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ON QUASI-AMBIVALENT GROUPS

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1. Introduction. The prototype for applications of group theory to physics, and to mathematical physics, is the quantum theory of angular momentum [**1**]; the use of such techniques is now almost universal, and familiarly (through somewhat imprecisely) known as "Racah algebra". To categorize, group theoretically, those characteristics which underlay this applicability to physical problems, Wigner [**30**] isolated two significant conditions, and designated groups possessing these properties as *simply reducible.*

The two conditions for simple reducibility are:

(a) *Every element is equivalent to its reciprocal, i.e., all classes are ambivalent.*

(b) *The Kronecker (or "direct") product of any two irreducible representations of the group contains no representation more than once.*

A group possessing only the first condition is called "ambivalent"; if it admits only the second condition it is termed "multiplicity free", (abbreviated m.f.).

Condition (b) has a direct connection with physical applications [**18**; **30**–**32**]. It implies that the "correct linear combinations" of products of basis functions are determined to within phase factors; that is to say, the solution of the physical problem is uniquely determined from symmetry arguments. It is important to realize, however, that condition (b) has the nature of a sufficient, but not necessary, condition for this uniqueness.

The role of condition (a) in physical problems is based upon the fact that the ground field in such problems is that of the complex numbers; hence conjugation is defined and significant in the resulting representation theory. For an ambivalent group, all characters are real; hence complex conjugation takes an irreducible representation into itself. This property is equivalently expressed by the fact that for an ambivalent group the Schur-Frobenius invariant [**15**; **16**] is ± 1 only.

Mackey [**21**] has pointed out that it is possible to generalize the concept of ambivalence; a multiplicity-free group satisfying also this weaker condition possessed, as he showed, many of the desirable properties of a simply reducible group. Mackey's generalization replaced condition (a) by condition (a′):

(a′) *The group admits an involutory anti-automorphism which preserves classes.* (*Quasi-ambivalence condition.*)

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That ambivalence implies quasi-ambivalence is easily seen by using the mapping $g \rightarrow g^{-1}$, $g \in G$, as the required involutory anti-automorphism.

In his thesis, Sharp [**26**] investigated in detail the basis for Racah algebra, the structure used in applications to physics of simply reducible groups. He demonstrated that the role of ambivalence in Racah algebra is simply to permit a consistent definition of the phase $(-1)^j$, where $(-1)^{2j} \equiv C_j$, the Schur-Frobenius invariant [**15**; **16**] for the irreducible representation labelled by j. The property of quasi-ambivalence is sufficient to permit both a generalization of the Schur-Frobenius invariant, and the corresponding phase [**26**]. (This generalized Schur-Frobenius invariant has the property that it is equal to ± 1 for all irreducible representations of a quasi-ambivalent group, and vanishes for at least one irreducible representation if the group is not quasi-ambivalent.) For quasi-ambivalent, multiplicity-free groups there exists, as discussed by Sharp [**26**], a generalized form of Racah algebra.

Experience with the construction of the Racah algebra [**3**] for the semi-simple Lie group $SU(3)$ (which is quasi-ambivalent as will be proved in the following section) shows that it is also possible to construct the Racah algebra for this group, although $SU(3)$ is not multiplicity-free, i.e., does not satisfy condition (b) [**5**; **6**; **20**]. This example suggests that in the construction of a Racah algebra, the quasi-ambivalent condition is more essential than the (difficult and restrictive) property of being multiplicity-free. Whether or not this surmise is true is the subject of current research. ($SU(3)$ satisfies the weaker condition of being a simple phase group [**10**; **13**; **14**; **28**; **29**]. It is not known as to whether the simple phase property is a prerequisite to construct a Racah algebra [**12**].)

However, the property of quasi-ambivalence for a group is sufficiently important in the framework of Racah algebra to justify a separate investigation. The present paper is devoted to such a discussion.

2. Quasi-ambivalence of simple Lie groups. In this section we record some elementary properties of quasi-ambivalent groups and demonstrate that many familiar Lie groups are quasi-ambivalent.

