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# Schunk classes are nilpotent product closed

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The following result is proved. Let $\underline{X}$ be a Schunk class and $k$ a positive integer. Then the $k$-th nilpotent product of any two groups in $\underline{X}$ is in $\underline{X}$ .

Let $\underline{X}$ be a class of finite soluble groups. Then $\underline{X}$ is a *Schunk class* if the following two conditions hold.

(i) Every epimorphic image of a group in $\underline{X}$ is in $\underline{X}$ .

(ii) Let $G$ be a finite soluble group such that every epimorphic image of $G$ having a faithful primitive permutation representation is in $\underline{X}$ . Then $G$ is in $\underline{X}$ .

In his lectures to the Summer Research Institute of the Australian Mathematical Society at Canberra in January 1969, W. Gaschütz gave a proof of the fact that Schunk classes are direct product closed, using a lemma of Itô about maximal subgroups of a direct product. Here we offer a proof of a more general result:

THEOREM. *Let* $\underline{X}$ *be a Schunk class and* $k$ *a positive integer. Then the* $k$-*th nilpotent product of any two groups in* $\underline{X}$ *is in* $\underline{X}$ .

Proof. Let $A_1$ and $B_1$ be any groups in $\underline{X}$ and $G_1$ their $k$-th nilpotent product. We shall show that every primitive epimorphic image of $G_1$ is an epimorphic image of $A_1$ or of $B_1$ , and that will be more than enough to ensure that $G_1$ is in $\underline{X}$ . Let $G$ be an epimorphic image of $G_1$ which is a primitive permutation group on a set $\Omega$ , and let $A$ and $B$ stand for the images of $A_1$ and $B_1$ under the epimorphism. If $A$ or $B$

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is $1$ , then $G$ is $B$ or $A$ ; so we may assume that $A$ and $B$ are both non-trivial. In this situation it will appear that $G$ is cyclic of prime order so that we have the stronger conclusion that $G = A = B$ .

Firstly suppose that the mutual commutator subgroup $[A , B]$ is not $1$ . Then it follows from the definition of $k$-th nilpotent product (see [1]) that $[A , B]$ contains non-trivial elements of the centre of $G$ . But then a subgroup $Z$ of prime order in the centre of $G$ is transitive on $\Omega$ since it is normal in $G$ ([2], Theorem 8.8), and regular since it is transitive and abelian. This means that $G = Z$ , otherwise $G$ would contain abelian subgroups containing $Z$ strictly, which is evidently not possible.

Finally suppose that $[A , B] = 1$ . Then $A$ and $B$ are normal in $G$ since they generate $G$ ; and they are therefore transitive on $\Omega$ . However, by ([2], Exercise 4.5'), the centralizer of a transitive subgroup is semiregular, which yields in our case that $A$ and $B$ are both regular. Since $G$ is soluble (and this is the one point at which solubility is used) it is monolithic, and the monolith $M$ is regular and abelian. As normal subgroups of $G$ , $A$ and $B$ must contain $M$ , so that $A = B = M$ since these three groups are of the same order. We have proved that $G = M$ ; and, as in the preceding paragraph, we conclude that $G$ is of prime order. This completes the proof.

## References

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[2] Helmut Wielandt, *Finite permutation groups* (Academic Press, New York, 1964).

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