Canad. Math. Bull. Vol. **19** (2), 1976

ON RINGS WITH MANY ENDOMORPHISMS

BY
JOSEPH NEGGERS

ABSTRACT. All rings have an identity, all homomorphisms map identities to identities, all homomorphisms on algebras over fields are algebra homomorphisms. A ring R is *a quotient-embeddable ring* (*a QE-ring*) if for any proper ideal a of R there is an endomorphism of R whose kernel is the ideal a. A QE-ring U is a *receptor* of R if for any proper ideal a of R there is a homomorphism from R to U whose kernel is the ideal a.

THEOREM. *A ring R has a receptor if and only if it is a K-algebra over some field K contained in the center of R. If R is a commutative K-algebra of this type, then it has a commutative receptor.*

In the following we shall only concern ourselves with rings which contain an identity 1. All homomorphisms of rings will have the property that they map the identity to the identity. If a ring R is a K-algebra over a field K, then $K=K1$ is contained in the center of R. Furthermore, when all rings involved are K-algebras, then homomorphisms will generally be K-algebra homomorphisms.

We shall call a ring R a *quotient-embeddable ring* (*a QE-ring*) if for any proper ideal a of R, there is an endomorphism of R whose kernel is the ideal a.

Given a ring R, a QE-ring is a *receptor* of R if for any proper ideal a of R, there is a homomorphism from R to U whose kernel is the ideal a.

In this note we prove the following theorem:

THEOREM. *A ring R has a receptor if and only if it is a K-algebra over some field K. If R is a commutative K-algebra it has a commutative receptor.*

Let $R=K\{X_i \mid i \in I\}$ be a free associative algebra over the field K, and let F be a family of proper ideals. Construct a new free associative algebra $W= K\{X_i(a) \mid i \in I,\ a \in F\}$, and let N be the ideal of W generated by the polynomials $P(X_i(a))$ (a fixed) such that $P(X_i)$ is an element of a. Finally take $U=W/N$.

LEMMA 1. *With the definitions of R and U as above, given any ideal $a \in F$, there is a homomorphism $\phi: R \to U$ such that* $\ker \phi = a$.

Received by the editors January 6, 1975 and, in revised form, June 20, 1975.

AMS subject classifications. Primary 16A06; Secondary 13B20, 13B25.

Key words and phrases. Free associative algebras, polynomial rings, algebras, homomorphisms, quotient-embeddable rings, receptors.

https://doi.org/10.4153/CMB-1976-030-3 Published online by Cambridge University Press

Proof. Since R is a free associative algebra, we define a homomorphism $\phi: R \to U$ by taking $\phi(X_i) = X_i(a) + N$ and extending to R. Certainly, if $P(X_i) \in a$, then $\phi(P(X_i)) = P(X_i(a)) + N = 0$ (in U), i.e., $a \subseteq \ker \phi$.

Hence we must show that $\ker \phi \subseteq a$, or equivalently, if $P(X_i(a)) \in N$, then $P(X_i) \in a$. Suppose that $P(X_i(a)) \in N$, i.e., $P(X_i) \in \ker \phi$, and write

$$(1) \qquad P(X_i(a)) = \sum \lambda_j M_j P_j(X_i(a_j)) N_j,$$

where the terms M_j and N_j are monomials in W, where λ_j is an element of K, and where $P_j(X_i)$ belongs to a_j for a_j an element of F. Suppose furthermore that the number of indices involved in the expression for $P(X_i(a))$ in (1) is as small as possible.

Suppose A is the collection of all indices j such that $M_j N_j$ is not a monomial in the variables $X_i(a)$ alone, and consider

$$(2) \qquad Q = \sum_{j \in A} \lambda_j M_j P_j(X_i(a_j)) N_j.$$

Since W is a free associative algebra, and since $P(X_i(a))$ is a sum of monomials in the variables $X_i(a)$ alone (including the constant term), it follows from (1) and (2), that Q is identically zero, whence by the minimality condition we may take $A = \phi$.

Now consider a fixed pair of monomials M_{j_0} and N_{j_0}, and list the distinct ideals which occur among the ideals a_j, say $b_1, \ldots, b_k$. Then as part of the expression for $P(X_i(a))$ in (1) we generate a term

$$(3) \qquad T = M_{j_0}(\sum \lambda_j P_j(X_i(b_l))) N_{j_0},$$

where j runs over all indices such that $M_j = M_{j_0}$, $N_j = N_{j_0}$ and $a_j = b_l$.

Hence, since $P(X_i) \in b_l$, and since b_l is an ideal, the minimality condition implies that there is precisely one term which goes between M_{j_0} and N_{j_0} per ideal in the expression for $P(X_i(a))$ given in (1).

