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Condition $C'_{\wedge}$ of Operator Spaces

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Abstract. In this paper, we study condition $C'_{\wedge}$, which is a projective tensor product analogue of condition C'. We show that the finite-dimensional OLLP operator spaces have condition $C'_{\wedge}$ and M_n $(n > 2)$ does not have that property.

1 Introduction

The theory of operator spaces is very recent. An operator space is a norm closed subspace of $B(H)$, the operator algebra of all bounded linear operators on the Hilbert space H. The theory of operator spaces is a very important tool in the study of operator algebras. Since the discovery of an abstract characterization of operator spaces by Ruan [19], there have been many more applications of operator spaces to other branches in functional analysis.

In the field of operator algebras, questions revolving around the local property have been a fruitful and important area of investigation. Archbold and Batty [1] introduced conditions C and C' for C*-algebras. Local reflexivity and condition C'' were introduced by Effros and Haagerup [5]. Exactness was defined by Kirchberg [13]. Subsequently, he proved that this condition is equivalent to condition C' [15]. Kirchberg also introduced the definition of LLP and LP for C*-algebra and proved a C*-algebra A has LLP if and only if $A \otimes_{\max} \mathbb{B} = A \otimes_{\min} \mathbb{B}$ [14]. With the development of the theory of operator spaces, the various versions of local properties have a similar impact on the field. Some local properties of operator spaces, such as local reflexivity, exactness, nuclear, and OLLP, were intensively studied in [4, 6–10, 12, 16, 17].

In this paper, we study a new local theory of operator spaces. The property is called condition $C'_{\wedge}$, which is a projective tensor product analogue of condition C'. In Section 1, we recall some notation for operator spaces. In Section 2, we give the definition of condition $C'_{\wedge}$. We show that finite-dimensional OLLP operator spaces have condition $C'_{\wedge}$ and that the operator space M_n $(n > 2)$ does not have condition $C'_{\wedge}$.

We refer the reader to [11, 18] for the basics on operator spaces. Only the concepts and results that are essential in the article will be recalled in this section.

Let $B(H)$ be the space of all bounded linear operator on a Hilbert space H. For each $n \in \mathbb{N}$, there is a canonical norm $\|\cdot\|_n$ on $M_n(B(H))$ given by identifying $M_n(B(H))$ with $B(H^n)$. We call this family of norms an operator space matrix norm on $B(H)$. An operator space is a norm closed subspace of $B(H)$ equipped with the operator space matrix norm inherited from $B(H)$. The morphisms in the category of operator spaces are completely bounded linear maps. Given operator spaces V

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and W, a linear map $\varphi: V \to W$ is completely bounded if the corresponding linear maps $\varphi_n: M_n(V) \to M_n(W)$ defined by φ assigning $[\varphi(x_{i,j})]$ to $\varphi_n([x_{i,j}])$ are uniformly bounded; *i.e.*, $\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$ is finite. A map is completely contractive (resp. completely isometric, or completely quotient) if $\|\varphi\|_{cb} \leq 1$ (resp. for each n in $\mathbb{N}$, φ_n is an isometry or a quotient map). We let $CB(V, W)$ be the space of all completely bounded linear maps from V to W. The dual space V^* of V has an operator space structure induced by natural isomorphisms from $M_n(V^*)$ onto $CB(V, M_n(\mathbb{C}))$. Let us suppose that we are given operator spaces V and W and a linear mapping $\varphi: V \to W$. Then φ is a complete isometry if and only if $\varphi^*: W^* \to V^*$ is an exact complete quotient mapping. If V and W are complete, then $\varphi: V \to W$ is a complete quotient mapping if and only if φ^* is a complete isometry. In the latter case, $\varphi^*(W^*)$ is weak* closed, and φ^* is a weak* homeomorphism in the topologies defined by V and W, respectively.

