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NOTES ON TOPPAIR, TOP$_*$ AND REGULAR FIBRATIONS, COGLUEING AND DUALITY

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This paper can be conceived of as an application of abstract homotopy theory to topological homotopy theory. Thus our starting point is to ask for conditions under which a map in Toppair (the category of topological pairs) is a fibration or cofibration in that category. Our motivation for this is to discover precise ingredients which will allow formal proofs of the Glueing and Coglueing theorems (3.2 respectively 3.6, cf. also [4: 7.5.7], [6], [9] and [15]) to carry over to the category Toppair and hence a fortiori to the subcategory Top$_*$ of based topological spaces. We have a special interest in this latter category, and we wish here to register a protest against the philosophy which fails to distinguish between free and based results. This manifests itself, for example, in the fact that the free results of [6] and [4: 7.5.7] have been quoted as if they were based; similar comments apply to the theory of homotopy pullbacks and pushouts. Nor, in our opinion, can the situation be satisfactorily resolved merely by requiring nondegenerate base points throughout as (a) many important cases are not covered, for example universal and functional fibrations are often excluded (functional fibrations are those whose total spaces are spaces of maps cf., for example, [1], [2] and [3]), (b) it is not immediately obvious, that the spaces considered do have non-degenerate points, (this is emphasised in the generalisation to Toppair, see the remark after 1.2), and (c) in the case of the Top$_*$ coglueing theorem they are not needed (cf. 2.14).

Further motivation is contained in the results and techniques themselves. We show, for example, that if $p:(E, E_0)\to(B, B_0)$ is a Toppair map in which the Top maps $p:E\to B$ and its restriction $p_0:E_0\to B_0$ are Hurewicz fibrations and (E, E_0) and (B, B_0) are closed with the HEP, then p is a Toppair fibration. (In particular, p is a fibration in the whole of Toppair and not just in the subcategory consisting of all pairs closed with the HEP.) Two of the results we particularly like are those (1.11 and 1.13) which free Top$_*$ fibrations of any condition on the total space, thus allowing for the cases mentioned above. As a spin off we obtain some connections between Top$_*$ and Regular fibrations, and the result which must surely be known, that every fibration over a LEC Space (in particular over a CW complex cf. [8]) is Regular. These latter considerations fit well into the level of generality of Toppair, and our working at this

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level answers the natural question as to whether our techniques carry the level of generality of [11]. (cf. 3.31).

After a preliminary section on Toppair exponential laws we give, in sections one and two, our main conclusions relating conditions under which a map is a fibration, respectively a cofibration in Toppair. Section two also deals with the question of existence of pushouts in Toppair—a non-trivial discussion as Example 2.1 shows. Finally, section three states, and makes reference to, the proofs of the theorems which precipitated our discussion. In this last section we also take the opportunity (3.3II) to amend an inadequate proof in [11].

§0. **Toppair exponential laws.** Let $f:(X, X_0)\to(Y, Y_0)$ be a map in Toppair. We denote by f_0 the restriction $f\mid X_0 : X_0\to Y_0$. The categories Top and Top_* can be regarded as subcategories of Toppair in the obvious way.

Let (X, X_0) and (Y, Y_0) be objects in Toppair, the Toppair function space is the pair

$$(X^Y, (X, X_0)^{(Y,Y_0)})$$

where X^Y is the space of continuous functions in Top, with the compact open topology, and where $(X, X_0)^{(Y,Y_0)}$ is the subspace of X^Y consisting of those maps which map Y_0 into X_0. If $(Y, Y_0)=(I, I)$, I being the unit interval, we see,

$$(X^Y, (X, X_0)^{(Y,Y_0)})=(X^I, (X, X_0)^{(I,I)})\cong(X^I, X_0^I).$$

We abuse notation and write this as $(X, X_0)^I$.

