^2.$$

From the Gauss equation, (1), (2), (8) and (9), we have

$$-2 < R(Iu, u)Iv, A_{h(v,v)}v > = |h(v, v)|^2 - 4|h(v, v)|^2|h(u, v)|^2.$$

From above equations, we get (16).

4. Proof of the Theorem. When $n = 1$, it follows easily from $|h| \leq 1$ that $f \leq \frac{1}{4}$, and the conclusion of Theorem is the consequence of Theorem 2.2 in **[4]**. So we need to consider the case $n > 1$. Assume the function f attains its maximum at $v \in UM_p$. If $f(v) = 0$, then M is totally geodesic. If $f(v) \neq 0$, we want to show that $f(v) \leq \frac{1}{4}$. To do this, let

$e_1, e_2 = Ie_1, \ldots, e_{2n-1}, e_{2n} = Ie_{2n-1}$ be an orthonormal basis of T_pM. By the assumption of Theorem, we have

$$n \geq \sum_{i,j=1}^{2n} |h(e_i, e_j)|^2 = 4|h(v, v)|^2 + 4\sum_{i=3}^{2n} |h(v, e_i)|^2 + \sum_{i,j=3}^{2n} |h(e_i, e_j)|^2.$$

From (9), we know that $A_{h(v,v)}v \parallel v$ and $A_{h(v,v)}Iv \parallel Iv$. Thus, for $i \geq 3$, we have

$$< A_{h(e_1,e_1)}e_i, e_1 > = < A_{h(e_1,e_2)}e_2, e_i > = 0,$$

and so when $i \geq 3$,

$$\begin{aligned}\sum_{j=3}^{2n} |h(e_i, e_j)|^2 &\geq \sum_{j=3}^{2n}(< h(e_i, e_j), \frac{h(v, v)}{|h(v, v)|} >^2 + < h(e_i, e_j), \frac{Ih(v, v)}{|Ih(v, v)|} >^2)\\ &= \frac{2}{|h(v, v)|^2}\sum_{j=3}^{2n} < h(e_i, e_j), h(v, v) >^2 = \frac{2}{|h(v, v)|^2}\sum_{j=3}^{2n} < A_{h(v,v)}e_i, e_j >^2\\ &= \frac{2}{|h(v, v)|^2}\sum_{j=3}^{2n} < A_{h(v,v)}e_i, e_j >^2 = \frac{2}{|h(v, v)|^2}\sum_{j=3}^{2n} |A_{h(v,v)}e_i|^2.\end{aligned}$$

Also, when $i \geq 3$, we have by Lemma 3.2

$$1 - 8|h(v, e_i)|^2 - \frac{4}{|h(v, v)|^2}|A_{h(v,v)}e_i|^2 \leq 0.$$

From the above equations, we obtain

$$\begin{aligned}n &\geq 4|h(v, v)|^2 + \sum_{j=3}^{2n}(4|h(v, e_i)|^2 + \frac{2}{|h(v, v)|^2}|A_{h(v,v)}e_i|^2)\\ &\geq 4|h(v, v)|^2 + \frac{2n-2}{2}.\end{aligned}$$

Thus $f(u) \leq \frac{1}{4}$ for any $u \in UM$. The theorem follows from Theorem 2.2 in [**4**]. This completes the proof of the theorem.

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