We use the notation $V \check{\otimes} W$ and $V \widehat{\otimes} W$ for the injective and projective operator space tensor products [2, 3]. The operator space tensor products share many of the properties of Banach space analogues. For example, we have the natural complete isometries $CB(V, W^*) = (V \widehat{\otimes} W)^*$, $CB(W, V^*) = (V \widehat{\otimes} W)^*$, and the completely isometric injection $V^* \check{\otimes} W \hookrightarrow CB(V, W)$.

2 Condition $C'_\wedge$ of Operator Spaces

Let V and W be operator spaces. Given a bounded linear function φ on $V \widehat{\otimes} W$ and $v_0 \in V$, we define the bounded linear function ${}_{v_0}\varphi$ on W by ${}_{v_0}\varphi(w) = \varphi(v_0 \otimes w)$ for $w \in W$. We define a linear map $\Phi^R_{V,W}: V \otimes W^{**} \to (V \widehat{\otimes} W)^{**}$ by

$$\Phi^R_{V,W}(v \otimes w^{**})(\varphi) = \langle {}_v\varphi, w^{**} \rangle_{W^*, W^{**}}$$

for $v \in V$, $w^{**} \in W^{**}$ and $\varphi \in (V \widehat{\otimes} W)^*$. It is clear that $\Phi^R_{V,W}$ is weak* continuous on the second component.

We denote by Φ the natural map form $V \widehat{\otimes} W \to V \check{\otimes} W$, and by Ψ the natural map from $V^* \widehat{\otimes} W^* \to (V \check{\otimes} W)^*$, which are both completely contractive. Then we have the following diagram:

$$\begin{array}{ccccccc}
V \otimes W^{**} & \xrightarrow{\Phi^R_{V,W}} & (V \widehat{\otimes} W)^{**} & \xrightarrow{\Phi^{**}} & (V \check{\otimes} W)^{**} & \xrightarrow{\Psi^*} & (V^* \widehat{\otimes} W^*)^* \\
\| & & & & & & \| \\
CB^\sigma_F(V^*, W^{**}) & & & \longrightarrow & & & CB(V^*, W^{**}),
\end{array}$$

where $CB^\sigma_F(V^*, W^{**})$ denote the space of weak*-continuous completely bounded linear maps from V^* to W^{**} with finite ranks. This diagram is commutative, since

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for any $v_0^* \in V^*$, $v \in V^*$,$w^{**} \in W^{**}$, $w_0^* \in W^*$ and $w_\alpha \to w^{**}$ in the weak*-topology,

$$\begin{aligned}\left\langle \Psi^* \Phi^{**} \Phi^R_{V,W}(v \otimes w^{**}), v_0^* \otimes w_0^* \right\rangle &= \left\langle \Phi^{**} \Phi^R_{V,W}(v \otimes w^{**}), \Psi(v_0^* \otimes w_0^*) \right\rangle \\ &= \left\langle {}_v(\Phi^*(v_0^* \otimes w_0^*)), w^{**} \right\rangle \\ &= \lim_{\alpha\to\infty} \left\langle \Phi^*(v_0^* \otimes w_0^*), v \otimes w_\alpha \right\rangle \\ &= \lim_{\alpha\to\infty} \langle v_0^* \otimes w_0^*, v \otimes w_\alpha \rangle = \left\langle \langle v, v_0^* \rangle w^{**}, w_0^* \right\rangle .\end{aligned}$$

Thus, the linear map $\Phi^R_{V,W}$ is injective, and the tensor product $V \otimes W^{**}$ may be given the operator space structure inherited from the operator space $(V \widehat{\otimes} W)^{**}$.

Definition 2.1 An operator space V satisfies *condition $C'_\wedge$* if the linear map

$$\Phi^R_{V,W}: V \widehat{\otimes} W^{**} \longrightarrow (V \widehat{\otimes} W)^{**}$$

is isometric for every operator space W.

