ON CARATHÉODORY'S THEOREM

P. Scherk

(received February 24, 1966)

The following proof of Carathéodory's Theorem, while not essentially new, seems to be natural and therefore of interest.

LEMMA 1. Let P denote a supporting hyperplane of the convex polytope K. Then $P \cap K$ is a convex polytope whose vertices are vertices of K.

Proof. Let $K = H(a_1, \ldots, a_m)$ be the convex hull of the points $a_1, \ldots, a_m$. Let $P = \{x \mid cx = \alpha\}$. Thus we have, say, $cx - \alpha \geq 0$ for every point $x \in K$; in particular

$$ca_i - \alpha \geq 0 \qquad (i = 1, \ldots, m).$$

Any point $b \in P \cap K$ can be represented in the form

$$b = \sum_1^m \lambda_i a_i \; ; \; \Sigma \lambda_i = 1 \; ; \; \lambda_1 \geq 0, \ldots, \lambda_m \geq 0 \; .$$

Since $b \in P$, we have

$$0 = c.b - \alpha = \Sigma \lambda_i \, . \, ca_i - \alpha \, \Sigma \lambda_i (ca_i - \alpha) \; .$$

Here $\lambda_i (ca_i - \alpha) \geq 0$ for each i. Hence

$$\lambda_i (ca_i - \alpha) = 0 \qquad (i = 1, \ldots, m).$$

If $\lambda_j > 0$, then $ca_j - \alpha = 0$, i.e., $a_j \in P$. Suppose, say, that

Canad. Math. Bull. vol. 9, no. 4, 1966

https://doi.org/10.4153/CMB-1966-056-5 Published online by Cambridge University Press

$a_1, \ldots, a_p$ are those a_i that lie in $P \cap K$. Then $\lambda_j = 0$ if $j > p$. Thus $b \in H(a_1, \ldots, a_p)$ and $P \cap K \subset H(a_1, \ldots, a_p)$. The inverse relation being trivial, we obtain

$$P \cap K = H(a_1, \ldots, a_p).$$

LEMMA 2. Every point of a convex polytope lies in a simplex whose vertices are also vertices of the polytope.

A short analytic proof of this lemma is implicitly contained in Eggleston's proof of Carathéodory's Theorem, [H.G. Eggleston, Convexity (Cambridge, 1958), pp. 34f.]

Alternate Proof. Let $K = H(a_1, \ldots, a_m)$ be the given polytope. The assertion is trivial if $\dim K \leq 1$. Suppose it is proved for $\dim K < d$ and let $\dim K = d$. By restricting our attention to the flat spanned by K, we may assume that K spans the whole space.

Let b denote a point on the boundary of K. Thus there exists a supporting hyperplane P of K through b. By Lemma 1, $P \cap K$ is a convex polytope whose vertices are vertices of K. Since $\dim (P \cap K) < d$, there exists a simplex containing b whose vertices are vertices of $P \cap K$ and hence of K.

Now let c be an interior point of K. Then there exists a point b on the boundary of K such that c lies on the segment connecting a_m with b. Construct a supporting hyperplane P of K through b and a simplex in P containing b as before. Since c and a_m do not lie in P, this simplex and a_m span a simplex containing c.

Let A be any non-void set in n-space. Let K_m denote the union of all the convex polytopes with m vertices belonging to A:

$$K_m = \bigcup_{\{a_1, \ldots, a_m\} \subset A} H(a_1, \ldots, a_m).$$

(The points $a_1, \ldots, a_m$ need not be mutually distinct.) Put

$$K = \bigcup_{1}^{\infty} K_m .$$

LEMMA 3. K is equal to the convex hull $H(A)$ of A (cf. Eggleston, l.c., p. 35).

Proof. Given any m-tuple $\{a_1, \ldots, a_m\} \subset A$, we have

$$H(a_1, \ldots, a_m) \subset H(A),$$

hence $K_m \subset H(A)$ and therefore $K \subset H(A)$.

Let x and y denote any two points of K, say

$$x \in H(a_1, \ldots, a_m), \quad y \in H(a_{m+1}, \ldots, a_p).$$

Then $H(a_1, \ldots, a_p)$ contains both x and y. Thus the segment $H(x, y)$ also lies in $H(a_1, \ldots, a_p) \subset K_p \subset K$. Hence K is convex. Since $A = K_1 \subset K$, $H(A) \subset K$.

CARATHEODORY'S THEOREM. Let A be a set in n-space. Then every point of $H(A)$ lies in a simplex whose vertices belong to A.

Proof. We may assume $A \neq \emptyset$. Let $x \in H(A)$. By Lemma 3, there exists an m such that $x \in K_m$. Thus x lies in some convex polytope $H(a_1, \ldots, a_m)$ with vertices in A. By Lemma 2, there exists a simplex containing x with vertices in A.

University of Toronto

https://doi.org/10.4153/CMB-1966-056-5 Published online by Cambridge University Press