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WHEN IS EVERY KERNEL FUNCTOR IDEMPOTENT?

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Introduction. All rings occurring are associative and possess a unity, which is preserved under subrings and ring homomorphisms. All modules are unitary right modules. We let $\mathscr{M}_R$ denote the category of right R-modules.

In recent years several authors have studied rings R by imposing restrictions on the torsion theories [**4**] of $\mathscr{M}_R$. (See for instance [**2**; **23**; **24**].) This paper offers another alternative to that trend, namely the study of rings R via their set of kernel functors $K(R)$.

The concept of kernel functor is by now well known, as it appears in [**12**]. We also know the similarities and differences that exist between the kernel functors of R and the torsion theories of $\mathscr{M}_R$. In particular, both concepts intersect at the hereditary torsion theories.

Any ring satisfies the following containment relationship: $\{0, \infty\} \subset I(R) \subset K(R)$; it is essentially proved in [**10**] that $\{0, \infty\} = I(R)$ if and only if R is a left perfect ring with a unique simple right R-module up to isomorphisms. In this paper we consider the other extreme case, i.e., when is $I(R) = K(R)$?

To study these rings we proceed as follows:

(a) We see first what happens if in addition R is assumed commutative. We settle the problem by proving the

THEOREM. *If R is commutative, $K(R) = I(R)$ if and only if R is a finite product of fields.*

We then analyze the consequences of this result.

(b) In the general case in which R is not commutative a complete characterization seems somehow distant at the moment. However, two particular instances are worth considering. The solutions we obtain show that V-rings and PCI-rings are called to play a central role in the study of the rings here examined. For an up to date account of results as well as open problems on PCI-rings the reader is referred to [**7**].

The particular cases we are referring to are described next.

We say a kernel functor σ *splits* whenever $\sigma(M)$ is a direct summand of M for every module M.

We say a ring R *has* (P) whenever $M \in \mathscr{M}_R$, $\sigma \in K(R)$, $\sigma \neq \infty$ implies $\sigma(M)$ is injective.

We say a ring R *has* (Q) whenever σ splits for every $\sigma \in K(R)$.

Clearly (P) $\Rightarrow$ (Q) $\Rightarrow K(R) = I(R)$, for any ring R. We obtain the following

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THEOREM. *R has* (P) *if and only if R is Morita equivalent to a right noetherian PCI-ring.*

Finally, a decomposition theorem for rings having (Q) is reached:

THEOREM. *R has* (Q) *if and only if R is Morita equivalent to* $D_1 \times \ldots \times D_n$, *where the* D_i*'s are simple V-domains having* (Q).

These theorems, besides being of interest in themselves, show that to obtain more definite results concerning the question posed in this paper further study of *PCI*-rings is necessary.

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Preliminaries. Given a ring R we will say that A_R is *large* (or essential) in $B_R(A_R \subset' B_R)$ whenever A intersects non-trivially with every non-zero submodule of B. Accordingly, $M \neq 0$ is called *uniform* whenever $N \subset' M$ for all non-zero $N_R \subset M_R$. For any module M we let $E(M)$ denote an injective hull of M. Given a ring R, a module M, a submodule $N \subset M$ and a non-empty set $S \subset M$, the right ideal $\{r \in R; S.r \subset N\}$ will be denoted by $(N:_R S)$ or by $(N:S)$ when no danger of confusion arises. The term ideal will mean a two-sided ideal. A ring is *simple* if it has exactly two ideals. A ring R is said to be *regular* (in the sense of Von Neumann) if every finitely generated right (left) ideal is generated by an idempotent.

Following Goldman [**12**] a functor $\sigma : \mathcal{M}_R \rightarrow \mathcal{M}_R$ is called a *kernel functor* if

(1) for all M_R, $\sigma(M)$ is a submodule of M;

(2) $f: M \rightarrow M'$ implies $f(\sigma(M)) \subset \sigma(M')$ and $\sigma(f)$ is the restriction of f to $\sigma(M)$; and

(3) $M' \subset M$ implies $\sigma(M') = M' \cap \sigma(M)$.