It is equivalent to suppose that $\Phi^R_{V,W}$ is a complete isometry, since, if the linear map $\Phi^R_{V,W}: V \widehat{\otimes} W^{**} \to (V \widehat{\otimes} W)^{**}$ is isometric for every operator space W, it is completely isometric for every operator space W, by the following isometric embedding

$$T_n(V \widehat{\otimes} W^{**}) = V \widehat{\otimes} T_n(W^{**}) \hookrightarrow (V \widehat{\otimes} T_n(W))^{**} = T_n(V \widehat{\otimes} W)^{**}.$$

For operator spaces V and W, we consider the following complete isometry

$$\theta: (V \widehat{\otimes} W)^* = CB(V, W^*) \longrightarrow CB(W^{**}, V^*) = (V \widehat{\otimes} W^{**})^*,$$

where $\theta(\varphi) = \varphi^*$. Then we have $\varphi^*(v \otimes w^{**}) = \langle \varphi, v \otimes w^{**} \rangle = \langle {}_v\varphi, w^{**} \rangle_{W^*, W^{**}}$ for any $v \in V$ and $w^{**} \in W^{**}$. Thus, for any $u \in V \widehat{\otimes} W^{**}$, $\varphi^*(u) = \langle \Phi^R_{V,W}(u), \varphi \rangle$.

Proposition 2.2 *The map $\Phi^R_{V,W}$ is completely contractive.*

Proof Suppose $u \in M_n(V \widehat{\otimes} W^{**})$, for any $\varphi \in M_n((V \widehat{\otimes} W)^*)$, and φ is completely isometric to $\varphi^* \in M_n((V \widehat{\otimes} W^{**})^*)$. Then

$$\begin{aligned}\|(\Phi^R_{V,W})_n(u)\| &= \sup_{\|\varphi\|_{cb} \le 1} \left\| \left\langle \langle (\Phi^R_{V,W})_n(u), \varphi \rangle \right\rangle \right\| = \sup_{\|\varphi\|_{cb} \le 1} \|\varphi_n^*(u)\| \\ &\le \sup_{\|\varphi^*\|_{cb} \le 1} \|\varphi_n^*(u)\| = \|u\|.\end{aligned}$$

Thus, $\Phi^R_{V,W}$ is a completely contractive map. ∎

For giving examples of operator spaces that have condition $C'_\wedge$, we recall an operator space V has OLLP if given any unital C*-algebra A with ideal $J \subseteq A$ and a complete contraction $\varphi: V \to A/J$, for every finite-dimensional subspace L of V, there exists a complete contraction $\widetilde{\varphi}: L \to A$ such that $\pi \circ \widetilde{\varphi} = \varphi|_L$, where $\pi: A \to A/J$ is the canonical quotient mapping.

Proposition 2.3 *If a finite-dimensional operator space has OLLP, then it has condition $C'_\wedge$.*

Proof Suppose L is a finite-dimensional operator space with OLLP; then for any $\varepsilon > 0$, there exists a completely bounded isomorphism $r: L \to Q$, where Q^* is a operator subspace of M_n, such that $\|r\|_{cb}\|r^{-1}\|_{cb} < 1+\varepsilon$ (see [16, theorem 2.5]). We have a commutative diagram

$$\begin{array}{ccc} T_n(W^{**}) & = & T_n(W)^{**} \\ \| & & \| \\ T_n\widehat{\otimes}W^{**} & = & (T_n\widehat{\otimes}W)^{**} \\ \downarrow & & \downarrow \\ Q\widehat{\otimes}W^{**} & \longrightarrow & (Q\widehat{\otimes}W)^{**}. \end{array}$$

The columns are complete quotient mappings, and the top row is a completely isometric isomorphism. Thus, $Q\widehat{\otimes}W^{**} = (Q\widehat{\otimes}W)^{**}$.

