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RINGS WITH FINITE NORM PROPERTY

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Introduction. A ring A has *finite norm property*, abbreviated *FNP*, if each proper homomorphic image of A is finite. In [**3**], Chew and Lawn described some of the structural properties of *FNP* rings with identity, which they called residually finite rings. The twofold aim of this paper is to extend the results of [**3**] to arbitrary rings with *FNP* and to give characterizations of *FNP* rings independent of the results of [**3**].

If A is a ring, let A^+ denote A regarded as an abelian group. In the first section of this paper, we explore the effects of *FNP* upon the structure of A^+. The following theorem is typical of the results in this section.

THEOREM. *If A is a ring with FNP, A^+ satisfies one of the following conditions:*
(1) *A^+ is torsion of bounded order;*
(2) *A^+ is torsion-free and reduced;*
(3) *A^+ is torsion-free and divisible.*

We consider only commutative rings in § 2. If A is an infinite non-null commutative ring with *FNP*, then A is an integral domain. Let A be an integral domain such that K, the quotient field of A, has multiplicative identity e. We prove that A has *FNP* if and only if A^*, the subring of K generated by A and e, has *FNP*. We give a new characterization of integral domains with identity that have *FNP*.

Finally, we consider conditions under which *FNP* is a hereditary property. We present necessary and sufficient conditions in order that all subrings of an integral domain have *FNP*, and we give sufficient conditions in order that the ring A have *FNP* if a subring of A has *FNP*.

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1. Additive group structure of rings with *FNP*.

In this section we use the notation and terminology of Fuchs [**6**]. If A is a ring, A^+ denotes A regarded as an abelian group.

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PROPOSITION 1.1. *Let A be a ring with FNP. A^+ is either torsion or torsion-free.*

Proof. The torsion subgroup T of A^+ is an ideal of A. If $T \neq 0$, A/T is a finite ring. Thus if $a \in A$, for some positive integer m, $ma \in T$. It follows that $a \in T$ and $A^+ = T$.

PROPOSITION 1.2. *If A is an infinite ring with FNP, either A is a prime ring, or $A^2 = 0$.*

Proof. If A has a multiplicative identity or if A^+ is torsion, this result is a consequence of the proof of [**3**, p. 92, Lemma 2.1]. The case where A^+ is torsion-free follows by considering $A/\text{Ann}(A)$.

When $A^2 = 0$, A is called a null ring. If A is a null ring, each subgroup of A^+ is an ideal of A. It follows that if A is a null ring with *FNP*, either A is finite or A^+ is an infinite cyclic group. Henceforth, we consider only *non-null rings*.

Definition. Let G be a group and let n be a positive integer. Then:

$$G(n) = \{x \in G \colon nx = 0\}$$
$$nG = \{x \in G \colon \text{for some } y \in G,\, x = ny\}.$$

PROPOSITION 1.3. *Let A be an infinite ring with FNP. If A^+ is torsion, then A^+ is a p-primary group for some positive prime integer p. Moreover, $A^+ = A^+(p)$.*

Proof. Let p be a prime integer such that $A^+(p) \neq 0$. If q is any prime integer distinct from p, $A^+(p) \cdot A^+(q) = (0)$. Since A is infinite, we conclude from Proposition 1.2 that $A^+(q) = 0$. Hence, A^+ is a p-primary group.

If $A^+ \neq A^+(p)$, there is an element $x \in A^+ \backslash A^+(p)$. Let i be the least positive integer such that $p^i x = 0$. Then $p^{i-1}x \neq 0$ and $px \neq 0$, but $(p^{i-1}x)(px) = 0$. By Proposition 1.2, A is finite, and the proof is complete.

COROLLARY 1.4. *Let A be a ring with FNP. If A^+ is torsion, then A^+ is bounded. In particular, A^+ is reduced.*

Proof. If A is finite, A^+ is bounded. If A is infinite, A^+ is a bounded group by Proposition 1.3. Any bounded group is reduced.