A kernel functor σ is said to be *idempotent* if for every M_R, $\sigma(M/\sigma(M)) = 0$.

The *trivial kernel functors* 0 and ∞ are defined by setting: $0(M) = 0$ and $\infty(M) = M$ for every R-module M.

Still borrowing from [**12**], if $\sigma \in K(R)$, M is called a *σ-torsion module* if $\sigma(M) = M$ and a *σ-torsion free module* if $\sigma(M) = 0$.

For any $\sigma \in K(R)$ the collection $C(\sigma)$ of all the σ-torsion modules is closed under arbitrary direct sums, submodules and homomorphic images. Conversely, for any collection of modules $\mathcal{C}$ closed under arbitrary direct sums, submodules and homomorphic images there exists a unique $\sigma \in K(R)$ such that $\mathcal{C} = C(\sigma)$. If a kernel functor σ is idempotent then $C(\sigma)$ is in addition closed under group extensions. Conversely, any collection $\mathcal{C}$ closed under submodules, arbitrary direct sums, homomorphic images and group extensions is of the form $C(\sigma)$ for a unique $\sigma \in I(R)$.

The map φ which sends $\sigma \in K(R)$ into

$$\mathscr{T}(\sigma) = \{I_R \subset R; R/I \text{ is } \sigma\text{-torsion}\}$$

establishes a one-to-one correspondence between $K(R)$ and the set of (Gabriel) topologizing filters of (right ideals of) R. A topologizing filter $\mathscr{T}$ is said to be *idempotent* if $I \in \mathscr{T}$, $J_R \subset I$, $(J:x) \in \mathscr{T}$ for every $x \in I$, implies $J \in \mathscr{F}$. Therefore φ induces by restriction a one-to-one correspondence between $I(R)$ and the set of idempotent topologizing filters of R.

For an excellent treatment of kernel functors the reader is referred to Goldman [**12**] and Gabriel [**9**]. The development of the subject can be found in Lambek [**18**].

We let $\mathscr{L}$ denote the filter of large right ideals of R and Z its associated kernel functor; consequently $Z(M)$ is the singular submodule of M. (See [**17**].)

The idempotent topologizing filter of dense (or rational) right ideals [**17**] of R will be indicated by $\mathscr{D}$. Therefore $\mathscr{D} \subset \mathscr{L}$ and $\mathscr{D} = \mathscr{L}$ if and only if $Z(R) = 0$.

We finally set $\mathscr{G}$ = Goldie's filter of R = smallest idempotent topologizing filter containing $\mathscr{L}$. We always have $\mathscr{D} \subset \mathscr{L} \subset \mathscr{G}$ and they all may differ.

We start with

LEMMA 1.1. *$\mathscr{L}$ is idempotent, i.e., $\mathscr{L} = \mathscr{G}$ if and only if $Z(R) = 0$.*

Proof. ($\Leftarrow$) $Z(R) = 0$ implies $\mathscr{L} = \mathscr{D}$, an idempotent topologizing filter.

($\Rightarrow$) We know that there exists a unique $G \in I(R)$ such that $\mathscr{G} = \mathscr{F}(G)$. Therefore for every M_R we have

$$G(M)/Z(M) = Z(M/Z(M)).$$

If $Z(R) \subset' R$ then $G(R)/Z(R) = R/Z(R)$ and so $G(R) = R$; since $\mathscr{L} = \mathscr{G}$ we conclude that $Z(R) = R$, an impossibility. Therefore $Z(R)$ is not large in R and so by Zorn's lemma there exists $A \neq 0$ such that $Z(R) \oplus A$ is large. Let $u \in Z(R) \oplus A$ an arbitrary element, say $u = z + a$ with $z \in Z(R)$ and $a \in A$. We have $(0:z) \subset (A:z) \subset (A:u)$ and since $(0:z) \in \mathscr{L}$, $(A:u) \in \mathscr{L}$. By assumption, $\mathscr{L}$ is idempotent and so $A \in \mathscr{L}$, i.e., $Z(R) = 0$ as asserted.

