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An Aspect of Icosahedral Symmetry

Dedicated to R.V. Moody on the occasion of his 60th birthday

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Abstract. We embed the moduli space Q of 5 points on the projective line S_5-equivariantly into $\mathbb{P}(V)$, where V is the 6-dimensional irreducible module of the symmetric group S_5. This module splits with respect to the icosahedral group A_5 into the two standard 3-dimensional representations. The resulting linear projections of Q relate the action of A_5 on Q to those on the regular icosahedron.

1 Introduction

The most famous source on icosahedral symmetry is probably the celebrated book [8] of Felix Klein on "The icosahedron and the solution of equations of degree five" which appeared for the first time in 1884. Its main objective is to show that the solution of general quintic equations can be reduced to that of (locally) inverting certain quotients of actions of the icosahedral group $G \cong A_5$ on spaces related to the usual geometric realisation of the regular icosahedron in three-space. Examples for such spaces are the standard three-dimensional representation of G on $\mathbb{R}^3$ or the induced action on the (unit-)two-sphere S^2 in $\mathbb{R}^3$ which may be identified with a projective linear action on the Riemann sphere $\mathbb{P}^1$, liftable to a two-dimensional complex representation of the binary icosahedral group $\tilde{G}$. Following preceding work of Hermite, Kronecker, Brioschi, Schwarz, and starting to build up his own comprehensive theory of elliptic modular and automorphic functions, Klein showed that this inversion can be effected in terms of elliptic or hypergeometric functions.

Maybe because of Klein's preoccupation with the above-mentioned transcendental theories we do not find in his writings any serious discussion of the work on quintic equations related to the study of binary quintic forms and the corresponding SL(2)-action as done for example in the later work by Hermite ([4]), a topic obtaining extensive interest by other invariant theorists of the time like Cayley, Silvester, Aronhold and Gordan (*cf. e.g.* [5]). However, later in the hands of the Chicago school (not without influence from Klein), especially by E. H. Moore and his student Slaught ([10, 15]), this connection was taken up again in a geometric framework. The study of binary forms of a given degree d up to equivalence under coordinate changes (*i.e.* essentially the group SL(2)) and scalar multiplication is equivalent to the study of the space of d-tuples of points on the projective line $\mathbb{P}^1$ up to simultaneous projective linear transformation (action of PGL(2)) and permutation by the symmetric group S_d. "Dividing out" the PGL(2)-action, this leads to a birational action of S_d on an affine space of dimension $d - 3$. In particular, for $d = 5$, we obtain a

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birational action of S_5 and the icosahedral group $G \cong A_5$ on a plane, given attention to already in [7].

Rather recently, in a redressing of the old works ([10, 15]) in terms of matrices and actions on them ([14]), a new look was given at the geometry of the above mentioned birational S_5-action (*cf.* the "real" picture of fixed point loci at the end of [15] and on p. 373 of [14]). Of course, as soon as the icosahedral group (*i.e.* the subgroup A_5) is involved, a "real icosahedron" might lurk behind the situation, and it was soon pointed out (*cf.* the "greek picture" of [16], in the same proceedings) that there was a combinatorial-numerical coincidence, equating the numbers of 2-, 3-, 5-fold intersections of Renner's involutory fixed point divisors to those of the real symmetry planes in the regular icosahedron. However, this has remained on a somewhat formal and mysterious level, up till now (*e.g.* the coincidence does not correctly respect the corresponding isotropy groups).

The purpose of this note is to show that there is a mathematically precise relation behind this coincidence. It comes about as follows. The moduli space Q, in the sense of Mumford's geometric invariant theory ([11]), of five points on $\mathbb{P}^1$ is a surface, birational to $\mathbb{P}^2$—in fact a del Pezzo surface of degree five—equipped with a natural, regular S_5-action. It admits an S_5-equivariant embedding into $\mathbb{P}^5$—the anti-canonical embedding of the del Pezzo surface. As an A_5-module, the corresponding linear space of anti-canonical sections $\mathbb{A}^6$, which is irreducible with respect to S_5, decomposes into two three-dimensional summands on which the icosahedral group acts by its two irreducible (and conjugate) representations of degree 3. Thus, the surface Q admits two A_5-equivariant projections (of degree 5) to the usual icosahedral projective planes, and the fixed point behaviour of the action of A_5 on Q is directly related to the "classical" picture in these planes.