PROPOSITION 1.5. *Let A be a ring with FNP. A^+ is either reduced or divisible.*

Proof. By Proposition 1.1, A^+ is either torsion or torsion-free. If A^+ is torsion, A^+ is reduced by Corollary 1.4. We assume that A^+ is torsion-free and not reduced. The maximal divisible subgroup D of A^+ is nonzero, and there is a reduced subgroup R of A^+ such that A^+ is the direct sum of R and D. Moreover, D is an ideal of A. Since R is isomorphic to $(A/D)^+$, R is a finite subgroup of A^+. Since A^+ is torsion-free, $R = (0)$. Hence $A^+ = D$, and our proof is complete.

We say that a ring A is *simple* if the only proper ideal of A is (0).

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PROPOSITION 1.6. *Let A have FNP. A^+ is torsion-free and A is simple if and only if A^+ is divisible.*

Proof. ($\Rightarrow$) Because A^+ is torsion-free, nA is a non-zero ideal of A for each nonzero integer n. Since A is simple, $nA = A$ for each non-zero integer n. Hence A^+ is divisible.

($\Leftarrow$) From Corollary 1.4 it follows that if A^+ is divisible, A^+ is torsion-free. If B is a proper homomorphic image of A, B^+ is a finite divisible group. Hence $B = (0)$. Thus A is simple.

In [**3**], Chew and Lawn defined an *FNP* ring A to be *proper* if A is not finite and not simple. From Proposition 1.6 we can conclude that an *FNP* ring A is proper only if A^+ is reduced. We now give a group theoretic criterion for distinguishing proper torsion-free *FNP* rings from proper torsion *FNP* rings.

LEMMA 1.7. *Let A be an FNP ring. If A^+ is torsion-free, then A^+ is homogeneous and the type of its elements is $(k_1, \ldots, k_n, \ldots)$ where k_n is either 0 or ∞ for each n.*

Proof. Since A has *FNP*, A satisfies the restricted minimum condition for ideals. From the proof of [**6**, p. 289, Theorem 75.3] it follows that A^+ has the required properties.

Definition. A group G is said to be hopfian if each epi-endormorphism of G is an automorphism.

PROPOSITION 1.8. *Let A be a proper FNP ring. A^+ is torsion-free if and only if A^+ is hopfian.*

Proof. ($\Rightarrow$) Let $a \in A\backslash\{0\}$. Since A^+ is reduced, there is a positive prime integer p such that

$$a \notin \bigcap_{i=1}^{\infty} p^i(A^+).$$

By Lemma 1.7, if $b \in A\backslash\{0\}$,

$$b \notin \bigcap_{i=1}^{\infty} p^i(A^+).$$

Hence

$$pA^+ \supset p^2A^+ \supset \ldots \supset p^nA^+ \supset \ldots$$

is a descending sequence of fully invariant subgroups of A^+ with hopfian quotient groups and

$$\bigcap_{i=1}^{\infty} p^i(A^+) = (0).$$

By [**1**, p. 332, Theorem 1], A^+ is hopfian.

(⇐) If A^+ is torsion, A^+ is a p-primary group and $A^+ = A^+(p)$ for some positive prime integer p. By [**10**, p. 17, Theorem 6],

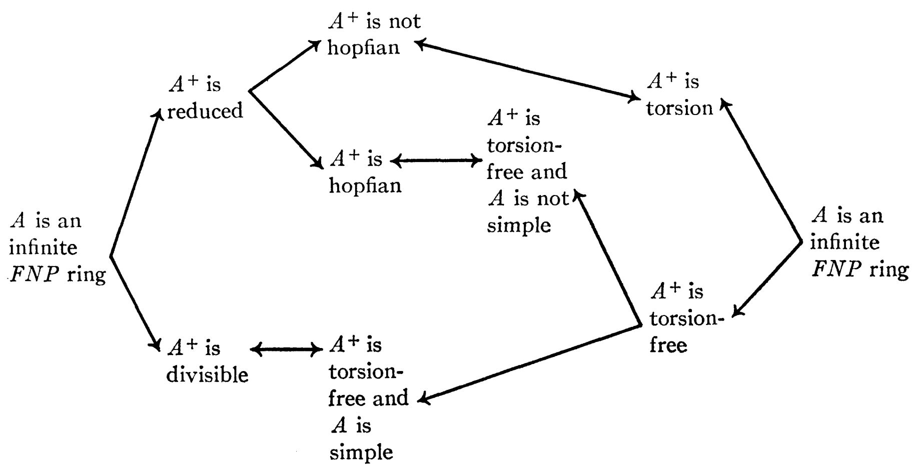
$$A^+ = \sum_{\lambda \in \Lambda} C(p)_\lambda,$$