Remark. This lemma tells us that either $\mathscr{D}, \mathscr{L}$ and $\mathscr{G}$ coincide or they all differ. It also shows that $K(R) = I(R)$ implies $Z(R) = 0$. Therefore throughout this paper we will be dealing with right non-singular rings.

The rings for which $K(R) = I(R)$.

The commutative case is considered first.

THEOREM 2.1. *Suppose R is commutative. Then $K(R) = I(R)$ if and only if R is a finite product of fields.*

Proof. ($\Leftarrow$) This is obvious.

($\Rightarrow$) Let I be an ideal. Then $\mathscr{F} = \{J_R; I \subset J\}$ is a topologizing filter which is idempotent by assumption. Therefore $I^2 = I$. Hence, R is regular. Assume R has a countably infinite set of orthogonal idempotents $\{e_i\}$. Put $I_k = (1 - e_k)R$

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and $I = \bigoplus_{i=1}^{\infty} (e_i)R$. Define next $\mathscr{F}'$ as the smallest topologizing filter containing I and I_k, $k \in \mathbf{N}$. Thus $J \in \mathscr{F}'$ if and only if there exist $r_1, \ldots, r_k$, $x_1, \ldots, x_m$ in R such that

$$J \supseteq (I_{n_1}:r_1) \cap \ldots \cap (I_{n_k}:r_k) \cap (I:x_1) \cap \ldots \cap (I:x_m).$$

Let σ be the kernel functor associated with $\mathscr{F}'$. We claim that I is σ-torsion. In fact, if $x \in I$, say $x = e_1\lambda_1 + \ldots + e_k\lambda_k$ it follows that $x.[\bigcap_{j=1}^{k} (I_j:\lambda_j)] = 0$, that is, $(0:x) \in \mathscr{F}'$. By assumption $\mathscr{F}'$ is idempotent and so the exact sequence

$$0 \to I \to R \to R/I \to 0$$

with both ends σ-torsion gives us that R is σ-torsion, i.e., $(0) \in \mathscr{F}'$. From this we obtain, for some k and some m

$$0 = (I_{n_1}:r_1) \cap \ldots \cap (I_{n_k}:r_k) \cap (I:x_1) \cap \ldots \cap (I:x_m)$$

which clearly contains $I_{n_1} \cap \ldots \cap I_{n_k} \cap I$. However, for any $j \neq n_1, \ldots, n_k$, $e_j \in I_{n_1} \cap \ldots \cap I_{n_k} \cap I$, a contradiction. We conclude that R does not admit infinitely many orthogonal idempotents. Therefore R is semisimple artinian and being commutative it is a finite direct product of fields.

Remarks. (a) It is obvious that R (not necessarily commutative) semisimple artinian implies $K(R) = I(R)$. We have just seen that the reverse implication is true when R is commutative. It will be shown later that this need not be the case when commutativity is removed.

(b) If R is semisimple artinian with exactly n simple modules (up to isomorphisms) the cardinality of $I(R)$ is 2^n. Hence, in the commutative case our approach of making $I(R)$ as large as possible curiously leads to only finitely many elements in $I(R)$ and does not take us far from the simple artinian rings.

If R is arbitrary $K(R) = I(R)$ implies $I^2 = I$ for all ideals of R and $Z(R_R) = 0$. By paralleling the proof of the last theorem we will show that the ring $R = \mathrm{End}_F(V)$, V a countably infinite dimensional vector space over the field F, has kernel functors which are not idempotents; however R is known to be a prime right non-singular ring in which every ideal equals its square.