We have a diagram

$$\begin{array}{ccc} L\widehat{\otimes}W^{**} & \xrightarrow{\Phi^R_{L,W}} & (L\widehat{\otimes}W)^{**} \\ {\scriptstyle r\otimes id}\downarrow & & \uparrow{\scriptstyle (r^{-1}\otimes \mathrm{id})^{**}} \\ Q\widehat{\otimes}W^{**} & \xrightarrow{\Phi^R_{Q,W}} & (Q\widehat{\otimes}W)^{**}. \end{array}$$

The diagram is commutative, since for any $l \in L, w^{**} \in W^{**}$, $\varphi \in (L\widehat{\otimes}W)^*$ and any $w_\alpha \in W$ such that $w_\alpha \to w^{**}$ in the weak* topology,

$$\begin{aligned} &\Big\langle (r^{-1}\otimes \mathrm{id})^{**}\circ \Phi^R_{Q,W}\circ (r\otimes \mathrm{id})(l\otimes w^{**}), \varphi\Big\rangle \\ &\quad = \langle \Phi^R_{Q,W}(r(l)\otimes w^{**}), (r^{-1}\otimes \mathrm{id})^*(\varphi)\rangle \\ &\quad = \Big\langle {}_{r(l)}((r^{-1}\otimes \mathrm{id}^*\varphi), w^{**}\Big\rangle = \lim_\alpha \Big\langle {}_{r(l)}((r^{-1}\otimes \mathrm{id})^*\varphi), w_\alpha\Big\rangle \\ &\quad = \lim_\alpha \Big\langle (r^{-1}\otimes \mathrm{id})^*\varphi, r(l)\otimes w_\alpha\Big\rangle = \lim_\alpha \langle \varphi, l\otimes w_\alpha\rangle \\ &\quad = \langle {}_l\varphi, w^{**}\rangle = \Big\langle \Phi^R_{L,W}(l\otimes w^{**}), \varphi\Big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|(\Phi^R_{L,W})^{-1}\|_{cb} &= \Big\| \Big((r^{-1}\otimes \mathrm{id})^{**}\circ \Phi^R_{Q,W}\circ (r\otimes \mathrm{id})\Big)^{-1}\Big\|_{cb} \\ &= \Big\| (r^{-1}\otimes \mathrm{id})\circ (\Phi^R_{Q,W})^{-1}\circ (r\otimes \mathrm{id})^{**}\Big\|_{cb} \\ &\le \|r^{-1}\|_{cb}\Big\|(\Phi^R_{Q,W})^{-1}\Big\|_{cb}\|r\|_{cb} < 1+\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $(\Phi^R_{L,W})^{-1}$ is a completely contractive. On the other hand, since $\Phi^R_{L,W}$ is completely contractive, $(\Phi^R_{L,W})^{-1}$ is a norm-increasing linear mapping. Thus, $(\Phi^R_{L,W})^{-1}$ is completely isometric; *i.e.*, L has condition $C'_\wedge$. ∎

For constructing examples of operator spaces that do not have condition $C'_\wedge$, we need a lemma first.

Lemma 2.4 ([11, corollary 14.5.2]) *There is a sequence of finite groups G_k and homomorphisms $\theta_k: F_n \to G_k$ such that* $\ker\theta_1 \supseteq \ker\theta_2 \supseteq \cdots$ *and* $\cap \ker\theta_k = \{e\}$.

We let λ_k be the regular representation of G_k on the Hilbert space $\mathbb{C}^{d(k)} = \ell_2(G_k)$, where $d(k)$ is the cardinality of G_k. We let

$$\pi_k = \lambda_k \circ \theta_k \colon F_n \longrightarrow M_{d(k)}$$

be the corresponding unitary representations of F_n, and we let I stand for the sequence $(d(k))$. These determine a unitary representation

$$\pi \colon F_n \longrightarrow \mathcal{M}_I = \prod_{k\in\mathbb{N}} M_{d(k)} \subseteq B(\oplus\mathbb{C}^{d(k)}),$$

where $\pi(g) = (\pi_k(g))$. We let $\beta\mathbb{N}$ be the spectrum of the C*-algebra $\ell_\infty(\mathbb{N})$, and we fix an element $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$, which corresponds to a free ultrafilter on $\mathbb{N}$. We can regard the elements of $\ell_\infty(\mathbb{N})$ as continuous functions on $\beta\mathbb{N}$, and given a bounded sequence $\alpha = (\alpha_k) \in \ell_\infty(\mathbb{N})$, we define $\lim_{k\to\omega} \alpha_k = \alpha(\omega)$. We let τ_m be the normalized trace on M_m. Owing to the fact that $\tau_{d(k)}$ is a state on $M_{d(k)}$,