We denote elements of a group $\mathscr{G}$ by $R, S, T, \ldots$. Our definition states that $\mathscr{G}$ is quasi-ambivalent if there is a mapping σ of $\mathscr{G}$ onto $\mathscr{G}$ such that

(1) $\sigma(RS) = \sigma(S)\,\sigma(R)$ for all R, S in $\mathscr{G}$;

(2) $\sigma(\sigma(R)) = R$ for all R in $\mathscr{G}$;

(3) given R in $\mathscr{G}$, there exists R' in $\mathscr{G}$ such that $\sigma(R) = R'^{-1} R R'$. One notes immediately that any ambivalent group is quasi-ambivalent (take $\sigma(R) = R^{-1}$) and that any abelian group is quasi-ambivalent (take $\sigma(R) = R$).

To every involutory anti-automorphism σ corresponds an involutory automorphism τ which is the mapping $R \rightarrow \tau(R) \equiv \sigma(R^{-1})$. To see this one need only note that $\sigma(R^{-1})\,\sigma(S^{-1}) = \sigma(S^{-1} R^{-1}) = \sigma((RS)^{-1})$ and $\sigma[(\sigma(R^{-1}))^{-1}] = \sigma(\sigma(R)) = R$. Conversely, to every involutory automorphism τ corresponds an involutory anti-automorphism σ (take $\sigma(R) = \tau(R^{-1})$). These remarks make it clear that a group $\mathscr{G}$ is quasi-ambivalent if and only if there exists an involutory automorphism τ, such that R and $\tau(R^{-1})$ are conjugate elements for all R in $\mathscr{G}$. If there exists an inner involutory automorphism with this property (in particular $\tau(R) = R$) then the group is ambivalent.

The connected compact simple Lie groups fall into the four infinite series A_n, B_n, C_n, and D_n ($n = 1, 2, \ldots$) of classical groups and the five exceptional groups E_6, E_7, E_8, F_4 and G_2. The groups of the series A_n are the special unitary groups $\mathrm{SU}(n + 1)$, those of the series B_n are the special orthogonal groups $\mathrm{SO}(2n + 1)$, those of the series C_n are the symplectic groups $\mathrm{Sp}(2n)$ and those of the series D_n are the special orthogonal groups $\mathrm{SO}(2n)$. It has been shown in the literature that the following of the above groups have real characters only and hence are ambivalent: $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$ as well as the exceptional Lie groups E_7, E_8, F_4 and G_2 and the groups $\mathrm{SO}(2n)$ if $n = 2k$ (cf. [**2**; **8**; **10**; **22–24**]).

The special unitary groups $\mathrm{SU}(n + 1)$, although not ambivalent, are quasi-ambivalent as we can see as follows. We consider the involutory anti-automorphism

$$\sigma(A) = A^T, \tag{1}$$

where A^T denotes the transpose of the matrix A. We shall prove now that this mapping preserves the classes of $\mathrm{SU}(n + 1)$. Every element A of $\mathrm{SU}(n + 1)$ can be brought to diagonal form by the similarity transformation

$$D = BAB^{-1}, \tag{2}$$

where B is also from $\mathrm{SU}(n + 1)$. From (2) we have

$$A = B^{-1}DB,$$

hence

$$\begin{aligned} \sigma(A) &= \sigma(B)\,\sigma(D)\,\sigma(B^{-1}) \\ &= B^T\, D^T(B^{-1})^T = B^T\, D\,(B^{-1})^T \\ &= B^T\, B\, A\,(B^T\, B)^{-1}, \end{aligned} \tag{3}$$

i.e., $\sigma(A)$ and A lie in the same class.

Next we consider the groups $\mathrm{SO}(2n)$ with $n = 2k + 1$. These groups are quasi-ambivalent as well. An element A of $\mathrm{SO}(2n)$ with $n = 2k + 1$ can by means of a similarity transformation be brought into the form

$$D = BAB^{-1} = \begin{bmatrix} C_1 & 0 & \cdots & & 0 \\ 0 & C_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots & & C_{2k+1} \end{bmatrix}$$

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where C_i are matrices of the form

$$(5)\qquad C_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \quad \text{(see [25]; p. 229).}$$

This transformation can be brought about with a matrix B, which belongs to the group $SO(2n)$.