R can be viewed as the ring of all row-finite matrices with entries in F. Let $\{e_{ij}\}_{1 \leqq i, j \leqq \infty}$ denote the matrix units of R having the unity element of F in the (ij)th position and zeros elsewhere and let e_i denote the idempotents e_{ii} for $i = 1, 2 \ldots$

Observe that $e_i.r = i$th row of r, for any r in R. Set $I_k = (1 - e_k)R$ and $I = \sum_{i \in \mathbf{N}} (Re_i)$, i.e., $I = \mathrm{soc}(R)$. It is known that I is the unique non-trivial ideal of R and that $I^2 = I$. (See [**14**].)

As before set $\mathscr{F}$ = the smallest topologizing filter containing I and the I_k's, for all $k \in \mathbf{N}$. If $\mathscr{F}$ is assumed idempotent, as before we obtain that $(0) \in \mathscr{F}$. Notice that for any $r_1, \ldots, r_k \in R$, $I \subset \bigcap_{i=1}^{n} (I:r_i)$.

Claim: For arbitrarily given $x_{\nu_1}, \ldots, x_{\nu_n}$,

$$0 \neq I \cap (I_{\nu_1}:x_{\nu_1}) \cap \ldots \cap (I_{\nu_n}:x_{\nu_n}).$$

In fact, we may assume $\nu_1 \leqq \nu_2 \leqq \ldots \leqq \nu_n$. We can find a natural number N such that $N > \nu_n$ and such that the first ν_n rows of $x_{\nu_1}, \ldots, x_{\nu_n}$ lie in the block

$$\nu_n \begin{bmatrix} * & \cdots & 0 & \cdots \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ * & \cdots & 0 & \cdots \end{bmatrix}_{N}$$

Consider next $z = e_{N+1,1}$. By construction the first ν_n rows of $x_{\nu_1} \cdot z, \ldots, x_{\nu_n} \cdot z$ all vanish. In particular, for $j = 1, \ldots, n$, $x_{\nu_j} \cdot z \in I_j$; since it is clear that $z \in I$ our claim is proved.

In other words, we have shown that for arbitrarily given $r_1, \ldots, r_m, x_{\nu_1}, \ldots, x_{\nu_n}$ in R

$$0 \neq (I_{\nu_1} : x_{\nu_1}) \cap \ldots \cap (I_{\nu_n} : x_{\nu_n}) \cap (I : r_1) \cap \ldots \cap (I : r_m),$$

which tells us that $(0) \notin \mathscr{F}$. Therefore $\mathscr{F}$ is not idempotent.

R_R is a *V-ring* if every simple R-module is injective. (See [**6**].) A module M_R is called *proper cyclic* if it is cyclic and non-isomorphic to R. Consequently R_R is a *PCI-ring* whenever its proper cyclic modules are injective. (See [**7**].)

We will write $R \sim S$ to indicate that R and S are Morita equivalent rings. (See [**19**].)

PROPOSITION 2.2. *Having* (P), (Q) *or* $K(R) = I(R)$ *is a Morita invariant.*

Proof. Suppose $R \sim S$ via $F: R \to S$ and $G: S \to R$. Assume $K(R) = I(R)$. If $\lambda \in K(S)$ define $\sigma \in K(R)$ such that M_R is σ-torsion if and only if $F(M)$ is λ-torsion. It is routine to check that in fact σ is in $K(R)$. To show that $\lambda \in I(S)$ we start with a sequence of S-modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where A and C are λ-torsion and must conclude that B is so. Since $R \sim S$ this sequence is (up to isomorphisms) the result of applying F to an exact sequence of R-modules

$$0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0.$$

It follows that A' and C' are σ-torsion and since σ is idempotent by assumption, B' is σ-torsion. Therefore B is λ-torsion, as we wanted to show. One proceeds analogously if R has (P) or (Q) after observing that $\lambda \neq \infty$ implies $\sigma \neq \infty$.

We are now ready to characterize rings with (P).

