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METRICAL PROPERTIES OF CONTINUED FRACTION AND LÜROTH SERIES EXPANSIONS

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The theory of continued fractions is an extremely useful tool in approximating irrational numbers by rational numbers. Any number $x \in \mathbb{R}\setminus\mathbb{Q}$ can be uniquely represented by a continued fraction of the form

$$x = a_0(x) + \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}} = [a_0(x); a_1(x), a_2(x), \ldots],$$

where $a_n(x) \in \mathbb{Z}, a_n(x) \geq 1$ for $n \geq 1$, is known as the nth partial quotient of x. The classical theory of continued fractions shows that the convergents of the partial quotients of x give exactly the best rational approximation of x (see [11, Theorems 16 and 17]. The nth convergent is given by

$$\frac{p_n}{q_n} := [a_0(x); a_1(x), \ldots, a_n(x)],$$

where $p_n, q_n \in \mathbb{Z}$ are coprime and $q_n \geq 1$. The speed of approximation for any irrational number x is related to the size of the partial quotients by

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} \quad \text{for all } n \in \mathbb{N}.$$

Kleinbock and Wadleigh [12] showed that Dirichlet's theorem is optimal in a precise sense. For any nonincreasing function $\psi : \mathbb{N} \to \mathbb{R}_+$, define the set of ψ-Dirichlet

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improvable numbers by

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{l} \exists N \text{ such that the system } |qx - p| < \psi(t), |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\}.$$

Then, Kleinbock and Wadleigh showed for $x \in [0, 1) \setminus \mathbb{Q}$ that:

(i) $x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \le \psi(q_n)/4$ for all sufficiently large n;
(ii) $x \notin D(\psi)$ if $a_{n+1}(x)a_n(x) > \psi(q_n)$ for infinitely many n.

The metric theory for the set $D(\psi)$ is fully characterised in the papers [2, 8, 9].

My thesis contains results on the metric theory of continued fraction and Lüroth series expansions. The first result gives metrical properties of the product of partial quotients in the plane. Let $\Psi : \mathbb{N} \to \mathbb{R}_+$ be a function. Define the set, for $(t_1, \dots, t_m) \in \mathbb{R}_+^m$,

$$\Lambda(\Psi) := \left\{ (x, y) \in [0, 1]^2 : \max \left\{ \prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y) \right\} \ge \Psi(n) \text{ for all } n \ge 1 \right\}.$$

For the one-dimensional analogue of this set, the Hausdorff dimension (for $m = 2$) was determined in [1] and can also be deduced from [10]. In my thesis, I prove the following two-dimensional result. Throughout, $\dim_H$ is the Hausdorff dimension.

Theorem 1 [5]. *Let Ψ be a positive function. Then,*

$$\dim_H(\Lambda(\Psi)) = \frac{2 + \tau}{1 + \tau} \quad \textit{where } \log \tau = \limsup_{n \to \infty} \frac{\log \log \Psi(n)}{n}.$$

For a nondecreasing function $\varphi : \mathbb{N} \to [2, \infty)$ and $\ell \in \mathbb{N}$, define the set

$$\mathcal{F}_\ell(\varphi) := \left\{ x \in [0, 1) : \begin{array}{l} a_j(x) \cdots a_{j+\ell-1}(x) \ge \varphi(n) \\ a_k(x) \cdots a_{k+\ell-1}(x) \ge \varphi(n) \end{array} \text{ with } 1 \le j < k \le n \text{ for i.m. } n \in \mathbb{N} \right\},$$

where 'i.m.' stands for 'infinitely many'. The set $\mathcal{F}_\ell(\varphi)$ arises in the determination of laws of large numbers for partial quotients. Phillip [13] proved that there is no reasonable function $\sigma : \mathbb{N} \to \mathbb{R}_+$ such that $(a_1(x) + a_2(x) + \cdots + a_n(x))/\sigma(n)$ converges almost everywhere as $n \to \infty$. However, Diamond and Vaaler [6] showed that such a relation holds if we omit the largest partial quotient. Hu *et al.* [7] extended this further by proving the case for the sum of products of two consecutive partial quotients and omitting the largest product. They proved that almost every $x \in [0, 1)$ satisfies

$$\lim_{n \to \infty} \frac{1}{n \log^2 n} \left(\sum_{j=1}^{n} a_j(x)a_{j+1}(x) - \max_{1 \le j \le n} a_j(x)a_{j+1}(x) \right) = \frac{1}{2 \log 2}. \tag{1}$$

This led Tan *et al.* in [14] and Tan and Zhou in [15] to find a zero-one law for the Lebesgue measure of $\mathcal{F}_1(\varphi)$. We extend this work to $\mathcal{F}_3(\varphi)$.