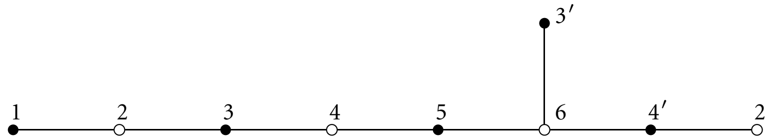
The results presented in this note are essentially contained in the first author's Diplomarbeit ([13], compare this also for a more detailed account) written, under the (partial) guidance of the second named author, at the University of Hamburg.

2 The Icosahedral Group and Some Linear Representations

As an abstract group, the icosahedral group G is the smallest non-abelian simple group, isomorphic to the alternating group A_5 on five letters. It decomposes into the following conjugacy classes:

- one class C_1 consisting of the neutral element,
- one class C_2 consisting of 15 conjugate involutions,
- one class C_3 consisting of 20 conjugate elements of order 3,
- and two classes C_5, C_5', each consisting of 12 conjugate elements of order 5 ($C_5 \cup C_5'$ forming one conjugacy class inside S_5).

All complex irreducible representations of the icosahedral group $G \cong A_5$ are most easily remembered by regarding the McKay graph associated with the binary icosahedral covering group $\tilde{G} \cong \tilde{A}_5$ of order 120, *i.e.* the extended Coxeter-Dynkin diagram of type $\tilde{E}_8$ parametrizing the irreducible complex characters of $\tilde{G}$ (*cf.* *e.g.* [8], p. 268, and references there):



Here, the attached indices yield the degrees of the representations, and the black nodes indicate those representations that descend to the icosahedral group:

- (1), the trivial representation,
- (3) and (3′), the *basic* three-dimensional representations, defined over $\mathbb{Q}(\sqrt{5})$ and conjugate under the non-trivial field automorphism of $\mathbb{Q}(\sqrt{5})$ (alternatively by an outer automorphism of A_5 realised by conjugation with an element of $S_5 \setminus A_5$),
- (5), which we won't touch in this article, and
- (4′), defined over $\mathbb{Q}$, which is obtained as the restriction of the irreducible four-dimensional permutation $\mathbb{Q}[S_5]$-module inside the standard permutation module on five elements. Its second exterior power $\Lambda^2(4')$ decomposes over $\mathbb{Q}(\sqrt{5})$ into the two basic representations (3) and (3′).

We shall also have to deal with the action of the symmetric group S_5 of order 120, having seven conjugacy classes. Here is the list of its complex irreducible representations, all defined over $\mathbb{Q}$, again suggestively denoted by their dimensions:

- (1) and (1′), the trivial and the *sign* representation,
- (4) and $(4') = (4) \otimes \text{sign}$,
- (5) and $(5') = (5) \otimes \text{sign}$,
- $(6) \cong (6) \otimes \text{sign} = \Lambda^2(4) = \Lambda^2(4')$.

The representations of A_5 and S_5 which will be of fundamental concern to us can all be constructed in a neat direct way.

Let us first consider the icosahedral group $G = A_5$. We fix once and for all one of the conjugacy classes C of G containing 12 elements of order 5. Define $\mathcal{F}_{\mathbb{Q}}$ as the 6-dimensional quotient module of the space of all functions $\mathcal{F}(C,\mathbb{Q})$ divided by the subspace $\mathcal{F}(C,\mathbb{Q})^+$ of all functions invariant with respect to inversion on C (alternatively, as the subspace $\mathcal{F}(C,\mathbb{Q})^-$ of inversion-anti-invariant functions, alternatively, as the induced representation $\operatorname{Ind}_D^G(\varepsilon)$ from a dihedral subgroup $D \subset G$ of order 10, $\varepsilon\colon D \to \mathbb{Z}_2$). Over the field $\mathbb{Q}$, this yields an irreducible representation of G. However, it is not absolutely irreducible. After extension of scalars to $\mathbb{Q}(\sqrt{5})$, it splits as a G-module into the sum of the two irreducible three-dimensional representations related to the regular icosahedron in three-space.