Since the ideals b_l are proper ideals, it follows that the polynomials $P_j(X_i)$ have positive degree. Hence, if S_j is the leading term of $P_j(X_i)$ with respect to a suitable ordering of the variables and the monomials, then for $b_l \neq a$ the term

$$(4) \qquad S = M_{j_0} S_{j_0}(X_i(b_l)) N_{j_0}$$

of T as in (3) cannot be cancelled by any other term, since M_{j_0} and N_{j_0} are monomials in the variables $X_i(a)$.

Again by the minimality condition, this implies that the term T given in (3) does not occur at all if $b_l \neq a$.

Therefore the only terms which can survive are those for which $b_l = a$, i.e.,

$$(5) \qquad P(X_i(a)) = \sum \lambda_j M_j P_j(X_i(a)) N_j$$

where $P_j(X_i) \in a_j = a$. Hence, by the minimality condition and the fact that the monomials M_jN_j involve only the variables $X_i(a)$, it follows that in fact the right hand side consists of a single term, which must be $P(X_i(a))$ itself. Thus $P(X_i) \in a$ and the lemma follows.

If $F = \{a\}$, then $W = R$, and $N = a$, whence $U = R/a$, and the mapping ϕ constructed in the lemma is precisely the natural map.

If we take F to be the family of all proper ideals, then U satisfies part of the definition of receptor. The problem is to show that we may arrange for U to be a *QE-ring*.

Lemma 2. *If R is a free associative algebra then R has a receptor.*

Proof. Let $R = W_0$ be the family of all ideals of W_0. By Lemma 1 we construct a free associative algebra W_1' and an ideal N_1' with the property that $U_1 = W_1'/N_1'$ is as given in Lemma 1. Let $W_0 = U_0$, $N_0 = 0$. If $\pi_{0,1}: U_0 \to U_1$ is defined by $X_i \to X_i(0) + N_1'$, then since $P(X_i) = 0$ if and only if $P(X_i(0)) \in N_1'$, the mapping $\pi_{0,1}$ is well-defined and an embedding. Now, construct a free associative algebra W_1 above U_1 by taking generators $Y_i(a)$, $i \in I$, $a \in F$, and let $\varepsilon_1: W_1 \to U_1$ be given by $Y_i(a) \to X_i(a) + N_1'$. Thus $\ker \varepsilon_1 = N_1$ is the ideal generated by all elements $P(Y_i(a))$, where $P(X_i) \in a$.

Let F_1 be the family of all ideals of W_1 containing N_1, and use Lemma 1 to obtain an algebra U_2 such that all ideals in F_1 are kernels of homomorphisms from W_1 to U_2. If we let $\pi_{1,2}$ be obtained by factoring the homomorphism with kernel N_1 through ε_1, then $\pi_{12}: U_1 \to U_2$ is an injection. Repeat the same process with respect to W_2, N_2 and ε_2, etcetera, to obtain a sequence

$$
\begin{array}{ccccccc}
R = W_0 & & W_1 & & & & W_n \\
\downarrow{\scriptstyle \varepsilon_0} & & \downarrow{\scriptstyle \varepsilon_1} & & & & \downarrow{\scriptstyle \varepsilon_n} \\
R = U_0 & \xrightarrow[\pi_{0,1}]{} & U_1 & \longrightarrow \cdots \longrightarrow & & & U_n \xrightarrow[\pi_{n,n+1}]{} \cdots,
\end{array}
\tag{6}
$$

where $\pi_{i,i+1}$ is obtained by factoring the homomorphism $\phi: W_i \to U_{i+1}$ whose kernel is $N_i = \ker \varepsilon_i$ through the mapping ε_i.

In particular every proper ideal a of U_i is the kernel of a homomorphism $\phi: U_i \to U_{i+1}$ obtained as a factorization of a homomorphism $\psi: W_i \to W_{i+1}$ whose kernel is $\varepsilon_i^{-1}(a)$, an element of F_i, the collection of all ideals of W_i containing N_i. If we let $\pi_{i,i+j} = \pi_{i,i+1} \circ \cdots \circ \pi_{i+j-1,i+j}$, then we obtain a direct system of inclusions and we let $U = \lim_n \mathrm{U_n} = \bigcup \mathrm{U_n}$, where the proper identifications have been made. U is a *QE*-ring which will serve as a receptor of R.