where $C(p)_\lambda$ is a cyclic group of order p, and Λ is an infinite ordinal. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a basis for A^+. The following mapping of this set of generators of A^+ onto itself extends to an epi-endomorphism of A^+ which is not an automorphism:

$$x_\lambda \rightarrow \begin{cases} x_\lambda & \text{if } \lambda = 1 \text{ or } \lambda \text{ is a limit ordinal} \\ x_{\lambda-1} & \text{if } \lambda \text{ has a predecessor.} \end{cases}$$

This completes the proof of the proposition.

The following diagram summarizes the preceding results of this section. In the diagram, the condition at the tail of an arrow implies the condition at the tip, statements joined by a doubleheaded arrow (↔) are equivalent, and the condition at the vertex of the symbol $<$ implies that one of the mutually exclusive conditions appearing at one of the two arrowheads occurs.



2. Commutative rings with *FNP***.** In this section we assume that all rings are commutative. Thus if A is an infinite commutative (non-null) FNP-ring, then A is an integral domain by Proposition 1.2.

Definition. Let A be an integral domain with quotient field K. Let e be the multiplicative identity of K. The ring A^* is the subring of K generated by A and e.

THEOREM 2.1. *Let A be an integral domain with quotient field K. A has FNP if and only if A^* has FNP.*

Proof. ($\Leftarrow$) Assume A^* has *FNP*. If B is a non-zero ideal of A, $B = BA^*$ [**8**, p. 25, Proposition 2.2]. Since $A/B = A/BA^*$ is a subset of A^*/BA^*, A/B is finite. Hence A has *FNP*.

($\Rightarrow$) We assume that A has *FNP*. Let a be a non-zero element of A. Then since A/a^2A is finite, there exists an integer $m \neq 0$ such that $ma = a^2b$ for some b in A. Hence $me = ab \in A$ and the factor group A^*/A is finite. If B^* is a non-zero ideal of A^*, $B^* \cap A$ is a non-zero ideal of A. Thus $A/B^* \cap A$ is finite. Let $|S|$ denote the cardinality of the set S. Then we see that

$$|A^*/B^*| \leqq |A^*/B^* \cap A|.$$

But

$$|A^*/B^* \cap A| = |A/B^* \cap A| \cdot |A^*/A|$$

because A^*/A is isomorphic to $(A^*/B^* \cap A)/(A/B^* \cap A)$. Hence $|A^*/B^* \cap A|$ is finite. It follows that A^*/B^* is finite and A^* has *FNP*. This completes the proof of the theorem.

Clearly each finite ring has *FNP*. Using Proposition 1.2 and Theorem 2.1 we see that an infinite ring A has *FNP* if and only if A^* has *FNP*. Hence to determine the conditions under which an arbitrary commutative (non-null) ring has *FNP*, it is sufficient to know those conditions under which a commutative ring with identity has *FNP*.

PROPOSITION 2.2. *Let A be a commutative ring with identity. A has FNP if and only if A satisfies one of the following conditions:*

(a) *A is a field;*

(b) *A is a Noetherian integral domain and each non-zero prime ideal of A has finite index;*

(c) *A is a finite ring.*

Proof. By [**3**, p. 93, Theorem 2.3].