It is illuminating, though we won't need that later, to regard this situation in more detail on the background of the icosahedral interpretation of G.

We may identify C with the 12 vertices of an abstract (*i.e.* graph theoretical) icosahedron, two vertices $g, g' \in C$ being neighbours, *i.e.* being connected by one of the 30 edges, if and only if the product gg' has order 3. Then every $g \in C$ has 5 neighbours $N(g) = \{g_1, \dots, g_5\}$, an antipode g^{-1} and 5 antipodes of $N(g)$, *i.e.* neighbours $N(g^{-1}) = \{g_1^{-1}, \dots, g_5^{-1}\}$ of the antipode g^{-1}. The operator

$$\Delta\colon \mathcal{F}_{\mathbb{Q}} \to \mathcal{F}_{\mathbb{Q}}, \quad \Delta(F)(g) = \sum_{g' \in N(g)} F(g'),$$

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commutes with G and satisfies $\Delta^2 = 5 \cdot \mathrm{Id}$. This yields

$$\mathrm{End}_{\mathbb{Q}[G]}(\mathcal{F}_{\mathbb{Q}}) \cong \mathbb{Q}[\Delta]/(\Delta^2 - 5) \cong \mathbb{Q}(\sqrt{5}).$$

The three-dimensional representations of G are then realised in the eigenspaces of Δ, defined over $\mathbb{Q}(\sqrt{5})$. Geometrically, the existence of a square root of 5 in the ground field $k \supset \mathbb{Q}$ or, alternatively, that of a *golden section* $\tau = \frac{1+\sqrt{5}}{2}$, allows the construction of a regular icosahedron inside k^3 (the main constructive problem being that of a regular plane pentagon). We may then map the abstract icosahedron introduced above onto this regular icosahedron, *i.e.* map the elements of C onto its 12 vertices preserving the neighbourhood and the G-action. Extending by linearity, this provides us with a non-zero G-equivariant map from $\mathcal{F}_k$ onto one of the standard 3-dimensional $k[G]$-modules (3), the kernel being the conjugate module $(3')$. Alternate choices in the above construction may lead to the conjugate situation.

Note that a basis for $\mathcal{F}_{\mathbb{Q}}$ may be obtained by six elements of G (identified with the corresponding δ-functions) representing all elements of C up to inversion), *e.g.* $C_+ = \{g\} \cup N(g)$ for a fixed $g \in G$. (For the linear G-action, we then have to interpret the inverse elements in $C_- = C \setminus C_+$ as the negatives of those in C_+).

Later, we shall also encounter an analogous construction of the 6-dimensional representation (6) for the full symmetric group S_5. For that, fix the conjugacy class $\tilde{C}$ consisting of all 24 order-5-elements in S_5. (Note that $\tilde{C}$ is equivariantly stable under taking second powers of elements, $s\colon g \mapsto g^2$, and that $s^2\colon g \mapsto g^4$, equals inversion). In analogy to the icosahedral case, we may now consider the 12-dimensional $\mathbb{Q}$-S_5-module $\mathcal{F}(\tilde{C}, \mathbb{Q})^-$, a $\mathbb{Q}$-basis of which may be provided by the elements of the icosahedral classes C_+ and $s(C_+)$. This shows that $\mathcal{F}(\tilde{C}, \mathbb{Q})^-$ decomposes as a $\mathbb{Q}[A_5]$-module into two copies of the above described 6-dimensional $\mathbb{Q}$-irreducible A_5-representation, which implies that, as a $\mathbb{Q}[S_5]$-module, it decomposes into two copies of the irreducible 6-dimensional S_5-representation, $\mathcal{F}(\tilde{C}, \mathbb{Q})^- \cong (6) + (6)$. In particular, any non-trivial sub- or quotient module of $\mathcal{F}(\tilde{C}, \mathbb{Q})^-$ is isomorphic to the 6-dimensional S_5-module (6). (Note, by the way, that s induces a complex structure on $\mathcal{F}(\tilde{C}, \mathbb{R})^-$.)