THEOREM 2.3. R_R *has* (P) *if and only if* $R \sim S$, S_S *a noetherian right PCI-ring.*

Proof. ($\Leftarrow$) It is enough to see that S has (P). Given $\sigma \neq \infty$, $\sigma \in K(S)$ and $M = \sigma(M)$ write $M = \sum_{m \in M} (mS)$. Clearly mS is proper cyclic for every $m \in M$ and hence injective. Thus $\bigoplus_{m \in M} (mS)$ is injective since R (and so S) is right noetherian. By a result of Faith [**7**], S is right hereditary and therefore M is injective by the exactness of

$$\bigoplus_{m \in m} (mS) \rightarrow \sum_{m \in m} (mS) \rightarrow 0.$$

($\Rightarrow$) Set $\sigma(M) = \operatorname{soc}(M)$ for all $M \in \mathscr{M}_R$.

(a) If $\sigma = \infty$ then R is semisimple artinian, and so a right noetherian PCI-ring.

(b) If $\sigma \neq \infty$ semisimple R-modules are injective and so by Kurshan [**16**], R_R is a noetherian V-ring. We claim that in this case R is a simple ring. In fact, if I is an ideal of R set $E = E(R/I)$ and $\lambda = \tau_E$. (See [**12**, p. 33].) We claim that $\lambda(I) = I$. In fact, assume there exists a non-zero R-homomorphism $f: I \rightarrow E$. Since $R/I \subset' E$ there exists $x \in I$ such that $f(x) \in R/I$ and $f(x) \neq 0$. Inasmuch as R_R is a V-ring $(xR) = (xR)^2$ and we express $x = xy$ where $y \in I$. It follows that $0 \neq f(x) = f(xy) = f(x)y = 0$ since I is an ideal, a contradiction. Therefore $\lambda(I) = I$ and $\lambda \neq \infty$ because I is non-trivial. By hypothesis I is injective and so there exists $A_R \subset R$ such that $I \oplus A = R$. Observe that

$$IA = (IA)^2 = I(AI)A = 0.$$

It follows that A is a non-trivial ideal and so it will be right injective since the argument used to deal with I applies. We infer that R_R is noetherian, injective and non-singular. Hence R is a semisimple artinian ring [**22**, Theorem 1.6, p. 115], a contradiction. We conclude that R is simple, as claimed. We proceed to show that R_R is hereditary. In fact, if X_R is injective and $g: X \rightarrow M$ is onto then $M = Z(M) \oplus M/Z(M)$ since $Z(M)$ is injective by hypothesis. But $M/Z(M)$ is a non-singular image of an injective module and thus it is injective, by [**26**]. Therefore M is injective and consequently R is right hereditary.

Next pick a uniform right ideal I. By Goldie [**11**], $S = \operatorname{End}_R(I)$ is a domain. Inasmuch as R_R is simple noetherian hereditary I is a finitely generated projective generator in $\mathscr{M}_R$. Therefore $R \sim S$. By (2.2) S inherits (P). If $0 \neq J_S \subset S$ is given then necessarily $J \subset' S$ and so $S/J = Z(S/J)$ is injective. It is clear that S is also right noetherian.

Remarks. (a) An alternative proof can be provided by considering the injectivity of all singular modules [**13**] instead of the injectivity of the semisimples.

(b) In [**21**] B. Osofsky furnished examples of right noetherian PCI-rings with infinitely many non-isomorphic simple modules. If R is such a ring and $\{S_\nu\}_{\nu \in \mathscr{M}}$ are all the non-isomorphic simple R-modules then $K(R) - \infty$ is in one-to-one correspondence with

$$C_{\mathscr{T}} = \left\{M; M = \bigoplus_{\nu \in \mathscr{T}} S_\nu\right\}$$

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for all $\mathscr{T} \subset \mathscr{A}$, if we agree that the direct sum taken over the empty set is (0). Hence we see that unlike the commutative case $K(R) = I(R)$ does not imply that $K(R)$ has finitely many elements.

