THE LIMIT THEOREM FOR APERIODIC DISCRETE RENEWAL PROCESSES

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1. Introduction

A discrete renewal process is a sequence $\{X_i\}$ of independently and identically distributed random variables which can take on only those values which are positive integral multiples of a positive real number δ. For notational convenience we take $\delta = 1$ and write

$$p_n = Pr\{X_i = n\}, \qquad n \geqq 1 \tag{1.1}$$

where

$$p_n \geqq 0, \qquad \sum_{n=1}^{\infty} p_n = 1.$$

The greatest common divisor, d, of those n for which $p_n > 0$ is called the period of the renewal process. If $d = 1$ the renewal process is said to be aperiodic. In this paper only aperiodic discrete renewal processes are considered. The limit theorem for periodic discrete renewal processes, that is $d > 1$, is easily deducible from that for aperiodic processes.

In renewal theory the random variables X_i are the successive lifetimes of items which are renewed at the instants $S_n = \sum_{i=1}^{n} X_i$. If u_n, $n \geqq 1$, is the probability that a renewal occurs at the instant n then the limit theorem of discrete renewal theory states that

$$\lim_{n\to\infty} u_n = \mu^{-1} \tag{1.2}$$

where

$$\mu = \sum_{n=1}^{\infty} n p_n \tag{1.3}$$

is the mean life of an item and the right-hand side of (1.2) is interpreted at 0 if the series in (1.3) diverges.

This result is due to Erdös, Feller and Pollard [3]. It can be deduced also from a result of Kolmogorov [5] on the ergodicity of Markov chains. Another proof has been given recently by Feller [4]. In this paper a new proof of the theorem is given. The ideas underlying the proof are similar to those of Doob [1]. The renewal process is regarded as a particular Markov chain and ergodicity is established for that chain. In the next section we prove two lemmas which are special cases of general results on Markov chains. It seems logically more appropriate, however, to prove them for the particular case of this paper than to refer to general theory.

Part of the interest of the methods of this paper is the ease with which they can be generalised to continuous renewal processes and it is hoped to present this generalisation in a subsequent paper.

2. The Markov chain associated with a renewal process

Let $\{X_i\}$, $i \geqq 1$, be a discrete aperiodic renewal process with lifetime distribution given by (1.1) Introduce the following notation

$$r_n = \sum_{m=n}^{\infty} p_m, \qquad n \geqq 1, \tag{2.1}$$

$$q_n = p_n r_n^{-1}, \qquad n \geqq 1, \quad q_0 = 1. \tag{2.2}$$

The following equations are easily verified.

$$\sum_{n=1}^{\infty} r_n = \mu, \tag{2.3}$$

where μ is given by (1.3),

$$\prod_{j=1}^{n-1} (1-q_j) = r_n \tag{2.4}$$

$$r_i q_n \prod_{j=1}^{n-1} (1-q_j) = p_n. \tag{2.5}$$

Consider the homogeneous Markov chain with one-step transition probabilities p_{ij}, $i \geqq 1, j \geqq 1$ defined by

$$p_{ij} = \begin{cases} q_i & \text{if } \; j = 1 \\ 1-q_i & \text{if } \; j = i+1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Let f_{i1}^n be the conditional probability of first entry into state 1 in n steps when it is given that the initial state is i. It follows from (2.6) that

$$f_{i1}^n = q_{n+i-1} \prod_{j=i}^{n+i-2} (1-q_j). \tag{2.7}$$

In particular, $f_{11}^n = p_n$ and thus the state 1 is recurrent and its recurrence time distribution is the lifetime distribution of the renewal process.