3 Five Points on the Projective Line

Let us look at five ordered points $\{p_1, \dots, p_5\}$ on the complex projective line $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ and classify such five-tuples up to simultaneous transformation under the projective linear group $\mathrm{PGL}(2) = \mathrm{PGL}(2, \mathbb{C})$. Let $\tilde{U}$ be the subset of $X = (\mathbb{P}^1)^5$ consisting of 5-tuples with at least three distinct points. Then the action of PGL(2) is free on $\tilde{U}$, and there is a geometric quotient $\tilde{q}\colon \tilde{U} \to \tilde{Q}$ of this action which is a PGL(2)-principal fibre bundle, however, not over an ordinary but over a non-separated scheme $\tilde{Q}$ (*cf.* the discussion in [12], Chapter 2). To remedy this, one has to restrict to the slightly smaller open subset U of stable points in X consisting of those 5-tuples which do not contain three pairwise identical points (note, due to 5 being an odd number, this is also the set of semistable points in X). Then Mumford's general theory ([11, 12]) provides us with a geometric quotient $q\colon U \to Q$ onto a projective scheme Q, by simple dimension reasons, an algebraic surface, now.

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Of course, Q contains as a dense open affine subset the quotient space Q' of the set U' of 5-tuples in X consisting of pairwise distinct points, which is the space considered in [14, 16] and which may be identified with the complement of the seven lines $\{x=0\}\cup\{x=1\}\cup\{x=\infty\}\cup\{y=0\}\cup\{y=1\}\cup\{y=\infty\}\cup\{x=y\}$ in $\mathbb{P}^1\times\mathbb{P}^1$. Since the action of $\mathrm{PGL}(2)$ on X commutes with the permutation action of S_5 on the ordering of points, $\sigma(p_1,\dots,p_5)=(p_{\sigma^{-1}(1)},\dots,p_{\sigma^{-1}(5)})$, the last action descends to a regular S_5-action on these quotient spaces, $\tilde{Q}, Q, Q'$, thereby realising the birational plane action of the icosahedral group $G=A_5$ mentioned in the introduction.

The basis for a finer description of the structure of the *moduli space* Q and its symmetry is a concrete S_5-equivariant embedding into projective 5-space $\mathbb{P}^5$ which will be dealt with in this and the next section.

In [12], Chapter 2, Theorem 1, an ample S_5-equivariant invertible sheaf $\mathcal{L}$ on Q is constructed which yields a realisation of Q as the projective spectrum of the graded $\mathbb{C}$-algebra A of all global sections $\bigoplus_{k\in\mathbb{N}} H^0(Q,\mathcal{L}^k)$. More precisely, $\mathcal{L}$ is obtained by descending (in an elementary way) the $\mathrm{PGL}(2)$-linearised sheaf $\tilde{\mathcal{L}}=\bigotimes_{i=1,\dots,5}\mathrm{pr}_i^*\,\mathcal{O}_{\mathbb{P}^1}(2)$ from $U\subset(\mathbb{P}^1)^5$ to the quotient Q. In our situation, here, one can improve on that by explicitly showing that $\mathcal{L}$ is in fact very ample, the global sections $H^0(Q,\mathcal{L})$ already providing the searched for projective embedding.

For that let us switch to the slightly more algebraic language of *classical invariant theory*. The moduli problem for the action of $\mathrm{PGL}(2)$ on $(\mathbb{P}^1)^5$ is the same as that for the action of the product $\mathrm{SL}(2)\times T$, $T\cong(\mathbb{C}^*)^5$, acting, via both factors individually, on $(\mathbb{C}^2\setminus 0)\otimes\mathbb{C}^5$. Let $\delta\colon\ \mathrm{SL}(2)\times T\to\mathbb{C}^*$ denote the character $\delta\big(g,(t_i)\big)=(\prod_i t_i)^2$. Then $H^0(Q,\mathcal{L}^k)$ is the subspace $\mathbb{C}[\mathbb{C}^2\times\mathbb{C}^5]_{k\delta}$ of all polynomials on $\mathbb{C}^2\times\mathbb{C}^5$ which transform under $\mathrm{SL}_2\times T$ with respect to the character $k\delta$. In particular, such elements are invariant under SL_2. Note that, due to the S_5-invariance of δ, we obtain an S_5-action on the graded algebra A and all its homogeneous components $H^0(Q,\mathcal{L}^k)$.