There is a Toppair exponential law. Let Y be locally compact then there is a bijection

$$\text{Toppair}((Z, Z_0)\times(Y, Y_0), (X, X_0))\cong\text{Toppair}((Z, Z_0), (X^Y, (X, X_0)^{(Y,Y_0)})).$$

Here $(Z, Z_0)\times(Y, Y_0)=(Z\times Y, Z_0\times Y_0)$. In case $(Y, Y_0)=(I, I)$ we write $(Z, Z_0)\times(Y, Y_0)$ as $(Z, Z_0)\times I$ or as $(Z\times I, Z_0\times I)$. In this case the exponential law reduces to

$$\text{Toppair}((Z, Z_0)\times I, (X, X_0))\cong\text{Toppair}((Z, Z_0), (X, X_0)^I).$$

We note that our exponential law does not generalize the usual one in Top_* associated with the smash product. There is, however, a Toppair exponential law which is motivated by the Top_* situation, namely that which is associated with the "tensor product" of pairs

$$(X, X_0)\otimes(Y, Y_0):=(X\times Y, X\times Y_0\cup X_0\times Y).$$

The corresponding function space pair is

$$(Y, Y_0)\pitchfork(Z, Z_0):=((Z, Z_0)^{(Y,Y_0)}, Z_0^Y),$$

and the Top_* situation is recovered by considering X_0, Y_0, Z_0 as singletons, and identifying the subspace of the tensored product to a point.

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This second rule has been generalized to the category of M-ads in [4]. Furthermore *our* exponential law also generalizes to this category. The first rule in Toppair is however sufficient for our purposes. In fact our proofs in Toppair will a fortiori give us proofs in Top_*. These proofs however, because the Toppair exponential law does not generalize the Top_* one, are rather different from those which would attempt formal proofs in Top_*. Our considerations allow a reasonable adjointness of homotopies via the cylinder object (I, I) in Toppair (cf. [13: 1.5, 1.6]), and enable us to set up the machinery which mimics the abstract homotopy situation making available formal proofs of 3.7 and 3.6.

§1. **Fibrations and the CHP**

Definition 1.1. Let $p:(E, E_0)\to(B, B_0)$ be a map then p is said to have the covering homotopy property (CHP) for (X, X_0) if given any commutative diagram

$$\begin{array}{ccc}(X, X_0) & \xrightarrow{\ f\ } & (E, E_0)\\ {\scriptstyle \varepsilon_0}\big\downarrow & & \big\downarrow{\scriptstyle p}\\ (X, X_0)\times I & \xrightarrow[F]{} & (B, B_0)\end{array}$$

where $\varepsilon_0(x)=(x, 0)$, then there exists a map $\tilde{F}:(X, X_0)\times I\to(E, E_0)$ with $\tilde{F}\varepsilon_0=f$ and $p\tilde{F}=F$. If p has the CHP for all (X, X_0) then p is said to be a Toppair fibration, or simply a fibration where no confusion arises.

We list some elementary properties; Isomorphisms and projections are fibrations, the composition of fibrations is a fibration, the pullback (see below) of a fibration is a fibration, finally if $p: E\to B$ is a fibration in Top and $E_0=p^{-1}(B_0)$, then $p:(E, E_0)\to(B, B_0)$ is always a fibration.

Let $f:(X, X_0)\to(B, B_0)$, $g:(E, E_0)\to(B, B_0)$ then the pullback of f and g exists in Toppair, and can be written $(X\sqcap E, X_0\sqcap E_0)$ where $\sqcap$ denotes the standard topological pullback. The two possible topologies on $X_0\sqcap E_0$ coincide, so there is no confusion.

The usual Top proof of the factorisation lemma easily adapts to prove

Proposition 1.2. *Any map $f:(E, E_0)\to(B, B_0)$ can be factored as $(E, E_0)\xrightarrow{u}(E\sqcap B^I, E_0\sqcap B_0^I)\xrightarrow{\theta}(B, B_0)$ where u is a homotopy equivalence and θ is a fibration.*

Here $E\sqcap B^I$ is the mapping track in Top consisting of $\{(e, \lambda)\in E\times B^I \mid \lambda(0)=f(e)\}$. We call the pair $(E\sqcap B^I, E_0\sqcap B_0^I)$ the mapping track of f. For *arbitrary* f it is not clear that this pair has the HEP even if (E, E_0) and (B, B_0) do. This raises one of the difficulties encountered in a formal proof of 3.2, cf. 1.7 and the remark preceding 1.8.

Given a map $p:(E, E_0) \to (B, B_0)$ there is an induced map $q = q_p : (E, E_0)^I \to (E \sqcap B^I, E_0 \sqcap B_0^I)$ defined by $q(\lambda) = (\lambda(0), p\lambda)$. The following characterization essentially generalizes [14].