Denote by p_{ij}^n the n-step transition probability of the Markov chain defined by (2.6). The probability p_{ij}^n can be interpreted as the conditional probability, given that at time 0 the first lifetime of the renewal process has lasted as long as i, that at time n the current lifetime has lasted at least as long as j. The interpretation can be verified easily by noting that it implies

$$(2.8) \qquad p_{ij}^{n+1} = \begin{cases} \sum_{k=1}^{\infty} p_{ik}^{n} q_k, & j = 1, \quad i \geqq 1, \\ p_{ij-1}^{n}(1-q_{j-1}), & j > 1, \quad i \geqq 1. \end{cases}$$

Comparison with (2.6) shows that this set of equations can be written in the standard form

$$(2.9) \qquad p_{ij}^{n+1} = \sum_{k=1}^{\infty} p_{ik}^{n} p_{kj}.$$

Note that from (2.8) we have

$$(2.10) \qquad p_{ij}^{n+1} = p_{i1}^{n+2-j} r_j, \qquad i, j \geqq 1.$$

In particular

$$(2.11) \qquad p_{ij}^{n} \leqq r_j, \qquad n, i, j \geqq 1.$$

We shall use later the following result, namely,

$$(2.12) \qquad \sum_{i=1}^{\infty} p_{ij}^{n} r_i = 1, \qquad n \geqq 1.$$

In order to prove (2.12) we require

$$(2.13) \qquad \sum_{k=1}^{n+1} p_{11}^{n+1-k} r_k = 1.$$

Equation (2.13) is well-known and is proved easily by induction on n. Note that

$$(2.14) \qquad p_{i1}^{n+1} = \sum_{k=1}^{n+1} f_{i1}^{k} p_{11}^{n+1-k}$$

To prove (2.12) we substitute from (2.5) and (2.7) into (2.14) to obtain

$$\sum_{i=1}^{\infty} p_{i1}^{n+1} r_i = \sum_{k=1}^{n+1} r_k p_{11}^{n+1-k}.$$

Equation (2.12) then follows from (2.13).

The following two lemmas are particular cases of general results on Markov chains.

LEMMA (2.1). *If the renewal process is aperiodic then for each $i \geqq 1$ the greatest common division of those n for which $p_{ii}^{n} > 0$ is unity.*

PROOF. Suppose that the positive p_j occur at $j = n_i$, $i = 1, 2, \cdots$ then it is easily deducible from the form of the 1-step transition probabilities (2.6) that $p_{11}^n > 0$ if and only if $n = a_1 n_1 + \cdots + a_k n_k$ where $a_1, \cdots a_k$, are non-negative integers and k is a positive integer. Since the g.c.d. of the n_i is unity it follows that the g.c.d. of those n for which $p_{11}^n > 0$ is also unity. Since 1-step transitions of the chain are either of the form $i \to 1$ or $i \to i+1$ the same is true of those n for which $p_{ii}^n > 0$.

LEMMA (2.2). *If the renewal process is aperiodic then given an integer $L > 1$, there is an integer $m = m(L)$ such that*

$$p_{l1}^m > 0, \text{ for all } l \leqq L. \tag{2.15}$$

PROOF. In order to prove this lemma we require the following well-known result, e.g. Doob [2].

"If S is a set of positive integers, with g.c.d. unity, such that if $n, m \in S$ then $n+m \in S$ then all sufficiently large integers belong to S."

By lemma (2.1) the g.c.d. of those n for which $p_{ll}^n > 0$ is unity. Since $p_{ll}^{n+n'} \geqq p_{ll}^n p_{ll}^{n'}$ it follows from the result quoted above that $p_{ll}^n > 0$ for all sufficiently large n. Thus to each integer l there corresponds an integer $m_1(l)$ such that $p_{ll}^n > 0$ for all $n \geqq m_1(l)$. Write

$$m_2(l) = \min_{n_i \geqq l} (n_i + 1 - l)$$

where the n_i are defined in the proof of lemma (2.1). Then $p_{l1}^{m_2} > 0$ for $m_2 = m_2(l)$, thus

$$p_{l1}^m \geqq p_{ll}^{m-m_2(l)} p_{l1}^{m_2(l)} > 0$$

for all $m \geqq m(l) = m_1(l) + m_2(l)$. Writing

$$m(L) = \max_{l \leqq L} m(l)$$

we obtain the lemma.