$$|\tau_{d(k)}(\alpha_k)| \le \|\alpha_k\|.$$

We define a trace τ_0 on $\mathcal{M}_I$ by letting $\tau_0(\alpha) = \lim_{k\to\omega} \tau_{d(k)}(\alpha_k)$. The set

$$\mathcal{J}_\omega = \{\alpha \in \mathcal{M}_I : \tau_0(\alpha^*\alpha) = 0\}.$$

is a closed two-sided ideal in $\mathcal{M}_I$, and we let π denote the quotient mapping of $\mathcal{M}_\omega = \mathcal{M}_I/\mathcal{J}_\omega$. We can prove that the C*-algebra $\mathcal{M}_\omega$ is a Π_1 factor [11].

Recall an operator space W is $\mathcal{T}$-locally reflexive if for any $L \subseteq T_n$, $n \in \mathbb{N}$, every complete contraction $\varphi\colon L^* \to W^{**}$ is the point weak* limit of a net of linear mappings $\varphi_\alpha\colon L^* \to W$ with $\|\varphi_\alpha\|_{cb} \le 1$. The following two lemmas are only small modifications of [4, theorem 5.2 and corollary 5.4].

Lemma 2.5 *Suppose that W is an operator space. Then the following are equivalent:*

(i) *W is $\mathcal{T}$-locally reflexive.*
(ii) *For any $L \subseteq T_n$, $n \in \mathbb{N}$, we have the isometry $L^*\widehat{\otimes}W^* = (L\check{\otimes}W)^*$.*
(ii)$'$ *For any $L \subseteq T_n$, $n > 2$, we have the isometry $L^*\widehat{\otimes}W^* = (L\check{\otimes}W)^*$.*
(iii) *For any $n \in \mathbb{N}$, we have the isometry $M_n\widehat{\otimes}W^* = (T_n\check{\otimes}W)^*$.*
(iii)$'$ *For any $n > 2$, we have the isometry $M_n\widehat{\otimes}W^* = (T_n\check{\otimes}W)^*$.*

Proof (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) have been proved in [4, theorem 5.2]. We could also prove (ii)$'\Leftrightarrow$(iii)$'$ by mimicking the proof of (ii)$\Leftrightarrow$(iii) in [4, theorem 5.2].

(ii)$\Rightarrow$(ii)$'$: This is obvious.

(ii)$'\Rightarrow$(ii): For any subspace $L \subseteq T_2$, the mapping $L \hookrightarrow T_n$ $(n \ge 3)$ is a completely isometric embedding. So $T_n^* \to L^*$ $(n \ge 3)$ is a complete quotient mapping. We have the commutative diagram

$$\begin{array}{ccc} T_n^*\widehat{\otimes}W^* & = & (T_n\widehat{\otimes}W)^* \\ \downarrow & & \downarrow \\ L^*\widehat{\otimes}W^* & \longrightarrow & (L\widehat{\otimes}W)^*. \end{array}$$

The top row is a completely isometric isomorphism, and the columns are complete quotient mappings. We have $L^*\widehat{\otimes}W^* = (L\check{\otimes}W)^*$. ∎

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Lemma 2.6 *For any $n > 2$, we have that $(M_n \widehat{\otimes} W)^{**} = M_n \widehat{\otimes} W^{**} \Leftrightarrow W^*$ is $\mathcal{T}$-locally reflexive.*

Proof Sufficiency: This is from [4, corollary 5.4].