Before studying rings with (Q) we pause for a moment to consider a dual of the previous result.

We say a ring R *has* (PD), that is (P) dual, whenever $\sigma \in K(R)$, $\sigma \neq 0$ implies $M/\sigma(M)$ is projective for all M_R.

It is clear that if R has (PD), R has (Q). Our next result shows rather easily that if R has (PD) then R is semisimple artinian. More precisely, we have

PROPOSITION 2.4. *If for all M_R, $(M/\text{soc}(M))$ is projective then R is semisimple artinian. In particular if R has* (PD) *R is semisimple artinian.*

Proof. It follows easily that R is a right noetherian V-ring; it decomposes as $R = R_1 \times \ldots \times R_n$ the R_i's being simple right noetherian V-rings. (See [**20**] or [**5**, p. 342].) It follows that each R_i satisfies our hypothesis. We may thus assume that R is simple. Set $Q = Q_{\max}(R)$, the maximal ring of quotients of R_R. (See [**25**; **15**].)

If $\text{soc}_R(Q) \neq 0$ then $\text{soc}(R) \neq 0$ and therefore R is simple artinian. If, on the other hand, $\text{soc}_R(Q) = 0$ then $Q_R = Q/\text{soc}(Q)$ is projective. Since R is a simple ring, Q turns out to be a generator of $\mathscr{M}_R$, and so R_R is injective, that is, $R = Q$. But R is right noetherian and regular [**15**] and so simple artinian in this case also.

Remark. A different proof, suggested by the referee, is provided next.

Proof. It follows that semisimple right R-modules are injective, hence $R_R = S_R \oplus T_R$, where $S_R = \text{Soc}(R)$. Now $(ST)^2 = 0$ so $ST = 0$ since R is a right V-ring and R has no nilpotent (right) ideals, thus $R = S \oplus T$ is a ring direct sum. Clearly, T is a V-ring with $M/\text{soc}(M)$ T-projective for all right T-modules M_T, hence if $\text{soc}(M_T) = 0$, M_T is T-projective. Since $\text{soc}(T_T) = 0$, it follows that any direct product of copies of T_T is T-projective, so by S. U. Chase (*Direct product of modules*, Trans. Amer. Math. Soc. *97* (1960), 457–73), $T/J(T)$, $J(T)$ the Jacobson radical of T, is a semisimple ring with minimum condition. As T is a V-ring, $J(T) = 0$ and the proposition follows.

Our next goal is to prove a decomposition theorem for rings with (Q). To prepare the ground, assume we have a ring decomposition $R = R_1 \times \ldots \times R_n$. Given $\mathscr{F}$, a collection of right ideals of R, set $\mathscr{F}_i = \{IR_i; I \in \mathscr{F}\}$ for $i = 1, \ldots, n$. As usual we have $1 = e_1 + \ldots + e_n$ where the e_i's are central orthogonal idempotents and $e_i \in R_i$ for $i = 1, \ldots, n$.

LEMMA 2.5. *With the notation as above we have:*
(1) $\mathscr{F} = \mathscr{F}_1 \times \ldots \times \mathscr{F}_n$.
(2) *$\mathscr{F}$ is a topologizing filter if and only if each $\mathscr{F}_i$ is so.*
(3) *If σ and σ_i denote the kernel functors associated with $\mathscr{F}$ and the $\mathscr{F}_i$'s*

respectively then σ *splits* (respectively, is idempotent) *if and only if each* σ_i *splits* (respectively, is idempotent).

Proof. The proof is straightforward.

For a ring R and a module M we use the following notations:

h. $\dim(M) = \inf\{n; \operatorname{Ext}^{n+1}(M, -) = 0\}$

r. gl. $\dim(R) = \sup\{$h. $\dim(M); M \in \mathscr{M}_R\}$

as they appear in [**1**].