THEOREM 2.3. *Let A be an integral domain with identity and with quotient field K. Let L be a finite algebraic extension of K. Let A′ be a ring such that $A \subseteq A' \subseteq L$. If A has FNP, then A′ has FNP.*

Proof. Let A' be a subring of L containing A. If A' is a field, A' has *FNP*. So we assume that A' is not a field. Let P' be any non-zero prime ideal of A'. By [**8**, p. 100, Lemma 9.1], $P' \cap A$ is a non-zero prime ideal of A. Since A is of Krull dimension 1, by the theorem of Krull-Akizuki [**12**, p. 115, Theorem 33.2], A' is Noetherian of Krull dimension 1, and A'/P' is a module of finite length over $A/(P' \cap A)$. Hence A'/P' is a finite-dimensional vector space over a finite field. It follows that A'/P' is finite. By Proposition 2.2, A' has *FNP*, and our proof of Theorem 2.3 is completed.

We mention a special case of Theorem 2.3 as a corollary.

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COROLLARY 2.4. *Let A be an integral domain with identity and quotient field $K \neq A$. Let $\bar{A}$ be the integral closure of A in K. If A has FNP, $\bar{A}$ is a Dedekind domain with FNP.*

Proof. By Theorem 2.3, $\bar{A}$ has *FNP*. Then $\bar{A}$ is an integrally closed, Noetherian domain of Krull dimension 1. By definition, $\bar{A}$ is Dedekind.

If we remove the hypothesis that L is a finite algebraic extension of K, then the conclusion of Theorem 2.3 is no longer valid.

Example 2.5. In this example we show that if L is not an algebraic extension of K, the quotient field of A, an integral domain with *FNP*, then the conclusion of Theorem 2.3 may be false. Let x be an indeterminate and let $L = Q(x)$. $Z[x]$ is a domain such that $Z \subset Z[x] \subset L$. $Z[x]$ is of Krull dimension 2. It follows that $Z[x]$ fails to have *FNP*, while Z has *FNP*.

Example 2.6. In this example we show that if L is a non-finite, algebraic extension of K, the quotient field of A, an integral domain with *FNP*, then the conclusion of Theorem 2.3 may be false. Let L be the algebraic closure of Q. If $\bar{Z}_L$ is the integral closure of Z in L, $Z \subset \bar{Z}_L \subset L$. $\bar{Z}_L$ is a non-Noetherian domain; hence $\bar{Z}_L$ is not an *FNP*-ring.

THEOREM 2.7. *Let A be an integral domain with identity and quotient field K. Let $\bar{A}$ be the integral closure of A in K. A has FNP if and only if $\bar{A}$ is a Dedekind domain and for each maximal ideal P of A, the quotient ring A_P has FNP.*

Proof. ($\Rightarrow$) By Corollary 2.4, $\bar{A}$ is Dedekind. From Theorem 2.3, we conclude that A_P has *FNP* for any prime ideal P of A.

($\Leftarrow$) If $\bar{A}$ is a Dedekind domain, A is of Krull dimension 1 and each non-zero element of A is a non-unit in only finitely many valuation overrings of A [**2**, p. 119, Theorem 4]. Thus each non-zero element of A lies in only finitely many maximal ideals of A. If P is any maximal ideal of A, A_P is a Noetherian domain. Hence A is Noetherian. Moreover, for each non-zero prime ideal P of A,

$$A/P = A_P/PA_P;$$

hence P has finite index in A. By Proposition 2.2, A has *FNP*.

COROLLARY 2.8. *Let A be a Dedekind domain with quotient field K. A has FNP if and only if each valuation overring of A has finite residue field.*

Proof. ($\Rightarrow$) This is an immediate consequence of Theorem 2.3.

($\Leftarrow$) This follows from Theorem 2.7.

Let A be an integral domain with identity and with integral closure $\bar{A}$. Clearly, from $\bar{A}$ having *FNP* it follows that A has Krull dimension 1, each non-zero prime ideal of A has finite index in A, and each non-zero ideal of A

is contained in only finitely many maximal ideals of A [**2**, p. 118, Lemma 1]. However, it is not in general true that A has *FNP* if $\bar{A}$ does. To prove this, William Heinzer constructed the following example.