We now define the matrix R as follows:

$$R = \begin{bmatrix} R_1 & 0 & \cdots & \cdots & 0 \\ 0 & R_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & R_{2k+1} \end{bmatrix}$$

where

$$(7)\qquad R_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider the involutory anti-automorphism

$$(8)\qquad \sigma(A) = R\,A^T R.$$

Observe that although R does not belong to $SO(2n)$ ($\det R = -1$), $R\,A^T\,R$ belongs to $SO(2n)$. Next we prove that the mapping σ preserves the classes of $SO(2n)$. From equation (4) we have

$$A = B^{-1} D\,B,$$

hence

$$(9)\qquad \begin{aligned} \sigma(A) &= \sigma(B)\,\sigma(D)\,\sigma(B^{-1}) \\ &= R\,B^T R\,R\,D^T R\,R(B^{-1})^T R \\ &= R\,B^T R\,D\,R(B^{-1})^T R \\ &= R\,B^T R\,B\,A\,B^{-1} R(B^{-1})^T R \\ &= (R\,B^T R\,B)\,A\,(R\,B^T R\,B)^{-1}. \end{aligned}$$

Because $R\,B^T R\,B$ is an element of $SO(2n)$, $\sigma(A)$ and A lie in the same class.

Thus we have proved that all connected compact simple Lie groups (except for the exceptional group E_6) are quasi-ambivalent.

To conclude this section we make a few remarks. The groups $U(n)$ and $GL(n)$ are also quasi-ambivalent. For $U(n)$ the proof is similar as for $SU(n)$. For $GL(n)$ we can also take the anti-automorphism

$$(10) \quad \sigma(A) = A^T.$$

By means of a similarity transformation A and A^T can always be transformed into the same Jordan normal form (cf. [**25**, p. 34]).

A careful investigation shows that the proper orthochronous Lorentz group $SO(2, 1)$ (which is not ambivalent) is quasi-ambivalent. A detailed proof of this assertion will not be given in this paper.

3. Criteria for Quasi-ambivalence. Here we shall present some criteria for a group to be non-quasi-ambivalent and give examples of non-quasi-ambivalent groups. Most of the criteria are valid for compact groups, but we shall formulate them for finite groups only.

THEOREM 1. *If a finite group $\mathscr{G}$ is quasi-ambivalent and if τ is an involutory automorphism such that R and $\tau(R^{-1})$ are in the same class for all elements R, of $\mathscr{G}$ then*

$$(11) \quad \frac{1}{g} \sum_{R \in \mathscr{G}} \chi^{(j)}(R) \chi^{(j)}(\tau(R)) = 1$$

for all irreducible representations j with character $\chi^{(j)}$.

Proof. Because R and $\tau(R^{-1})$ are in the same class, there exists an element $S \in \mathscr{G}$ such that $\tau(R^{-1}) = S R S^{-1}$. Hence $\chi^{(j)}(\tau(R)) = \chi^{(j)*}(\tau(R^{-1})) = \chi^{(j)*}(S R S^{-1}) = \chi^{(j)*}(R)$ and

$$\frac{1}{g} \sum_{R \in \mathscr{G}} \chi^{(j)}(R) \chi^{(j)}(\tau(R)) = \frac{1}{g} \sum_{R \in \mathscr{G}} \chi^{(j)}(R) \chi^{(j)*}(R) = 1$$

(* denotes complex conjugation).

COROLLARY 1. *If a finite group $\mathscr{G}$ has a fixed irreducible representation j, such that for all involutory automorphisms τ of $\mathscr{G}$ we have*

$$(12) \quad \frac{1}{g} \sum_{R \in \mathscr{G}} \chi^{(j)}(R) \chi^{(j)}(\tau(R)) = 0,$$

then $\mathscr{G}$ is a non-quasi-ambivalent group.