3. The limit theorem

We prove the following

THEOREM. *In the renewal process with lifetime distribution $\{p_j\}$ is aperiodic then the following limit exists and has the stated value*

$$\lim_{n \to \infty} p_{ij}^n = r_j \mu^{-1}, \qquad i, j \geqq 1, \tag{3.1}$$

where r_j is given by (2.1), *μ by* (1.3) *and the right-hand side of* (3.1) *is intorpreted as* 0 *when $\mu = \infty$.*

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PROOF. Write

$$(3.2)\qquad \begin{cases} \bar{p}_{ij} = \limsup_{n\to\infty} p_{ij}^n \\ \underline{p}_{ij} = \liminf_{n\to\infty} p_{ij}^n . \end{cases}$$

From equations (2.8) we obtain

$$(3.3)\qquad \bar{p}_{ij} = \begin{cases} \sum_{k=1}^{\infty} \bar{p}_{ik} q_k, & j = 1, \quad i \geqq 1, \\ \bar{p}_{i1} r_j, & j \geqq 1, \quad i \geqq 1, \end{cases}$$

$$(3.4)\qquad \underline{p}_{ij} = \begin{cases} \sum_{k=1}^{\infty} \underline{p}_{ik} q_k, & j = 1, \quad i \geqq 1, \\ \underline{p}_{i1} r_j, & j \geqq 1, \quad i \geqq 1. \end{cases}$$

The second equations of (3.3) and (3.4) follow easily from the second equations of (2.8). The first equation of (3.3) follows from the second equation since

$$\sum_{k=1}^{\infty} \bar{p}_{ik} q_k = \bar{p}_{i1} \sum_{k=1}^{\infty} r_k q_k = \bar{p}_{i1}.$$

The first equation of (3.4) is proved in the same way.

The equations (3.3) can be written as

$$(3.5)\qquad \bar{p}_{ij} = \sum_{k=1}^{\infty} \bar{p}_{ik} p_{kj}$$

and it follows by induction that

$$(3.6)\qquad \bar{p}_{ij} = \sum_{k=1}^{\infty} \bar{p}_{ik} p_{kj}^m, \qquad m \geqq 1.$$

Let e be an arbitrarily small positive number and choose a $K = K(e)$ such that

$$(3.7)\qquad \sum_{j=K+1}^{\infty} r_j \leqq e,$$

when $\sum r_j$ converges, that is $\mu < \infty$, and such that

$$(3.8)\qquad \sum_{j=1}^{K} r_j > e^{-1}$$

when $\sum r_j$ diverges, that is $\mu = \infty$.

By Lemma (2.2) there is an $m \geqq m(K)$ such that $p_{k1}^m > 0$ for $k \leqq K$. Let T_m be the set of integers, t, such that $p_{t1}^m > 0$ and let U_m be the set of integers, u, such that $p_{u1}^m = 0$. Then T_m contains the integers not exceeding K and if $\mu < \infty$

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(3.9) $$\sum_{j \in U_m} r_j \leqq e$$

whilst if $\mu = \infty$

(3.10) $$\sum_{j \in T_m} r_j > e^{-1}.$$

Let $S_{m,i}$ be a subsequence of integers $\{n\}$ such that

$$p_{i1}^{n+m} \to \bar{p}_{i1} \quad \text{as} \quad n \in S_{m,i} \to \infty.$$

Consider the following set of inequalities.

$$\begin{aligned} p_{k1}^m \liminf_{n \in S_{m,i}} p_{ik}^n &= \liminf_{n \in S_{m,i}} [p_{i1}^{n+m} - \sum_{j \neq k} p_{ij}^n p_{j1}^m] \\ &\geqq \bar{p}_{i1} - \sum_{j \neq k} \bar{p}_{ij} p_{j1}^m \\ &\geqq \bar{p}_{ik} p_{k1}^m, \quad \text{by (3.6)}. \end{aligned}$$