The determination of the invariants of the $\mathrm{SL}(2)$-action is a direct consequence of the so-called First Fundamental Theorem for the special linear group (*cf. e.g.* [19], II.6) which implies, in our situation, that $\mathbb{C}[\mathbb{C}^2\otimes\mathbb{C}^5]^{\mathrm{SL}(2)}$ is generated by all 2×2-minors (i,j) formed from two column vectors p_i and p_j, $i,j=1,\dots,5$ (here we consider $\mathbb{C}^2\otimes\mathbb{C}^5$, in matrix interpretation, as the set of 5-tuples of 2×1-column vectors p_i; we will adhere to the obvious convention $(i,j)=-(j,i)$). The Second Fundamental Theorem (*cf. loc. cit.*) implies that the only relations satisfied by these minors are the Plücker relations (for indices $i,j,k,l=1,\dots,5$):

$$(i,j)(k,l)+(i,k)(l,j)+(i,l)(j,k)=0.$$

Note that $T\times S_5$ acts naturally on the free polynomial ring generated by the minors (i,j) and that the ideal generated by the Plücker relations is $T\times S_5$-stable.

From a geometric point of view, this construction realises the partial quotient of $\big((\mathbb{C}^2\setminus 0)\otimes\mathbb{C}^5\big)_{\text{stable}}$ by $\mathrm{SL}(2)$ as an open subset of the affine cone over the Grassmannian $\mathbb{G}_{2,5}$ of 2-planes in $\mathbb{C}^5$. To obtain our moduli space Q, we still have to form the quotient of that set by the action of T in a further step, which is achieved in the following way.

If m is a monomial product $\prod_{\ell=1,\dots,m}(i_\ell,j_\ell)$ of minors in $\mathbb{C}[\mathbb{C}^2\otimes\mathbb{C}^5]^{\mathrm{SL}(2)}$, then it transforms under T with respect to $k\delta$, *i.e.*, it determines a section of $H^0(Q,\mathcal{L}^k)$,

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exactly when each column $i \in [1, \ldots, 5]$ occurs exactly $2k$-times among the indices i_ℓ, j_ℓ. In particular, we then have $m = 5k$. It is now an exercise in elementary combinatorics (performed in [13], Chap. 4) to show that the algebra $A = \bigoplus_{k \in \mathbb{N}} H^0(Q, \mathcal{L}^k)$ is generated by its lowest degree terms, *i.e.*, by the space of global sections $H^0(Q, \mathcal{L})$. Thus we obtain:

Theorem 1 *The anticanonical sheaf* $\tilde{\mathcal{L}} = \bigotimes_{i=1,\ldots,5} \operatorname{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(2)$ *on* $X = (\mathbb{P}^1)^5$ *admits a unique* $\mathrm{PGL}(2)$*-linearisation. It descends from* $U \subset (\mathbb{P}^1)^5$ *to a very ample sheaf* $\mathcal{L}$ *on the moduli space* Q*. In particular,* Q *is an irreducible algebraic surface admitting a projective embedding into* $\mathbb{P}\big(H^0(Q, \mathcal{L})^*\big)$.

Remark Since X is a principal PGL_2-bundle over Q, it is easily seen that $\mathcal{L}$ is the anticanonical sheaf on Q.

4 Equivariant Embedding of the Moduli Space

According to the last section, we obtain a projective embedding of Q into the projective space $\mathbb{P}\big(H^0(Q, \mathcal{L})^*\big)$, which is equivariant with respect to the given S_5-actions.

Theorem 2 *As a* $\mathbb{C}$-S_5*-module,* $H^0(Q, \mathcal{L})$ *is isomorphic to the unique irreducible* 6*-dimensional* S_5*-module discussed in Section 2. In particular, as an* A_5*-module,* $H^0(Q, \mathcal{L})$ *decomposes into the direct sum of the two three-dimensional "icosahedral" representations,* $V = (3) + (3')$.