Example 2.9. Let k be a finite field of characteristic p, and let $\{y_i\}^\infty{}_{i=1}$ be a collection of power series in $xk[[x]]$, all algebraicially independent over k. Assume $y_1 = x$. Let

$$V = k(\{y_i\}^\infty{}_{i=1}) \cap k[[x]]$$

and let

$$W = k(\{y_i^p\}^\infty{}_{i=1}) \cap k[[x]].$$

Both V and W are rank one discrete valuation rings and $W \subset V$. Moreover, V and W both have *FNP* since their residue fields are imbedded in k. Finally, V is integral over W. If $R = W[\{y_i\}^\infty{}_{i=1}]$, then $W \subseteq R \subseteq V$, and V is the integral closure of R in its quotient field. Since each $y_i \in xk[[x]]$, all the y_i generate a proper ideal of V. Hence $R \neq V$. Let $L = k(\{y_i^p\}^\infty{}_{i=1})$. For any j,

$$[L(y_j):L] = [L(\{y_i\}^\infty{}_{i=1}) : L(\{y_i|i \neq j\}) = p > 1$$

since $\{y_i\}^\infty{}_{i=1}$ is a set of algebraically independent elements. If B denotes the ideal of R generated by $\{y_i\}^\infty{}_{i=1}$, B is a proper ideal R because $1 \notin B$. It follows from [**5**, Lemma 3.3, p. 338] that B is not finitely generated. Thus R is a ring that fails to have *FNP*, while its integral closure V does have *FNP*.

If $\bar{A}$ does have *FNP*, to show that A has *FNP*, it is sufficient to find conditions on A which insure that each non-zero ideal contains a product of non-zero prime ideals since each non-zero prime ideal of A has finite index. Because each non-zero ideal of A is contained in only finitely many maximal ideals, each ideal is a finite intersection of primary ideals. Thus A will have *FNP* if each primary ideal of A contains a power of its radical.

There are other sufficient conditions that will imply that A has *FNP* if $\bar{A}$ has *FNP*. For example, we know that A will be Noetherian, and hence have *FNP*, if $\bar{A}$ is a finite A-module or if C, the conductor of $\bar{A}$ relative to A, is non-zero. In fact, when $\bar{A}$ has *FNP* these two conditions are equivalent, as we show in the following proposition.

PROPOSITION 2.10. *Let A be an integral domain with identity and with integral closure $\bar{A}$. Let C be the conductor of $\bar{A}$ relative to A. If $\bar{A}$ has FNP, then $\bar{A}$ is a finite A-module if and only if $C \neq (0)$.*

Proof. ($\Rightarrow$) If $\bar{A}$ is a finite A-module,

$$\bar{A} = Ax_1 + Ax_2 + \ldots + Ax_n,$$

where $x_i = a_i/a_i'$ for each i, with $a_i, a_i' \in A\backslash\{0\}$ for each i, $1 \leqq i \leqq n$. Clearly,

$$0 \neq \prod_{i=1}^{n} a_i' \in C,$$

so $C \neq (0)$.

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($\Leftarrow$) We assume $C \neq (0)$. If B is a non-zero ideal of A, $(0) \neq BC \subseteq B$. If A/BC is finite, A/B is finite. Now BC is an ideal in A and in $\bar{A}$. Hence $A/BC \subseteq \bar{A}/BC$. It follows that A/BC is finite and that A has *FNP*. Since A is Noetherian, $\bar{A}$ is a finite A-module by [**11**, problem 41, p. 46].

This completes the proof of Proposition 2.10.

Definition. Let A be an integral domain with quotient field K. If A' is a ring such that $A \subseteq A' \subseteq K$, then A' is an *overring* of A. If $A \neq A'$, A' is a *proper overring* of A.