Proposition 1.3. *Let $p:(E, E_0) \to (B, B_0)$ be a map, then the following are equivalent*

(i) *p is a fibration,*
(ii) *the map q_p has a section (i.e. a right inverse),*
(iii) *any map $h:(X, X_0) \to (E \sqcap B^I, E_0 \sqcap B_0^I)$ lifts over q_p,*
(iv) *p has the CHP with respect to the "test pair" $(E \sqcap B^I, E_0 \sqcap B_0^I)$.*

The proof is an easy exercise in the exponential law and is left to the reader.

Corollary 1.4. *Any Top_* fibration is a Toppair fibration and if $p: E \to B$ is a Top fibration then $p:(E, \varnothing) \to (B, \varnothing)$ is a Toppair fibration.*

We merely observe that if E_0 and B_0 are singletons (empty) then so is $E_0 \sqcap B_0^I$ a singleton (empty).

Corollary 1.5. *If $p:(E, E_0) \to (B, B_0)$ is a Toppair fibration then $p: E \to B$ and p_0, are Top fibrations (see §0 for the notation p_0).*

Proof. If $s:(E \sqcap B^I, E_0 \sqcap B_0^I) \to (E^I, E_0^I)$ is a section to q_p then s and $s_0 : E_0 \sqcap B_0^I \to E_0^I$ are sections to the Top maps "q_p" and q_{p_0}.

In the opposite direction we have

Proposition 1.6 (cf. also [11: p. 1153]). *If $p:(E, E_0) \to (B, B_0)$ is such that $p: E \to B$, $p_0 : E_0 \to B_0$ are Top fibrations then p has the Toppair CHP with respect to all pairs (X, X_0) that are closed with the HEP.*

The proof involves lifting F_0 of 1.1 in Top to $\tilde{F}_0$ at f_0 by p_0, then using Strøm's relative lifting Theorem [18: Th. 4] to lift F extending $f \times 0 \cup \tilde{F}_0 : X \times 0 \cup X_0 \times I \to E$.

In view of 1.6 and characterization 1.3(iv) we investigate conditions under which the test pair $(E \sqcap B^I, E_0 \sqcap B_0^I)$ is closed with the HEP (in Top). We prove more generally

Proposition 1.7. *Let $p:(X, X_0) \to (B, B_0)$, $q:(Y, Y_0) \to (B, B_0)$ be maps in Toppair such that the maps p, p_0, q, q_0 are Top fibrations. If the pairs (X, X_0), (Y, Y_0) and (B, B_0) are closed and have the HEP then the pair $(X \sqcap Y, X_0 \sqcap Y_0)$ is closed with the HEP.*

We would like to acknowledge the comments of the referee of an earlier manuscript which enabled us to dispense with a condition on (B, B_0).

Proof. The proof depends on two results of Strøm.

Firstly [19] that if $p: E \to B$ is a fibration and $B_0 \hookrightarrow B$ a closed cofibration then so also is $p^{-1}(B_0) \hookrightarrow E$; secondly [20] if in the composite $A \xrightarrow{i} B \xrightarrow{j} C$, j and ji are cofibrations then so also is i. Writing $\tilde{X}_0 = p^{-1}(B_0)$, $\tilde{Y}_0 = q^{-1}(B_0)$ we deduce from the above that the following are closed cofibrations $X_0 \hookrightarrow \tilde{X}_0 \hookrightarrow X$, and $Y_0 \hookrightarrow \tilde{Y}_0 \hookrightarrow Y$. Furthermore from the pull-back diagrams,

$$\begin{array}{ccc} X_0 \sqcap Y_0 & \to & \tilde{X}_0 \sqcap Y_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \tilde{X}_0 \end{array} \qquad \begin{array}{ccc} \tilde{X}_0 \sqcap Y_0 & \to & \tilde{X}_0 \sqcap \tilde{Y}_0 \\ \downarrow & & \downarrow \\ \tilde{Y}_0 & \longrightarrow & \tilde{Y}_0 \end{array} \qquad \begin{array}{ccc} \tilde{X}_0 \sqcap \tilde{Y}_0 & \to & X \sqcap Y \\ \downarrow & & \downarrow \\ B_0 & \longrightarrow & B \end{array}$$

in which one easily verifies that the right hand vertical maps are fibrations, we deduce that each of the inclusions

$$X_0 \sqcap Y_0 \hookrightarrow \tilde{X}_0 \sqcap Y_0 \hookrightarrow \tilde{X}_0 \sqcap \tilde{Y}_0 \hookrightarrow X \sqcap Y$$

is a closed cofibration. The result follows.