Proof As explained above, the space of global sections $V = H^0(Q, \mathcal{L})$ may be viewed as a subspace of $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^5]^{\mathrm{SL}(2)}$, which is a quotient of the free polynomial ring $M = \mathbb{C}[(i, j)]$ on 10 formal minors (i, j) by the Plücker relations. The set of all monomials m (of degree 5 in the minors (i, j)) mapping to a generating set of V may be obtained by the following construction:

(i) Let g be one of the 24 elements of order 5 in S_5. Define

$$m_g = \big(g(1), g(2)\big) \cdot \big(g(2), g(3)\big) \cdot \big(g(3), g(4)\big) \cdot \big(g(4), g(5)\big) \cdot \big(g(5), g(1)\big).$$

(ii) Let g be one of the 20 elements of order 3 in S_5. If i and $j \in \{1, \ldots, 5\}$ are the fixed letters of g, k one of the other three, put

$$m_g = (i, j)^2 \cdot \big(k, g(k)\big) \cdot \big(g(k), g^2(k)\big) \cdot \big(g^2(k), k\big).$$

Up to sign, this gives $(6+6)+10$ distinct monomials. Exploiting the Plücker relations, the S_5-equivariance, $\sigma m_g = m_{\sigma g \sigma^{-1}}$, between conjugacy classes and monomials, and the explicit module constructions in Section 2, one obtains the result.

Remarks

(1) As a by-result of the proof above, one gets an explicit basis of V, *e.g.* the set of (residue classes of the) monomials m_g where g runs through the set C_+ of 6 elements in A_5 marked in Section 2. Enumerating these elements by $g_0, \ldots, g_5$,

one thus obtains the following 'explicit' definition of the projective embedding $\Phi\colon Q \to \mathbb{P}(V^*) = \mathbb{P}^5$ of Q:

$$\Phi(q) = \left(m_{g_0}(q) : \cdots : m_{g_5}(q) \right).$$

(2) This explicit form of the map Φ, together with local arguments, can of course be used to give a direct, elementary proof of the projectivity of Q. Note, that the set of stable points $U \subset (\mathbb{P}^1)^5$ is covered by open subsets of the form $U_{i,j,k}$ consisting of those 5-tuples with p_i, p_j, p_k pairwise distinct. The quotient $Q_{i,j,k}$ of such an open set is isomorphic to the complement of the three 'diagonal' points $(0:0), (1:1), (\infty, \infty)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and it is easily seen that Φ realises Q as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in these points. In other words, Q is obtained by blowing up three generic points on $\mathbb{P}^1 \times \mathbb{P}^1$ or, alternatively, of four generic points in the projective plane $\mathbb{P}^2$. Thus Q is a smooth del Pezzo surface of degree 5 embedded via Φ by its anti-canonical system (*cf. e.g.* [3, 9, 18]). A different identification of Q with a del Pezzo surface of degree 5 is given in [2]. It is based on the system of 10 exceptional curves of the first kind on Q.

(3) In (1) above, we have chosen the basis (of classes) $\{m_g \,;\, g \in C_+\}$ for V. Another basis, realising the same matrix form for the action of A_5, is given by (the classes of) $\{m_g \,;\, g \in s(C_+)\}$, leading to a rational and A_5-invariant 6×6-matrix $S = \big((S_{g,h})\big)_{g,h \in C_+}$

$$m_{s(g)} = \sum_{h \in C_+} S_{g,h} m_h.$$

It corresponds to an A_5-equivariant "induced" endomorphism $\bar{s}$ of V, defined over $\mathbb{Q}$. However, since s does not respect the kernel of the projection from $\mathcal{F}(\tilde{C}, \mathbb{Q})^-$ to V, it is highly non-canonical (depending on the explicit choice an A_5-equivariant lift of V to $\mathcal{F}(\tilde{C}, \mathbb{Q})^-$). For example, it cannot satisfy anymore the relation $\bar{s}^2 = -1$ satisfied by s. Explicit matrix entries for S can, of course, be determined by means of the Plücker relations.

Any element $m \in H^0(Q, \mathcal{L}^k)$ determines a form of degree k on $H^0(Q, \mathcal{L})^*$, in particular, for $g \in C_+$, $m_g m_{s(g)}$ determines a quadratic form. It is invariant under A_5 since it consists (up to sign) of exactly all 10 minors (i, j) (if g is visualised as a pentagon with edges $g(1), \ldots, g(5)$, then $s(g)$ corresponds to the associated pentagram). Moreover, the different quadrics $m_g m_{s(g)}$, $g \in C_+$, are transformed into each other by A_5. Thus they all must equal each other. This provides already the first half of the following theorem:

Theorem 3 *As a subvariety of $\mathbb{P}\big(H^0(Q, \mathcal{L})^*\big)$, the moduli space Q is defined by the 15 quadratic equations*

$$m_g . m_{s(g)} = m_h . m_{s(h)},$$

where g and h run over all distinct pairs of elements in C_+.