From Theorem 2.3 we conclude that if A' is any overring of a domain A with *FNP*, then A' has *FNP*. However, it is not in general true that each subring of a ring with *FNP* also has *FNP*. We now offer necessary and sufficient conditions for each subring of a domain with identity to possess *FNP*.

THEOREM 2.11. *Let A be an integral domain with identity and with quotient field K of characteristic* 0. *Each subring of A has FNP if and only if K is a finite extension of Q, the field of rational numbers.*

Proof. ($\Rightarrow$) If each subring of A has *FNP*, each subring of A is Noetherian. By [**7**, p. 131, Theorem 3], K is a finite extension of Q.

($\Leftarrow$) Let K be a finite extension of Q. Let S be a subring of A. Without loss of generality, we can assume that S contains 1. If L is the quotient field of S, L is a finite extension of Q. Since $Z \subseteq S \subseteq L$, S has *FNP* by Theorem 2.3, and the proof is complete.

THEOREM 2.12. *Let A be an infinite integral domain with identity and with quotient field $K \neq A$ of characteristic $p \neq 0$. Let Π_p be the prime field of K. Each subring of A has FNP if and only if the transcendence degree of K over Π_p is* 1, *and K is a finite extension of a purely transcendental extension of Π_p.*

Proof. ($\Rightarrow$) This is an immediate consequence of [**7**, p. 131, Theorem 4].

($\Leftarrow$) Let S be a subring of A. Without loss of generality, we assume $1 \in S$. Let L be the quotient field of S. If L is algebraic over Π_p, $L = S$ and S has *FNP*. We assume that L is of transcendence degree 1 over Π_p. There exists $x \in S$ such that x is transcendental over Π_p. Clearly, $\Pi_p[x]$ has *FNP*. Since $\Pi_p[x] \subseteq S \subseteq K$, and K is a finite extension of $\Pi_p(x)$, S has *FNP* by Theorem 2.3. Thus the proof of Theorem 2.12 is complete.

Even if the quotient field of an *FNP* domain A fails to satisfy the conditions of either Theorem 2.11 or Theorem 2.12 we can still conclude that certain subrings of A will have *FNP*.

PROPOSITION 2.13. *Let A be an integral domain with identity and with quotient field K. Let $\bar{A}$ be the integral closure of A and let L be any subfield of K. If $\bar{A}$ has FNP, $\bar{A} \cap L$ has FNP.*

Proof. This follows immediately from Corollary 2.4 and [**9**, p. 117, Lemma 4].

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References

1. G. Baumschlag, *Hopficity and abelian groups*, in Topics in abelian groups; proceedings of the symposium held June 4–8, 1962, edited by J. M. Irwin and E. A. Walker, pp. 331–336 (Scott, Foresman, Chicago, Illinois, 1963).
2. H. S. Butts and W. W. Smith, *On the integral closure of a domain*, J. Sci. Hiroshima Univ. Ser. A-I Math. *30* (1966), 117–122.
3. K. L. Chew and S. Lawn, *Residually finite rings*, Can. J. Math. *22* (1970), 92–101.
4. P. M. Eakin, *The converse to a well known theorem on Noetherian rings*, Math. Ann. *177* (1968), 278–282.
5. P. M. Eakin and W. Heinzer, *Nonfiniteness in finite dimensional Krull domains*, J. Algebra *14* (1970), 333–340.
6. L. Fuchs, *Abelian groups*, 3rd ed. (Pergamon, London, 1960).
7. R. Gilmer, *Integral domains with noetherian subrings*, Comment. Math. Helv. *45* (1970), 129–134.
8. ——— *Multiplicative ideal theory*, Vol. I (Queens University Press, Kingston, Ontario, 1968).
9. O. Goldman, *On a special class of Dedekind domain*, Topology *3* (1964), 113–118.
10. I. Kaplansky, *Infinite abelian groups* (University of Michigan Press, Ann Arbor, Michigan, 1964).
11. ——— *Commutative rings* (Allyn and Bacon, Boston, Mass., 1970).
12. M. Nagata, *Local rings* (Interscience, New York, 1962).

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