Since I is compact and (B, B_0) is closed with the HEP we deduce from [20] that (B^I, B_0^I) is closed with the HEP and conclude from 1.3(iv):

Corollary 1.8. *Let $p:(E, E_0) \to (B, B_0)$ be a Toppair map with p and p_0 Top fibrations: suppose further that (E, E_0) and (B, B_0) are closed with the HEP then p is a Toppair fibration.*

Corollary 1.8 gives very simple conditions which will allow a formal proof of 3.2 in Toppair; in Top_* we can do even better, as we are able to free ourselves of conditions on the total space. To do this we need to consider the relation of Top_* and regular fibrations [14]. First some notation.

For each topological space X, let $\delta: X \to X^I$ denote the map which takes x into the constant path at x. If $p: E \to B$ is a map we can thus regard E as a subspace of E^I. It can further be regarded as a subspace of $E \sqcap B^I$ via the double pullback diagram

$$\begin{array}{ccccc} E & \hookrightarrow & E \sqcap B^I & \to & E \\ \downarrow & & \downarrow & & \downarrow p \\ B & \overset{\delta}{\hookrightarrow} & B^I & \longrightarrow & B \end{array}$$

Diagram 1.9

Now this situation gives us a Toppair map

$$q_p:(E^I, E) \to (E \sqcap B^I, E)$$

defined as in 1.3.

The reader is left to check the following definition against Hurewicz's original one [14].

Definition 1.10. The map $p: E \to B$ is a regular (Top) fibration if and only if there is a *Toppair* section s to the above map q_p.

A section $s:(E\sqcap B^I, E)\to(E^I, E)$ restricts to a section $(E\sqcap B^I, e_0)\to(E^I, e_0)$ for each choice of $e_0\in E$, using 1.3(ii) we have proved the first part of

PROPOSITION 1.11. *Any regular fibration $p:E\to B$ is a Top_* fibration $p:(E, e_0)\to(B, p(e_0))$ for all choices of $e_0\in E$. In particular this is the case if B is a CW complex, or if B is LEC* [8].

Tulley [21] has shown that if B is a zero set in B^I for some continuous function $\phi: B^I\to I$ then any fibration over B is regular. Strøm [20: p. 441], in a different context, notes the existence of such a ϕ for CW complexes, while the existence of a ϕ for LEC spaces is a simple consequence of the definition given by Dyer and Eilenberg [8], (who also show that a CW complex is LEC).

REMARK 1.12. I. Although we believe the result is known that, any fibration over a CW complex is regular, we know of no statement or proof of this fact in the literature.

II. Since B is always a deformation retract of B^I, the additional information of B being a zero set in B^I ensures that δ in 1.9 is a cofibration, we can thus deduce Tulley's theorem using our techniques.

III. The freeing of the results in Propositions 1.11 above and 1.13 below, from conditions on the total space is particularly useful when studying universal fibrations or functional fibrations, i.e. fibrations whose total spaces are spaces of maps (cf. for example [1], [2] and [3]), where non-degenerate base points in the total space are not generally available.

PROPOSITION 1.13. *Let $p:E\to B$ be a (Top) fibration and b_0 a closed non-degenerate point of B then $p:(E, e_0)\to(B, b_0)$ is a Top_* fibration for all choices of $e_0\in F$, the fibre of p over b_0.*

Proof. If we replace the left hand B in 1.9 by singleton $*$ then the left hand arrow of the corresponding double pullback diagram becomes $F\to *$. We conclude that the pair $(E\sqcap B^I, F)$ is closed with the HEP and hence there exists a section s to $(E^I, F)\to(E\sqcap B^I, F)$. Selecting the base point and restricting s gives the result.

§2. **Pushouts and cofiberings in Toppair.** The existence of pushouts in Toppair is not quite as simple as the pullback situation. One might expect the pushout of

$$(X, X_0)\xleftarrow{f}(A, A_0)\xrightarrow{g}(Y, Y_0)$$

to be $(X\sqcup Y, X_0\sqcup Y_0)$ where $\sqcup$ denotes topological pushout.

However this is not necessarily the case.

EXAMPLE 2.1. Let $(X, X_0) = (I, I)$, $(A, A_0) = (I, 0)$, $(Y, Y_0) = (*, *)$, the singleton pair, $f: (A, A_0) \to (X, X_0)$ the inclusion, $g: (A, A_0) \to (Y, Y_0)$ the unique map then $(X \sqcup Y, X_0 \sqcup Y_0) = (*, I)$ which is clearly not a pair.