Unfortunately, at this moment we don't have an elegant direct proof of the second half of this statement, *i.e.* that besides the points in Q there are no other solutions to

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the given quadratic equations. The proof in [13] rests on a verification using Computer Algebra (MACAULAY2).

Remarks

(1) Obviously, 5 of the 15 equations are sufficient (as are the custom for del Pezzo surfaces of degree 5).
(2) Using the m_g, $g \in C_+$, as linear coordinates and the matrix S, these equations can be put in coordinate form $\sum_{k \in C_+}(m_g S_{g,k} - m_h S_{h,k}) m_k = 0$.
(3) Using Computer Algebra and the explicit equations, one may also show independently of the theory of del Pezzo surfaces that Q has degree 5 in $\mathbb{P}^5$.

As a further application of Computer Algebra and the explicit equations, let us mention the following result which also follows from the realisation of Q as a del Pezzo surface of degree 5.

Theorem 4 *As a subvariety of $\mathbb{P}^5$, the moduli space Q contains exactly 10 projective lines, all having self-intersection -1 (exceptional curves of the first kind). Two such curves are disjoint or meet transversely in one point. There are 15 such intersection points. The group S_5 acts transitively on this configuration.*

Remark These 10 lines are the images of the 'diagonal' divisors on U where two points coincide. They make up the discriminant locus $D = Q \setminus Q'$. This locus does not enter, *i.e.* lies outside, the 'minimal' affine picture of [14, 16].

5 Icosahedral Projections of the Moduli Space

We shall have a closer look now at the position of Q inside $\mathbb{P}^5$. Since the points of Q generate an S_5-stable linear subspace of $\mathbb{P}^5 = \mathbb{P}(V^*)$, the S_5-irreducibilty of V implies that Q cannot lie inside any hyperplane.

Let $V = W_1 \oplus W_2$ denote the decomposition of V as an A_5-module into the two three-dimensional "icosahedral" representations. Then $\mathbb{P}(W_i)$ is an A_5-stable projective plane inside $\mathbb{P}^5$, and we have two natural A_5-equivariant projections $\mathbb{P}^5 \setminus \mathbb{P}(W_i) \to \mathbb{P}(W_j)$, $i \neq j$.

Theorem 5 *The intersections $Q \cap \mathbb{P}(W_i)$, $i = 1, 2$, are empty. In particular, there are two A_5-equivariant projections*

$$\pi_i \colon Q \to \mathbb{P}(W_i)$$

of degree 5.

Proof For $i = 1, 2$, the intersections $Q \cap \mathbb{P}(W_i)$ are A_5-stable subvarieties of $\mathbb{P}(W_i)$ which are permuted by the elements in $S_5 \setminus A_5$. If they were of dimension 2, they would be components of Q contradicting its irreducibility. If they were of dimension 0, they would each have to consist of at least 6 points (the minimal A_5-orbit in $\mathbb{P}(W_i)$, *cf.* [8], II4). However, they are intersections of quadrics, and thus have at most 4 points. Finally, assume they were of dimension 1. Then they would have to equal the fundamental A_5-stable conic C_i in $\mathbb{P}(W_i)$, *cf.* [8], II4, *i.e.* all the restrictions to W_i of the quadrics $m_g \cdot m_{s(g)} - m_h \cdot m_{s(h)}$ would have to be proportional to the corresponding non-degenerate A_5-invariant quadric on W_i. However, the restrictions of $m_g . m_{s(g)}$ to W_i are proportional to squares of linear forms (because of the A_5-equivariance of s on V and the irreducibility of W_i, we have for the restricted linear forms $m_{s(g)}|_{W_i} = \lambda_i \cdot m_g|_{W_i}$ for some $\lambda_i \in \mathbb{C}^*$), and their differences give degenerate conics (pairs of lines) in $\mathbb{P}(W_i)$. Thus the first claim. The second is then a trivial consequence.