Thus

(3.11) $$\liminf_{n \in S_{m,i}} p_{ik}^n \geqq \bar{p}_{ik}, \qquad k \in T_m.$$

The only step in the argument leading to (3.11) which requires justification is where we have used

$$\limsup_{n \in S_{m,i}} \sum_{j \neq k} p_{ij}^n p_{j1}^m \leqq \sum_{j \neq k} \bar{p}_{ij} p_{j1}^m.$$

This is justified by (2.11) and (2.12) since $p_{ij}^n p_{j1}^m \leqq r_j p_{j1}^m$ and $\sum r_j p_{j1}^m$ is convergent.

It follows from (3.11) that

(3.12) $$\lim_{n \in S_{m,i}} p_{ik}^n = \bar{p}_{ik}, \qquad k \in T_m.$$

Suppose now that $\mu < \infty$ we prove

(3.13) $$\sum_{j=1}^{\infty} \bar{p}_{ij} = 1.$$

To prove (3.13) we note firstly that

(3.14) $$1 = \limsup \sum_{j=1}^{\infty} p_{ij}^n \leqq \sum_{j=1}^{\infty} \bar{p}_{ij},$$

for $p_{ij}^n \leqq r_j$ and $\sum r_j$ converges since $\mu < \infty$.

But

$$\begin{aligned} 1 &\geqq \liminf_{n \in S_{m,i}} \sum_{j=1}^{\infty} p_{ij}^n \\ &\geqq \sum_{j \in T_m} \bar{p}_{ij} + \sum_{j \in U_m} \underline{p}_{ij}. \end{aligned}$$

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Using (3.3) and (3.4) we obtain

$$\sum_{j=1}^{\infty} \bar{p}_{ij} \leqq 1+(\bar{p}_{i1}-\underline{p}_{i1}) \sum_{j \in U_m} r_j .$$

Thus by (3.9) we have

$$\sum_{j=1}^{\infty} \bar{p}_{ij} \leqq 1+e. \tag{3.15}$$

Since e is arbitrary (3.14) and (3.15) together imply (3.13). It follows at once from (3.3) that $\bar{p}_{ij} = r_j \mu^{-1}$. An exactly similar argument shows that $\underline{p}_{ij} = r_j \mu^{-1}$ and this proves the theorem when $\mu < \infty$.

Suppose finally that $\mu = \infty$, then

$$\begin{aligned} 1 = \sum_{j=1}^{\infty} p_{ij}^n &\geqq \liminf_{n \in S_{m,i}} \sum_{j \in T_m} p_{ij}^n \\ &\geqq \sum_{j \in T_m} \bar{p}_{ij}, \quad \text{by (3.12)} \\ &\geqq \bar{p}_{i1} \sum_{j \in T_m} r_j . \end{aligned}$$

Thus by (3.10)

$$0 \leqq \bar{p}_{i1} < e.$$

Since e is arbitrarily small it follows that $\bar{p}_{i1} = 0$ and thus from (3.3) $\bar{p}_{ij} = 0,\ i \geqq 1, j \geqq 1$ and this proves the theorem.

References

[1] Doob, J. L., Renewal theory from the point of view of the theory of probability, *Trans. Amer. Math. Soc.* **63**, 422–38 (1948).

[2] Doob, J. L., *Stochastic Processes*, New York, Wiley (1953).

[3] Erdös, P., Feller, W., and Pollard, H., A theorem on power series, *Bull. Amer. Math. Soc.* **55**, 201–204 (1949).

[4] Feller, W., A simple proof for renewal theorems, *Comm. Pure Appl. Math.* **14**, 285–293 (1961).

[5] Kolmogorov, A., Anfangsgründe der Theorie der Markoffschen Ketten mit unendlich vielen möglichen Zuständen, *Mat. Sbornik.* N.S. **1**, 607–610 (1936).

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