We will see later (2.6.I) that even when $X_0 \sqcup Y_0$ is a subset of $X \sqcup Y$ the subspace and the pushout topologies need not coincide.

Pushouts, however, do exist in the category Toppair as we now demonstrate. Consider the unique *function* $f_0 \sqcup g_0 : X_0 \sqcup Y_0 \to X \sqcup Y$ induced by the universal properties. As example 2.1 shows $f_0 \sqcup g_0$ need not be injective. Define an equivalence relation $\sim$ on $X_0 \sqcup Y_0$ by $w_1 \sim w_2$ if and only if $(f_0 \sqcup g_0)(w_1) = (f_0 \sqcup g_0)(w_2)$. The function $f_0 \sqcup g_0$ induces an inclusion of the quotient set $(X_0 \sqcup Y_0)/\sim$ into $X \sqcup Y$.

PROPOSITION 2.2. *The pair* $(X \sqcup Y, (X_0 \sqcup Y_0)/\sim)$ *is a pushout of* $(X, X_0) \xleftarrow{f} (A, A_0) \xrightarrow{g} (Y, Y_0)$ *if we give* $(X_0 \sqcup Y_0)/\sim$ *the subspace topology with respect to* $X \sqcup Y$.

Proof. Maps $h: (X, X_0) \to (Z, Z_0)$, $k: (Y, Y_0) \to (Z, Z_0)$ compatible with f and g define a unique Top map $h \sqcup k: X \sqcup Y \to Z$. As a *function* the composite $(h \sqcup k) \circ (f_0 \sqcup g_0)$ factors through $(X_0 \sqcup Y_0)/\sim$, by the universal property of quotient sets. The above composite by assumption has its image in Z_0 so the factored "restriction" of $h \sqcup k$ is continuous by the universal property of the initial topology on Z_0.

Under certain circumstances the equivalence relation of Proposition 2.2 turns out to be trivial. We consider only cases in which either f or g is an inclusion. This condition is not sufficient as 2.1 again demonstrates. Without loss of generality we assume (A, A_0) is a subpair of (X, X_0). We can consider the *sets* $X_0 \sqcup Y_0$, $X \sqcup Y$ as $Y_0 \dot\cup (X_0 \backslash A_0)$ respectively $Y \dot\cup (X \backslash A)$. (Here $\dot\cup$ denotes the disjoint union). Clearly we have:

PROPOSITION 2.3. *In the above situation,* $X_0 \sqcup Y_0$ *is a subset of* $X \sqcup Y$ *if* $(X_0 \backslash A_0)$ *is a subset of* $(X \backslash A)$.

The condition of 2.3 is evidently not satisfied by 2.1.

COROLLARY 2.4. *Let* $f: (X, X_0) \to (Y, Y_0)$ *be a map then the Toppair suspension on* (X, X_0), *cone on* (X, X_0) *and mapping cylinder on* f *are respectively the pairs* (SX, SX_0), (CX, CX_0) *and* (M_f, M_{f_0}).

The suspension and cone can be taken in either the reduced or unreduced sense. M_g for example is the mapping cylinder, in Top, of a map g, and in each case the second space of the pair is given the subspace topology with respect to the first.

Proof. We prove only the mapping cylinder result. This follows from 2.3 since as sets $M_{f_0} = Y_0 \times 0 \cup X_0 \times (0, 1]$ while $M_f = Y \times 0 \cup X \times (0, 1]$.

Proposition 2.5. *If f and g in 2.2 are both inclusions then the equivalence relation is always trivial.*

This proposition is easily proved using the explicit description of the counter images of points under the identifications $X_0 \uplus Y_0 \to X_0 \sqcup Y_0$, $X \uplus Y \to X \sqcup Y$, together with the evident injection $X_0 \uplus Y_0 \rightarrowtail X \uplus Y$.

Remarks 2.6. I. We wish to emphasize that the subspace topology on $X_0 \sqcup Y_0$ need not coincide with the identification topology even when the equivalence relation of 2.2 is trivial. To see this consider the mapping cylinder of the inclusion $(X, A) \to (X, X)$ where $A = (0, 1]$, $X = [0, 1]$. It is shown in [7: 1.21] that the subspace topology (with respect to $X \times I$) on $X \times 0 \cup A \times I$ is not the same as the identification topology.

II. The Top_* and Toppair mapping cylinder of a map $f: (X, x_0) \to (Y, y_0)$ are not identical. This is clear since the mapping cylinder of $f_0 : \{x_0\} \to \{y_0\}$ is not a point.