Remarks

(1) A look at the fibres of these projections π_i (see *e.g.* below) quickly reveals that they cannot be (ramified) Galois covers.
(2) Under these projections, the 10 lines in D are mapped onto the 10 polars of the $10 = 20/2$ midpoints of the faces of the projective icosahedron inside $P^2(\mathbb{R}) \subset \mathbb{P}(W_i)$.

Theorem 5 is the basis for a detailed investigations of the fixed points of A_5 (and of S_5) on Q and their relation to the "usual" icosahedron. Let us just mention some results (for more details *cf.* [13]; it is quite instructive to derive these results by using the symplectic reduction of the PGL(2)-quotient of $(\mathbb{P}^1)^5$ to that of SO(3) acting on 5 points p_i on the 2-sphere satisfying the centre of gravity condition $\sum_i p_i = 0$, *cf.* [11], Appendix):

• There are 24 elements of order 5 in A_5. Each such element has exactly two fixed points on Q, and there is a total of 12 such points on Q. They map 2:1 onto the 6 vertices of the projective icosahedron inside $P^2(\mathbb{R}) \subset \mathbb{P}(W_i)$.

• There are 20 elements of order 3 in A_5. Each such element has exactly 4 fixed points on Q, two of which lie on the discriminant $D = Q \setminus Q'$. There is a total of 20 such points inside the "regular" part Q', mapping 2:1 onto the 10 midpoints of the faces of the projective icosahedron inside $P^2(\mathbb{R}) \subset \mathbb{P}(W_i)$. On each projective line in D there are 2 such further fixed points.

• There are 15 elements of order 2 in A_5. Each such element has a fixed line isomorphic to $\mathbb{P}^1$ and a single isolated fixed point on Q. There is a total of 15 such lines and 15 such points in Q, the points being the intersection points of the 10 lines in D. The 15 lines map to the polars of the $15 = 30/2$ midpoints of the edges of the projective icosahedron inside $P^2(\mathbb{R}) \subset \mathbb{P}(W_i)$.

6 Questions

Due to the formal limitations on a "Diploma"-project like this one, a number of questions which pose themselves quite naturally could not be addressed in ([13]).

• The most urgent problem would be to obtain a better and more detailed understanding of the ramified coverings π_i of Q onto the icosahedral planes $\mathbb{P}(W_i)$ and their effect on the fixed point behaviour which is well studied in $\mathbb{P}(W_i)$ (see *e.g.* [8], II4). We have already mentioned above that they cannot be Galois coverings.

• Does the interpretation of Q as a del Pezzo surface of degree 5 give any useful information in that respect (*cf.* [3, 9, 18])? It certainly illuminates the discussions in

[15].

• Though these coverings do not show up in the work [1], they might be related to them. Certainly, Q itself is realized as a ball quotient (see [2]), and thus also the further covering of degree 5 over $\mathbb{P}(W_i)$ should have an interpretation in terms of ball quotients, *i.e.* it should give a 2-dimensional analogue of Klein's original 1-dimensional situation (hypergeometric and elliptic modular functions) by means of the Appel-Picard hypergeometric functions and automorphic functions on the 2-ball (see [2, 6]).

• One might also try to establish a direct relation between Klein's original 1-dimensional approach and the variety Q. For example, is Klein's use of his "Hauptfläche" (see [8], II3) related to the quadric H obtained inside the direct product $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ as the product of the fundamental conics $C_1 \times C_2$? More generally, does the variety Q embed into the product $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ (via the two projections), and is there an analogue of Gordan's S_5-invariant theory ([8], II3, Sections 6–11) for this product. This would lead to a complete icosahedral interpretation of the $\mathrm{SL}(2)$-invariant forms of binary quintics (*cf. e.g.* [5], I7) as they are the same as the S_5-invariants on the affine cone over Q. For some observations in that respect *cf.* [13], Chapter 5. Finally, let us point out that Klein himself, led by line geometric reasoning, invokes the decomposition of the 6-dimensional S_5-module $\Lambda^2\mathbb{C}^4$ into the two 3-dimensional representations of A_5 in his concluding, general reflections on the resolution process (see [8], II5).

It seems to us that any progress on these questions would be worthwhile, exhibiting further *aspects of icosahedral symmetry*.

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