COROLLARY 2. *Let $\mathscr{G}$ be a finite non-ambivalent group. If for all involutory outer automorphisms τ of $\mathscr{G}$ we have*

$$(13) \quad \frac{1}{g} \sum_{R} \chi^{(j)}(R) \chi^{(j)}(\tau(R)) = 0$$

for some fixed irreducible representation j, then $\mathscr{G}$ is non-quasi-ambivalent.

Proof of Corollary 2. Consider an irreducible representation j_1 with a complex valued character and suppose that τ_1 is an involutory inner automorphism. Then

$$\frac{1}{g}\sum_{R\in\mathcal{G}} \chi^{(j_1)}(R)\,\chi^{(j_1)}(\tau_1(R)) = \frac{1}{g}\sum_{R\in\mathcal{G}} \chi^{(j_1)}(R)\,\chi^{(j_1)}(S\,R\,S^{-1})$$
$$= \frac{1}{g}\sum_{R\in\mathcal{G}} \{\chi^{(j_1)}(R)\}^2 = 0,$$

and from Corollary 1 $\mathcal{G}$ is non-quasi-ambivalent.

COROLLARY 3. *Let $\mathcal{G}$ be a finite non-ambivalent group. If $\mathcal{G}$ does not have an involutory outer automorphism, then $\mathcal{G}$ is non-quasi-ambivalent.*

We can apply Corollary 3 to complete groups, i.e. groups which have a trivial center and no outer automorphisms. If such a group is non-ambivalent it is non-quasi-ambivalent. Examples are the K-metacyclic groups for p = 5, 7, 11, . . . , which are defined by

(14) $S^p = T^{p-1} = E, \quad T^{-1}\,S\,T = S^r,$

where r satisfies $r^{p-1} \equiv 1 \pmod{p}$ and is primitive [**11**, p. 11]. These groups are complete [**27**, S. 126] and are non-ambivalent, hence also non-quasi-ambivalent. Note that from Corollary 3 it follows that for groups which have no involutory outer automorphisms the notions ambivalent and quasi-ambivalent coincide.

We shall prove now that there are many more non-quasi-ambivalent groups.

THEOREM 2. *A non-abelian group of odd order is non-quasi-ambivalent.*

Proof. Let $\mathcal{G}$ be a non-abelian group of odd order. Let us assume that $\mathcal{G}$ is quasi-ambivalent and let τ be an involutory automorphism, such that R^{-1} and $\tau(R)$ are conjugate elements for all R in $\mathcal{G}$ (cf. section 2). The class of the unit element E is the only ambivalent class of $\mathcal{G}$ [**9**, p. 294]. From this it follows that $\tau(R) \neq R$ for all $R \neq E$. This means that τ is a fixed-point-free automorphism of order 2. By [**17**, Theorem 1.4, Chapter 10] it follows that $\mathcal{G}$ is abelian. However, this contradicts the assumption that $\mathcal{G}$ is a non-abelian group and therefore $\mathcal{G}$ cannot be quasi-ambivalent.

Theorem 2 shows that the non-abelian group of order 21 and the two non-abelian groups of order 27 are non-quasi-ambivalent. This corrects and completes some statements made by two of the authors in [**4**, Chapter IV]. Furthermore the statement in [**4**], that the group of order 16, defined by

(15) $R^2 = S^2 = T^2 = E, \quad R\,S\,T = S\,T\,R = T\,R\,S$

is not quasi-ambivalent is incorrect: in fact

(16) $T \to R\,T\,R, \quad S \to R\,S\,R, \quad T \to S\,T\,S$

defines an involutory anti-automorphism that preserves the classes. Actually

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the K-metacyclic group for $p = 5$ and order 20 is the group of smallest order which is non-quasi-ambivalent.

4. Quasi-ambivalence for the alternating groups. It is often a difficult task to investigate if an arbitrary group is quasi-ambivalent or non-quasi-ambivalent. To illustrate this we shall study the case of the alternating groups A_n. We remark that all symmetric groups are ambivalent, hence quasi-ambivalent. For the alternating groups we have the following theorem.

THEOREM 3. *All alternating groups are non-quasi-ambivalent, except* A_2 *through* A_8, A_{10}, A_{12} *and* A_{14}.