THEOREM 2.6. *R has* (Q), *i.e., every kernel functor of* $\mathscr{M}_R$ *splits, if and only if* $R \sim D_1 \times \ldots \times D_n$, *where the* D_i's *are simple V-domains having* (Q).

Proof. ($\Leftarrow$) This is clear, according to (2.3) and (2.5).

($\Rightarrow$) Since $\operatorname{soc}(\cdot)$ is a splitting kernel functor by assumption, semisimple modules are injective. Thus, R_R is a noetherian V-ring. Consequently $R = R_1 \times \ldots \times R_n$ (a ring decomposition) where the R_i's are simple right noetherian V-rings. Since the singular submodule splits off, r. gl. $\dim(R) \leqq 2$ according to [**23**]. It is then clear that for all $i = 1, \ldots, n$, r. gl. $\dim(R_i) \leqq 2$. It is enough to show that for each i there exists a simple V-domain D_i having (Q) such that $R_i \sim D_i$. Inasmuch as R has (Q), R_i has (Q) for each i according to (2.5). We proceed now to work componentwise.

Assume (after changing notation) that R is a simple right noetherian V-ring having (Q) and such that r. gl. $\dim(R) \leqq 2$. By the Faith-Utumi theorem [**8**; **17**] there exists a subset S of R and a uniform right ideal U_R such that $U = \{r \in R; S.r = 0\}$. (Reason: Let Q be the right classical quotient ring of R. We know that $Q \cong F_n$, the ring of n by n matrices over a division ring F. The Faith-Utumi theorem says that there exists a complete set of matrix units $\{e_{ij}; 1 \leqq i, j \leqq n\}$ and an Ore domain $D = e_{11}Re_{11}$ with quotient field F such that $R \supseteq D_n = \sum e_{ij}D$. Set next $S = \sum_{j=2} \sum_{i=1} e_{ij}D$, i.e., S is the set of all matrices in Q with entries in D and with first column equal to zero. Since D is a domain with quotient field F, $\sum e_{ij}a_{ij} \in Q$ annihilates S on the right if and only if $a_{ij} = 0$ for all $i > 1$, i.e., $(0:_Q S) = e_{11}Q$, that is, the first rows of matrices in Q. In particular, $(0:_Q S)$ is indecomposable as an R-module and

$$U = (0:_R S) = (0:_Q S) \cap R \supseteq e_{11}Re_{11} = D \neq 0$$

must be a non-zero uniform right ideal). It follows from [**11**] that $U_R = (0:s)$ for some single element s of R. From the exact sequence

$$0 \rightarrow U \rightarrow R \rightarrow sR \rightarrow 0$$

and the fact that h. $\dim(sR) \leqq 1$ we conclude that U_R is projective. Since U is uniform, $D^* = \operatorname{End}_R(U)$ is a domain according to [**11**]. Since R is simple and right noetherian U_R is a finitely generated projective generator in $\mathscr{M}_R$ and so $R \sim D^*$. It is clear that D^* is a simple V-domain which has (Q), since (2.2) applies.

It is now easy to provide an example of a ring R which has (Q) and does not have (P). To that end consider $D_1 = D_2 = K[x; d]$, the ring of differential polynomials over a (Kolchin) universal field K, whose properties have been investigated by Cozzens [**3**].D_1 (and so does D_2) has exactly three kernel functors, 0, ∞ and Z; they are all idempotent kernels. D_1 (and so is D_2) is a simple V-domain having (Q). Form $R = D_1 \times D_2$. According to last theorem R has (Q). To see that R can not have (P) we use the fact that a PCI-ring is either semisimple artinian or a simple domain, according to [**7**]. We now quote (2.3) and observe that R is neither a domain nor a semisimple artinian ring [**3**].

Besides investigating the V-domains having (Q) this work should be carried further by studying how much the rings having (Q) and the rings for which $K(R) = I(R)$ differ.

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