We turn now to a discussion of Toppair cofibrations.

Definition 2.7. *Let $i: (A, A_0) \hookrightarrow (X, X_0)$* be an inclusion map in Toppair, then i is said to be a cofibration if the following diagram

$$\begin{array}{ccccc} a & \quad & (A, A_0) & \xrightarrow{\;\; i \;\;} & (X, X_0) \\ \big\downarrow & & \Big\downarrow {\scriptstyle \varepsilon_0} & & \Big\downarrow {\scriptstyle \varepsilon_0} \\ (a, 0) & & (A, A_0) \times I & \xrightarrow{\; i \times 1 \;} & (X, X_0) \times I \end{array}$$

is a weak pushout. That is given maps $f: (X, X_0) \to (Z, Z_0)$, $F: (A, A_0) \times I \to (Z, Z_0)$ *with $F\varepsilon_0 = fi$, then there exists a map $\tilde{F}: (X, X_0) \times I \to (Z, Z_0)$ with $\tilde{F}\varepsilon_0 = f$ and $\tilde{F}(i \times 1) = F$.*

Isomorphisms are cofibrations, the composition of cofibrations is a cofibration, the pushout of a cofibration is a cofibration, the inclusion of a pair into the disjoint union of itself with another pair is a cofibration.

The following propositions are formal generalizations of Topological ones.

Proposition 2.8. *Every map $f: (X, X_0) \to (Y, Y_0)$ factors as $(X, X_0) \overset{i}{\hookrightarrow} (M_f, M_{f_0}) \xrightarrow{h} (Y, Y_0)$ where i is a cofibration and h a homotopy equivalence.*

Proposition 2.9. *Let $i: (A, A_0) \hookrightarrow (X, X_0)$ be an inclusion with X locally compact. Then i is a cofibration if and only if*

$$i^* : (Z^X, (Z, Z_0)^{(X, X_0)}) \to (Z^A, (Z, Z_0)^{(A, A_0)})$$

is a fibration for all pairs (Z, Z_0).

The following characterization dualizes 1.3.

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PROPOSITION 2.10. *Let* $i:(A, A_0)\hookrightarrow(X, X_0)$ *be an inclusion the following are equivalent:*

(i) *i is a cofibration*

(ii) *The injection* $q_i:(M_i, M_{i_0})\to(X\times I, X_0\times I)$ *has a left inverse.*

(iii) *Any map* $h:(M_i, M_{i_0})\to(Z, Z_0)$ *extends over* $(X\times I, X_0\times I)$.

COROLLARY 2.11. *If* $i:(A, A_0)\to(X, X_0)$ *is a Toppair cofibration then* $i:A\hookrightarrow X$ *and* $i_0:A_0\hookrightarrow X_0$ *are cofibrations in Top.*

Proof. A retraction $r:(X\times I, X_0\times I)\to(M_i, M_{i_0})$ defines retractions $r:X\times I\to M_i$, $r_0:X_0\times I\to M_{i_0}$ and the proof follows from the characterization corresponding to 2.10 in Top (cf. [4]).

REMARKS 2.12. I. According to [19: p. 132] the existence of retractions r and r_0 as above ensures that the subspace topologies and pushout topologies on M_i and M_{i_0} coincide.

II. The existence of the retractions r and r_0 justify our considering only inclusion maps as cofibrations (cf. [18]).

III. The converse of 2.11 is false, for consider $i:(I, 0)\hookrightarrow(I, \dot{I})$. The identification map $I\to S'$ is homotopic as a Top$_*$ map $(I, 0)\to(S', 0)$ to the constant map, however as a Toppair map $(I, \dot{I})\to(S', 0)$ this is not the case.

The reader will note that in the example 2.12.III, in the usual terminology, $A\cap X\neq A$. Indeed it is an open question whether cofibrations of pairs $i:(A, A_0)\hookrightarrow(X, X_0)$ exist with $A\cap X_0\neq A_0$. We will consider as candidates for cofibrations only inclusions for which $A\cap X_0=A_0$. We justify this intuitively as follows. Consider all maps $(X, X_0)\to(Z, Z_0)$ for fixed X, Z, Z_0. Larger subspaces X_0 will, in general, result in a smaller total number of maps. Similarly for fixed A, A_0, X, Z, Z_0, larger X will allow fewer homotopies $(A, A_0)\to(Z, Z_0)$ to extend to homotopies $(X, X_0)\to(Z, Z_0)$.