Proof. The alternating groups A_2, A_5, A_6, A_{10} and A_{14} have real characters and therefore these groups are ambivalent. All other alternating groups have complex characters and are therefore non-ambivalent (cf. [**7**, Theorem 6.2, p. 223]). From now on we shall consider only the non-ambivalent alternating groups. In order to investigate whether such a group is quasi-ambivalent, we have to find an involutory outer automorphism that has the property that it maps a class with a complex character into its inverse class. (This is clear from equation (11) and the orthogonality relation

$$\frac{1}{g} \sum_{R \in \mathcal{G}} \chi^{(j)}(R)\, \chi^{(j)}(R^{-1}) = 1.)$$

The classes of A_n with complex characters are classes which together with the inverse class form one class of the symmetric group S_n.

Generating relations for A_n are given, e.g., in [**11**, p. 66] as follows:

$$(17) \quad V_i^3 = (V_i V_j)^2 = E, \quad 1 \leqq i \leqq j \leqq n-2.$$

A presentation for V_i is

$$(18) \quad V_i = (1\; i+1\; n).$$

All elements of A_n are also elements of S_n. Take in S_n the element (12) and calculate in S_n:

$$(19) \quad V_i' = (12)\, V_i (12)^{-1} = (12)\, V_i (12)$$

then in S_n we have

$$(20) \quad V_i'^3 = (V_i' \, V_j')^2 = E, \quad 1 \leqq i \leqq j \leqq n-2$$

for the transformation of equation (19) is an inner automorphism in S_n. But V_i' are also elements of A_n (V_i' is again an even permutation). Equations (20) are identical with equations (17), hence also V_i' generate A_n and equation (19) defines an outer automorphism of A_n.

Because A_n is a subgroup of index 2 of S_n one has

$$(21) \quad S_n = A_n + (12) A_n.$$

Consider now a class $\mathscr{C}_i$ of S_n which splits into $\mathscr{C}_i'$ and $\mathscr{C}_i''$ in A_n. Then

$$(22) \quad (12)\mathscr{C}_i'(12)^{-1} = \mathscr{C}_i''.$$

For suppose that $(12)\,\mathscr{C}_i'(12)^{-1} = \mathscr{C}_i'$. An arbitrary element from S_n is either in A_n or in $(12)\,A_n$. If $x \in S_n$ and $x \in A_n$, then $x\,\mathscr{C}_i'\,x^{-1} = \mathscr{C}_i'$, for $\mathscr{C}_i'$ is a class of A_n, but if $x \in S_n$ and $x \notin A_n$, then $x = (12)y$ with $y \in A_n$. If $(12)\,\mathscr{C}_i'\,(12)^{-1} = \mathscr{C}_i'$, then also $x\,\mathscr{C}_i'\,x^{-1} = \mathscr{C}_i'$, i.e., $\mathscr{C}_i'$ would also be a class of S_n, but we know that this is false. Hence equation (22) holds. Therefore equation (19) defines an outer automorphism of A_n with the property that each split class is transformed into its inverse class.

From [**7**, Theorem 6.2, p. 223] it can be seen that the characters of the split class of A_n are *all* complex-valued for $n = 3, 5, 7, 8$ and 12. This means that the groups A_3, A_4, A_7, A_8 and A_{12} are quasi-ambivalent for equation (19) defines an involutory automorphism such that $\tau(R)$ and R^{-1} lie in the same class.

From [**7**, Theorem 6.2, p. 223] it also follows that *some* of the characters of the split classes of A_n are complex-valued whereas the characters of the other split classes are real-valued for $n = 9$, 11, 13 and $n \geqq 15$. Clearly the involutory automorphism defined by equation (19) does not have the required property for quasi-ambivalence for these groups. We shall now prove that the automorphism group of A_n $n \geqq 4$ and $n \neq 6$ is generated by the automorphism defined by equation (19) and the inner automorphisms, which means that the automorphism group of $A_n (n \geqq 4$ and $n \neq 6)$ is the symmetric group S_n.