The free associative algebra W_{n+1} is generated by variables $Y_i^*(a_0, \ldots, a_n) = Y_i^*(a_0, \ldots, a_{n-1})(a_n)$, where $i \in I$ and $a_j \in F_j$, and U_{n+1} is accordingly generated by the elements $X_i(a_0, \ldots, a_n) = Y_i^*(a_0, \ldots, a_n) + N_{n+1}$. Now, let W be the free associative algebra over K generated by inderminates $Y_i(a_0, \ldots, a_t)$, where $a_j \in F_j$, for $y = -1, 0, 1, \ldots$ $(Y_i(a_{-1}) = Y_i)$.

https://doi.org/10.4153/CMB-1976-030-3 Published online by Cambridge University Press

Next, define a mapping $\varepsilon: W \to U$ by mapping $Y_i(a_0, \ldots, a_t)$ to $X_i(a_0, \ldots, a_t)$. Thus since the elements $X_i(a_0, \ldots, a_t)$ form a generating set for U_{t+1}, it follows that ε is a surjection. If $N = \ker \varepsilon$, then $U = W/N$. If $V_{t+1} \subseteq W$ is the free associative algebra generated by all $Y_i(a_0, \ldots, a_s)$, where $-1 \le s \le t$, then $W = \lim_t V_t = \bigcup V_t$.

Let A be an ideal of U. Let $B = \varepsilon^{-1}(A)$ and let $B_{t+1} = B \cap V_{t+1}$. Then $B_{t+1} \supseteq \ker \varepsilon^*_{t+1}$, where $\varepsilon^*_{t+1}: V_{t+1} \to U_{t+1}$ is obtained by mapping $Y_i(a_0, \ldots, a_t)$ to $X_i(a_o, \ldots, a_t)$.

Furthermore, $\ker \varepsilon^*_{t+1}$ contains all elements $Y_i(a_0, \ldots, a_s) - Y_i(a_0, \ldots, a_s, N_{s+1}, \ldots, N_t)$. This is so since $X_i(a_0, \ldots, a_s) = X_i(a_0, \ldots, a_s, N_{s+1}, \ldots, N_t)$ in U_{t+1} via the identifications obtained from the injection $\pi_{s+1,t+1}$.

Now define $\varphi: W \to U = W/N$ by:

$$\varphi'(Y_i(a_0, \ldots, a_t) = Y_i(a_0, \ldots, a_t, B^*_{t+1}) + N \tag{7}$$

where

$$B^*_{t+1} = \varepsilon^{-1}_{t+1}(\varepsilon^*_{t+1}(B_{t+1})) \in F_{t+1}.$$

We claim that $\ker \varphi = B = \varepsilon^{-1}(A)$. Thus φ induces a mapping $\varphi': U \to U$ with $\ker \varphi = A$. i.e., U is a QE-ring and therefore a receptor of R.

Suppose $P(Y_i(a_0, \ldots, a_t)) \in \ker \varphi$. Since $P \in V_{d+1}$ for some minimal d, we adjust all sequences to sequences $(a_0, \ldots, a_t, N_{t+1}, \ldots, N_d)$. This may be done since $\ker \varphi \supseteq B \supseteq N$.

Consider the corresponding element $P(Y^*_i(a_0, \ldots, N_d))$ of W_{d+1}.

Let $\phi_{s+1}: W_{s+1} \to U_{s+2}$ be defined by:

$$\phi_{s+1}(Y^*_i(a_0, \ldots, a_s)) = X_i(a_0, \ldots, a_s, B^*_{s+1}). \tag{8}$$

From Lemma 1 and the construction of W_{s+1}, $\ker \phi_{s+1} = B^*_{s+1}$. Now, $P(Y^*_i(a_0, \ldots, N_d)) \in \ker \phi_{d+1}$ if and only if $P(X_i(a_0, \ldots, N_d))) \in A_{d+1}$.

Since $\phi_{s+1}(Y^*_i(a_0, \ldots, a_s)) = \phi(Y_i(a_0, \ldots, a_s))$, we find that $P(Y_i(a_o, \ldots, a_t, N_{t+1}, \ldots, N_d)) \in \ker \phi$ if and only if $P(X_i(a_0, \ldots, N_d)) \in A_{d+1}$. Hence $P(Y_i(a_0, \ldots, a_s)) \in \ker \varphi$ if and only if $P(X_i(a_0, \ldots, a_s)) \in A$, i.e., $\ker \varphi = \varepsilon^{-1}(A) = B$.

The proof of the lemma is now complete.