PROPOSITION 2.13. *Let* i: $A\hookrightarrow X$ *be a Top cofibration, if* A_0 *is a subspace of* A *then* $(A, A_0)\hookrightarrow(X, A_0)$ *is a Toppair cofibration.*

Proof. A retraction $X\times I\hookrightarrow X\times 0\cup A\times I$ restricts to a "retraction" $A_0\times I\to A_0\times 0\cup A_0\times I$.

COROLLARY 2.14. *If* $i:A\hookrightarrow X$ *is a Top cofibration then* $(A, \varnothing)\hookrightarrow(X, \varnothing)$ *and, for each* $a_0\in A$, $(A, a_0)\hookrightarrow(X, a_0)$ *are Toppair cofibrations.*

Thus, in particular, we note that nondegenerate base points are not required for the pointed glueing theorem cf. 3.6.

PROPOSITION 2.15. *Let* A_0 *be a subspace of* X_0 *and of* A, *with* $X_0\cap A=A_0$ *and the inclusion* $A_0\hookrightarrow X_0$ *a Top cofibration then* $(A, A_0)\hookrightarrow(X_0\sqcup A, X_0)$ *is a Toppair cofibration.*

Proof. It is easy to see that $(A_0, A_0) \hookrightarrow (X_0, X_0)$ is a cofibration, and the diagram

$$\begin{array}{ccc} (A_0, A_0) & \rightarrow & (A, A_0) \\ \downarrow & & \downarrow \\ (X_0, X_0) & \rightarrow & (X_0 \sqcup A, X_0) \end{array}$$

is a pushout. The result follows.

COROLLARY 2.16. *Let $(Y; Y_1, Y_2)$ be a Mayer Vietoris Triad (cf. for example* [4; p. 240]), *if $Y_0 = Y_1 \cap Y_2$, then $(Y_1, Y_0) \hookrightarrow (Y, Y_2)$ and $(Y, Y_0) \hookrightarrow (Y, Y_1)$, and their symmetries, are Toppair cofibrations.*

Proof. For the first part take $A_0 = Y_0$, $X_0 = Y_2$ and $A = Y_1$, for the second take $A_0 = Y_0$ and $X_0 = Y_1$ and $A = Y$ in 2.15.

PROPOSITION 2.17. *Let $i : (A, A_0) \hookrightarrow (X, X_0)$ be an inclusion such that $(A \cap X_0) = A_0$, $X_0 \sqcup A = X_0 \cup A$, and $A_0 \hookrightarrow X_0$ and $X_0 \cup A \hookrightarrow X$ are Top cofibrations then i is a Toppair cofibration.*

Proof. The Map i can be factored as $(A, A_0) \hookrightarrow (X_0 \cup A, X_0) \hookrightarrow (X, X_0)$, the first map is a cofibration by 2.15 while the second is by 2.13.

COROLLARY 2.18. *If $A \hookrightarrow X$ and $B \hookrightarrow X$ are closed cofibrations and if $A \cap B \hookrightarrow X$ is a cofibration, then $(A, A \cap B) \hookrightarrow (X, B)$ is a cofibration in Toppair.*

Proof. By Lillig [16; cor. 2], the union $A \cup B \hookrightarrow X$ is a cofibration, by [20; Lemma 5] $A \cap B \hookrightarrow B$ is a cofibration, the result follows from 2.17.

The next corollary generalizes [17; p. 380].

COROLLARY 2.19. *If $B \hookrightarrow Y$ is a Top cofibration and $b \in B$ is non-degenerate in B or in Y then (b, b) is non-degenerate in (Y, B).*

Proof. Putting $A_0 = A = b$, $X_0 = B$, $X = Y$ we see that $A_0 \hookrightarrow X_0$ and $X_0 \cup A \hookrightarrow X$ are cofibrations (use [20; Lemma 5] if b is non-degenerate in Y).

§3. **Coglueing and Glueing Theorems.** In this section we give versions of the Coglueing and Glueing Theorems in the category Toppair and hence a fortiori in the category Top_*. We remark that despite quotations to the contrary the results of [6] and [4: 7.5.7] are for the free cases only. We know of no other publication with the based results.