Suppose $n \geqq 4$. As we know the elements $V_i (i = 1, 2, \ldots, n-2)$ constitute a set of generators for A_n with $V_i = (1\; i+1\; n)$. Let us consider an arbitrary automorphism of A_n. Let V_i be mapped onto V_i'. Then because V_i is of order 3, V_i' also has to be of order 3 and moreover because for $n \geqq 4$ all elements with the cycle structure $(a_1\, b_1\, c_1)\,(a_2\, b_2\, c_2) \ldots (a_n\, b_n\, c_n)$ lie in one class of A_n, all generators have to belong to one class (cf. [**7**, Theorem 2.1, p. 31]). We define $\mathscr{C}_k$ to be the class $(1^{n-3k}\, 3^k)$. The number of elements of this class equals

$$(23) \quad g_k = \frac{n!}{(n-3k)!\, 3^k k!}.$$

Under an automorphism of a group whole classes are mapped onto whole classes, hence the class of the generators V_i, which has g_1 elements can be mapped onto $\mathscr{C}_k$ only if $g_k = g_1$, or

$$\frac{n!}{(n-3k)!\, 3^k\, k!} = \frac{n!}{(n-3)!\, 3},$$

which gives

$$(24) \quad 3^{k-1}\, k! = (n-3)(n-4)(n-5) \cdots (n-3k+1).$$

This equation is satisfied for $k = 2$ and $n = 6$, but for no other values of n and k (cf. [**19**, p. 93]). Hence $g_k \neq g_1$, except for $n = 6$, then $g_2 = g_1$. From now on we shall therefore also suppose that $n \neq 6$. Then for an arbitrary automorphism the elements V_i can be mapped only onto the elements of the class $\mathscr{C}_1$.

Because of the relation $(V_i V_j)^2 = E$, all generators V_i must have 2 symbols in the same order in common (only then $(a_1 b_1 c_1)(a_2 b_2 c_2)$ is of order 2, as can be checked easily). Consider an arbitrary automorphism $V_i' \rightarrow V_i$. Then V_i' can be any arbitrary 3-cycle say $(a\ b\ c)$, hence any element from the class $(1^{n-3}\ 3)$ with $n!/(n-3)!\,3$ elements. $V_2' = (a\ b\ x)$ has to have 2 symbols in common with V_1', but for the third symbol x we have $n - 3$ choices. Similarly for $V_3' = (a\ b\ y)$ we have $n - 4$ choices for y, etc. The total number of choices is

$$\frac{n!}{(n-3)!\,3}(n-3)(n-4)\cdot\cdots\cdot 2\cdot 1 = \frac{n!}{3}.$$

However, instead of keeping a and b fixed, we can also keep a and c or b and c fixed, therefore we have to multiply our number of choices by 3 and we find, totally, $n!$. Hence the total number of automorphisms of A_n is $n!$. However, we know already $n!$ automorphisms of A_n: the elements of S_n. *Therefore S_n is the automorphism group of A_n for $n \geqq 4$, $n \neq 6$.* For the alternating groups this means that the groups are non-quasi-ambivalent for $n = 9, 11, 13$ and $n \geqq 15$.