Although the computations are somewhat messy, the proof is intuitively quite straightforward. Thus, given $R = U_0$, we apply Lemma 1 to the family of all ideals of U_0 to obtain $U_0 \subseteq U_1$. Again, we apply Lemma 1 to the family of all ideals of U_1 to obtain $U_0 \subseteq U_1 \subseteq U_2$, and we repeat this process ad infinitum to obtain an algebra $U = \lim_n U_n = \bigcup U_n$. Clearly, if A is an ideal of U, and if $A_i \cap U_i$, we have homomorphisms $\phi_i: U_1 \to U_{i+1}$ such that $\ker \phi_i = A_i$. The problem is to construct them all at once or in such a manner that the restriction of ϕ_{i+1} to U_i is precisely ϕ_i. The labor in the lemma concerned this construction.

https://doi.org/10.4153/CMB-1976-030-3 Published online by Cambridge University Press

LEMMA 3. *If R is a polynomial ring over a field K, then R has a commutative receptor.*

Proof. In the proof of Lemma 1 we take $W = K[X_i(a) \mid i \in I, a \in F]$ to be a polynomial ring and if we define N as the ideal generated by the polynomials $P(X_i(a))$ (a fixed) such that $P(X_i)$ is an element of $a \in F$, then $U = W/N$ is a commutative ring, and the expression (1) becomes

$$P(X_i(a)) = \sum M_j P_j(X_i(a_j)), \tag{9}$$

where M_j is a monomial in W, where $P_j(X_i)$ belongs to a_j for a_j an element of F and where we may assume that the number of indices is as small as possible. By the same argument as in Lemma 1, we again find that $P(X_i(a)) \in N$ if and only if $P(X_i) \in N$. Hence, if $\phi : R \to U$ is defined by $\phi(X_i) = X_i(a)$, which can be done since we are dealing with a free object in the category of commutative K-algebras, then $\ker \phi = a$, for $a \in F$. Thus Lemma 1 continues to hold.

Similarly, if we use the argument of Lemma 2, replacing free associative algebras everywhere by polynomial rings, then $U = \lim_n U_n = \bigcup U_n$, is itself a commutative ring and a receptor of R.

LEMMA 4. *Suppose that R is a ring such that for some ring U, given any proper ideal a or R, there is a homomorphism ϕ from R to U with* $\ker \phi = a$. *Then R contains a field K in its center.*

Proof. Let $Z1 \subseteq R$ be generated by the identity. Then, if $n \in Z1$, and if nR is a proper ideal, there is a homomorphism ϕ with kernel nR. This means that since $\phi(n) = n\phi(1) = n1 = 0$, then $nU = 0$ and hence $nR = 0$ as well. Thus, if n is the characteristic of R, we have $n = m_1 m_2$, where $(m_1, m_2) = 1$ implies $m_1 R$ and $m_2 R$ are proper, whence $m_1 R = m_2 R = 0$, and $R = 0$. If $n = p^r$, p a prime, then pR is proper, $pR = 0$, so that finite characteristic implies $Z1$ is a finite field. If no ideal nR is proper, then $nn^{-1} = 1$, where n^{-1} is in the center of R since n is in the center of R. Thus the center of R contains the field Q of rational numbers.

The theorem follows as an easy consequence of Lemmas 1, 2, 3, and 4, if we observe that if U is a receptor of R, then U is also a receptor of R/a for any proper ideal a. Thus, we write an arbitrary K-algebra R as a quotient $R = S/a$, where S is a free associative algebra, while if R is commutative we take S to be a polynomial ring.

We close with some remarks. If R is a QE-ring, then it is not difficult to see that $R \oplus \cdots \oplus R = R^n$, a direct sum of n-copies of R, is a QE-ring. Similarly, if R is a QE-ring, then R_n, the full matrix algebra of nxn-matrices over R, is a QE-ring. Simple algebras are QE-rings. If R is a QE-ring which is a domain and which as a K-algebra is finite dimensional over its center, then R is easily seen to be a division ring. The Weyl-algebra over a field of characteristic 0 is a

QE-domain which is simple but not a division ring. Commutative QE-domains are fields. One question which is suggested by a dearth of examples of QE-domains which are not simple algebras is the following: are QE-domains simple algebras?

Concerning the theorem proven in this note we have the following question: If R is a K-algebra satisfying a given polynomial identity, does it have a receptor satisfying the same polynomial identity? Do we need to put any restrictions on the types of identities used?

If $R = K\{y_1, \ldots, y_n\}$ and if U is a receptor of R, then any proper ideal of R has a generic zero in U^n. Thus, if a is a proper ideal, then there is an element $(u_1, \ldots, u_n)$ of U^n such that $P(u_1, \ldots, u_n) = 0$ if and only if $P(y_1, \ldots, y_n) \in a$. Thus in a sense a receptor behaves somewhat like an algebraic closure.

If a ring is a QE-ring and if it satisfies other conditions, then there is a tendency for the other conditions to become inherited by epimorphic images. Thus, e.g., if R is a QE-ring and if it contains no nilpotent elements, then the same is true for R/a since R/a can be embedded in R. Hence one might expect the class of complete rings to be reasonably small and quite suited to further analysis.

Reference

1. P. M. Cohn, *Free rings and their relations*, Academic Press, New York, 1971.

Department of Mathematics
The University of Alabama
University, Alabama 35486
U.S.A.