Consider the commutative diagram in Toppair

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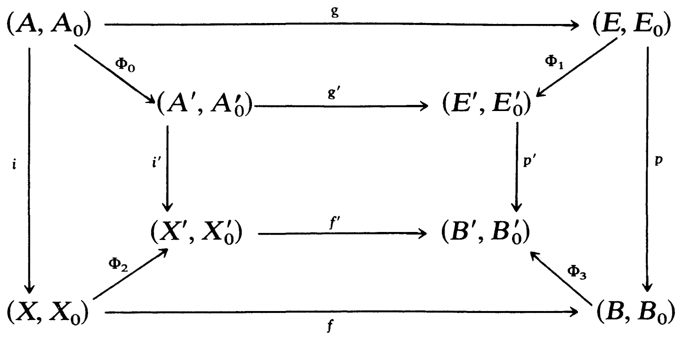

Diagram 3.1

THEOREM 3.2 (Toppair, Top_*, Coglueing Theorem). *If in* 3.1 *the inner and outer squares are pullbacks, Φ_1, Φ_2 and Φ_3 are homotopy equivalences and p and p' are fibrations in Toppair, then Φ_0 is also a homotopy equivalence. In particular the result holds if Φ_1, Φ_2, Φ_3 are homotopy equivalences and either*

(a) *if (E, E_0), (E', E_0'), (B, B_0), (B', B_0') are closed with HEP and p, p_0, p', p_0' are Top fibrations or*

(b) *if E_0, E_0', B_0, B_0' are singletons, B_0, B_0' closed non-degenerate base points and p and p' Top fibrations.*

We will not labour over a proof of 3.2. The conditions in (a) and (b) ensure that p are p' are Toppair fibrations. If this paper is read in conjunction with [11] it will be found that all the formal categorical homotopy theory necessary to mimic [4] is available. Alternatively one could verify that the Kan conditions of [15] are fulfilled, this note together with [11] and [15] would then also provide a proof of 3.2 (see also 3.3.II below in connection with the Kan conditions).

REMARKS 3.3. I. Both methods of proof mentioned above include a replacement argument. The precise condition needed for the replacement argument for f, for example, is that p has the Toppair CHP with respect to $(E \sqcap B^I, E_0 \sqcap B_0^I)$ and $(E \sqcap B^I, E_0 \sqcap B_0^I) \times I$. (Compare the proof of 2.4.1 of [12]). This fact excludes the generality of [11].

II. We take the opportunity to point out and correct an insufficiency in the proof of Lemma A.1 in [11] (an essential step in the proof of 1.1 of that paper and hence in 3.2 of ours). The last step to deduce the existence of a map M extending L would require a relative lifting theorem in the category Toppair. This is clearly the case for the examples of section two of that paper but may not be true in general. However the statement of A.1 in [11] is correct. The proof may be completed by simply lifting $K \simeq 0_f$ rel. end maps. Restricting this lift to the three edges $(X, X_0) \times 0 \times I$, $(X, X_0) \times I \times 1$, $(X, X_0) \times 1 \times I$ gives the required lifting.

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Corollary 3.4 (generalizing [6: 14]). *Consider the outer square of* 3.1. *If this square is a pullback, p a Toppair fibration and f a homotopy equivalence then so also is g.*

If $p: E \to B$ is a fixed Top_* fibration then Corollary 3.4 can be used, for example, to choose a classifying map k for p, and a universal fibration $p_\infty : E_\infty \to B_\infty$ such that k is an inclusion (this is required for Top fibrations in [9: p. 45]).

Corollary 3.5. *Consider the trapezoid to the right of* 3.1. *If p and p' are Toppair fibrations and Φ_1 and Φ_3 are homotopy equivalences then the induced map $\Phi_0 : (F, F_0) \to (F', F'_0)$, of the fibres (F, F_0) of $b \in B_0$ and (F', F'_0) of $\Phi_3(b) \in B'_0$, is a homotopy equivalence.*

Theorem 3.6 (Toppair, Top_* Glueing Theorem). *If in* 3.1 *the inner and outer squares are pushouts, Φ_0, Φ_1* and *Φ_2 are homotopy equivalences and i and i' are cofibrations in Toppair, then Φ_3 is also a homotopy equivalence.*

Theorem 3.6 can be proved in a way dual to 3.2 or, in a convenient category, by using the techniques of the main theorem of [10]. These techniques were generalized in [13] and fit the circumstances here.

We leave the reader to give the precise conditions corresponding to section two, under which 3.6 holds. We would however, point out that because of 2.14, it is not necessary, for the Top_* Glueing Theorem, to require non-degenerate base points throughout.

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