References

1. L. C. Biedenharn and H. van Dam (eds.), *Quantum theory of angular momentum* (Academic Press, New York-London, 1965).
2. L. C. Biedenharn, J. Nuyts, and H. Ruegg, *On generalizations of isoparity*, Comm. Math. Phys. *2* (1966), 231–250.
3. L. C. Biedenharn, A. Giovannini, and J. D. Louck, *Canonical definition of Wigner coefficients in U_n*, J. Mathematical Phys. *8* (1967), 691–700.
4. L. C. Biedenharn, W. Brouwer, and W. T. Sharp, *The algebra of representations of some finite groups*, Rice University Studies *54*, no. 2 (1968).
5. L. C. Biedenharn, J. D. Louck, E. Chacón, and M. Ciftan, *On the structure of the canonical tensor operators in the unitary groups. I, An extension of the pattern calculus rules and the canonical splitting in $U(3)$*, J. Mathematical Phys. *13* (1972), 1957–1984.
6. L. C. Biedenharn and J. D. Louck, *On the structure of the canonical tensor operators in the unitary groups. II, The tensor operators in $U(3)$ characterized by maximum null space*, J. Mathematical Phys. *13* (1972), 1985–2001.
7. H. Boerner, *Representations of groups*, 2nd ed. (North-Holland Publ. Comp., Amsterdam-London, 1970).
8. A. K. Bose and J. Patera, *Classification of finite-dimensional irreducible representations of connected complex semisimple Lie groups*, J. Mathematical Phys. *7* (1970), 2231–2234.
9. W. Burnside, *Theory of groups of finite order*, 2nd ed. (Dover Publ., New York, 1955).
10. P. H. Butler and R. C. King, *Symmetrized kronecker products of group representations*, Can. J. Math. *26* (1974), 328–339.
11. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for Discrete groups*, 2nd ed. (Springer Verlag, Berlin-Göttingen-Heidelberg-New York, 1965).
12. J.-R. Derome and W. T. Sharp, *Racah algebra for an arbitrary group*, J. Mathematical Phys. *6* (1965), 1584–1590.
13. J.-R. Derome, *Symmetry properties of the 3j-symbols for an arbitrary group*, J. Mathematical Phys. *7* (1966), 612–615.

14. ——— *Symmetry properties of the 3j-symbols for* SU (3), J. Mathematical Phys. *8* (1967), 714–716.

15. W. Feit, *Characters of finite groups* (W. A. Benjamin, New York-Amsterdam, 1967).

16. G. Frobenius and I. Schur, *Über die reellen Darstellingen der endlichen Gruppen, Sitzungsberichte der kön* (preuss. Ak. der Wissenschaften, Jahrgang 1906, S. 186, Berlin).

17. D. Gorenstein, *Finite groups* (Harper and Row Publ., New York-Evanston-London, 1968).

18. M. Hamermesh, *Group theory and its applications to physical problems* (Addison-Wesley Publ. Comp., Reading (Mass.), 1964).

19. A. G. Kurosh, *The theory of groups*, 2nd ed. vol. 1 (Chelsea Publ. Comp., New York, 1960).

20. J. D. Louck, *Recent progress toward a theory of tensor operators in the unitary groups*, Amer. J. Phys. *38* (1970), 3–42.

21. G. W. Mackey, *Multiplicity free representations of finite groups*, Pacific J. Math. *8* (1958), 503–510.

22. A. I. Mal'cev, *On semi-simple subgroups of Lie groups*, Izv. Akad. Nauk SSSR Ser. Mat. *8* (1944), 143–174 [in Amer. Math. Soc. Transl. no. *33* (1950)].

23. M. L. Mehta, *Classification of irreducible unitary representations of compact simple Lie groups. I*, J. Mathematical Phys. *7* (1966), 1824–1832.

24. M. L. Mehta and P. K. Srivastava, *Classification of irreducible unitary representations of compact simple Lie groups. II*, J. Mathematical Phys. *7* (1966), 1833–1835.

25. F. D. Murnaghan, *The theory of group representations* (Dover Publ., New York, 1963).

26. W. T. Sharp, *Racah algebra and the contraction of groups* (AECL-1098, Atomic Energy of Canada Limited, Chalk River, Ont., 1960; a revised version, edited by E. de Vries and A. J. van Zanten is to be published by University of Toronto Press).

27. A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung* 4^{te} *Auflage* (Birkhäuser Verlag, Basel-Stuttgart, 1956).

28. A. J. van Zanten, *Some applications of the representation theory of finite groups: a partial reduction method*, Ph.D. thesis, Groningen, 1972.

29. A. J. van Zanten and E. de Vries, *On the number of roots of the equation* $X^n = 1$ *in finite groups and related properties*, J. Algebra *25* (1973), 475–486.

30. E. P. Wigner, *On representations of certain finite groups*, Amer. J. Math. *63* (1941), 57–63 (reprinted in [**1**]).

31. ——— *On the matrices which reduce the Kronecker products of representations of S. R. groups* (Princeton, 1951, reprinted in [**1**]).

32. ——— *Group theory and its application to the quantum mechanics of atomic spectra* (Academic Press, New York-London, 1959).

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