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THEOREM 2 [4]. *Let $\varphi : \mathbb{N} \to [2, \infty)$ be nondecreasing. The Lebesgue measure λ of $\mathcal{F}_3(\varphi)$ is given by*

$$\lambda(\mathcal{F}_3(\varphi)) = \begin{cases} 0 & \text{if } \sum_{n\geq 1} \dfrac{n \log^4 \varphi(n)}{\varphi^2(n)} + \dfrac{\log \varphi(n)}{\varphi(n)} < \infty, \\ 1 & \text{if } \sum_{n\geq 1} \dfrac{n \log^4 \varphi(n)}{\varphi^2(n)} + \dfrac{\log \varphi(n)}{\varphi(n)} = \infty. \end{cases}$$

I further calculate the Hausdorff dimension for $\mathcal{F}_3(\varphi)$. Define $g_3 : \mathbb{R} \to \mathbb{R}$ by

$$g_3(s) := \frac{3s^3 - 5s^2 + 4s - 1}{s^2 - s + 1}.$$

For a function $\varphi : \mathbb{N} \to \mathbb{R}_+$, let B and b be defined by

$$\log B = \liminf_{n\to\infty} \frac{\log \varphi(n)}{n} \quad \text{and} \quad \log b = \liminf_{n\to\infty} \frac{\log \log \varphi(n)}{n}. \tag{2}$$

THEOREM 3 [4]. *Let $\varphi : \mathbb{N} \to [2, \infty)$ be nondecreasing. Then, the Hausdorff dimension of $\mathcal{F}_3(\varphi)$ is given by*

$$\dim_H \mathcal{F}_3(\varphi) = \begin{cases} 1 & \text{if } B = 1, \\ \inf\{s \geq 0 : P(T, -g_3(s) \log B - s \log |T'|) \leq 0\} & \text{if } 1 < B < \infty, \\ 1/(1 + b) & \text{if } B = \infty, \end{cases}$$

where $P(T, \cdot)$ is a pressure function.

The thesis also contains a result on the Lebesgue measure of a set associated with the Lüroth series expansion of a real number. Every $x \in (0, 1]$ has a Lüroth series expansion

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)d_2} + \frac{1}{d_1(d_1 - 1)d_2(d_2 - 1)d_3} + \cdots$$

with a unique sequence $(d_n)_{n\geq 1}$ of integers at least 2. Let $m \in \mathbb{N}$, $\mathbf{t} = (t_0, \ldots, t_{m-1}) \in \mathbb{R}_+^m$ and $\liminf_{n\to\infty} \Psi(n) > 1$. Define the set

$$\mathfrak{E}_{\mathbf{t}}(\Psi) := \Big\{x \in [0, 1) : \prod_{i=0}^{m-1} d_{n+i}^{t_i}(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\Big\},$$

and the numbers

$$t_{\min} := \min\{t_0, t_1, \ldots, t_{m-1}\}, \qquad t_{\max} := \max\{t_0, t_1, \ldots, t_{m-1}\}$$

and

$$\ell(\mathbf{t}) := \#\{j \in \{0, \ldots, m - 1\} : t_j = t_{\max}\}.$$

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THEOREM 4 [3]. *Let $m \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{R}_+^m$ be arbitrary. If* $\liminf_{n\to\infty} \Psi(n) > 1$, *then*

$$\lambda\left(\mathfrak{E}_{\mathbf{t}}(\Psi)\right) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/t_{\max}}} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/t_{\max}}} = \infty. \end{cases} \tag{3}$$

THEOREM 5 [3]. *Let B and b be given by (2). For any $m \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{R}_+^m$,*

$$\dim_H \mathfrak{E}_{\mathbf{t}}(\Psi) = \begin{cases} 1 & \text{if } B = 1, \\ 1/(b+1) & \text{if } B = \infty. \end{cases}$$

THEOREM 6 [3]. *Suppose $m = 2$. Let B and b be given by (2) and assume $1 < B < \infty$. For a given $\mathbf{t} = (t_0, t_1) \in \mathbb{R}_+^2$, define*

$$f_{t_0,t_1}(s) := \frac{s^2}{t_0 t_1 \max\{s/t_1 + (1-s)/t_0, s/t_0\}}.$$

Then, the Hausdorff dimension of $\mathfrak{E}_{\mathbf{t}}(\Psi)$ is the unique solution of

$$\sum_{d=2}^{\infty} \frac{1}{d^s (d-1)^s B^{f_{t_0,t_1}(s)}} = 1.$$

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