Necessity: Since $(M_n \widehat{\otimes} W)^{**} = (T_n \check{\otimes} W^*)^*$, we have $M_n \widehat{\otimes} W^{**} = (T_n \check{\otimes} W^*)^*$ for $n > 2$. By the above lemma, we get that W^* is $\mathcal{T}$-locally reflexive. ∎

Theorem 2.7 *For any $n > 2$, M_n does not have condition $C'_\wedge$.*

Proof Assume that M_n $(n > 2)$ has condition $C'_\wedge$, *i.e.*, $M_n \widehat{\otimes} W^{**} = (M_n \widehat{\otimes} W)^{**}$ for any operator space W and $n > 2$. We get that W^* is $\mathcal{T}$-locally reflexive. From Lemma 2.5, for $n \in \mathbb{N}$

$$(T_n \check{\otimes} W^*)^{**} = (T_n^* \widehat{\otimes} W^{**})^* = CB(T_n^*, W^{***}) = T_n \check{\otimes} W^{***}.$$

Let $W = \mathcal{M}_{I*}$; we have $(T_n \check{\otimes} \mathcal{M}_I)^{**} = T_n \check{\otimes} \mathcal{M}_I^{**}$. Since $\mathrm{MAX}\,\ell_1^n$ is the diagonal operator subspace of T_n, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**} & \longrightarrow & (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)^{**} \\ \downarrow & & \downarrow \\ T_n \check{\otimes} \mathcal{M}_I^{**} & = & (T_n \check{\otimes} \mathcal{M}_I)^{**}. \end{array}$$

The columns are completely isometric embeddings, and the bottom row is a completely isometric isomorphism. Thus $\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**} = (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)^{**}$. Let π be the quotient mapping form $\mathcal{M}_I \to \mathcal{M}_\omega$. The weak* closure $\bar{\mathcal{J}}_\omega$ of $\mathcal{J}_\omega$ is a closed two-sided ideal in the von Neumann algebra $\mathcal{M}_I^{**}$, and thus it has the form $\mathcal{M}_I^{**} e$ for some central projection e in $\mathcal{M}_I^{**}$. Since

$$\mathcal{M}_\omega^{**} = (\mathcal{M}_I / \mathcal{J}_\omega)^{**} \cong \mathcal{M}_I^{**} / \bar{\mathcal{J}}_\omega = \mathcal{M}_I^{**}(1 - e),$$

the complete quotient mapping $\pi^{**} : \mathcal{M}_I^{**} \to \mathcal{M}_\omega^{**}$ has a completely contractive lifting given by the canonical inclusion $\mathcal{M}_I^{**}(1 - e) \hookrightarrow \mathcal{M}_I^{**}$. It follows from [11, proposition 8.1.5] that $\mathrm{id} \otimes \pi^{**} : \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**} \to \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega^{**}$ is a complete quotient mapping. Since $\mathrm{MAX}\,\ell_1^n$ is finite-dimensional, we have $\ker(\mathrm{id} \otimes \pi) = \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega$ and $\ker(\mathrm{id} \otimes \pi^{**}) = \mathrm{MAX}\,\ell_1^n \check{\otimes} \bar{\mathcal{J}}_\omega$. Therefore, we obtain a complete isometry

$$(\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**}) / (\mathrm{MAX}\,\ell_1^n \check{\otimes} \bar{\mathcal{J}}_\omega) \cong \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega^{**}.$$

We have the complete isometry $\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**} = (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)^{**}$ and thus the complete isometries

$$\begin{aligned} &\big((\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I) / (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega)\big)^{**} \\ &\quad \cong \big((\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega)^\perp\big)^* \cong (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)^{**} / (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega)^{\perp\perp} \\ &\quad \cong (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)^{**} / \overline{(\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega)} \cong (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**}) / (\mathrm{MAX}\,\ell_1^n \check{\otimes} \bar{\mathcal{J}}_\omega). \end{aligned}$$

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It follows that the columns in the following diagram are completely isometric injections, and the bottom row is a completely isometric isomorphism:

$$\begin{array}{ccc} (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I)/(\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{J}_\omega) & \longrightarrow & \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega \\ \big\downarrow & & \big\downarrow \\ (\mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I^{**})/(\mathrm{MAX}\,\ell_1^n \check{\otimes} \bar{\mathcal{J}}_\omega) & \longrightarrow & \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega^{**}, \end{array}$$

and thus the top row is a complete isometry. So $\mathrm{id} \otimes \pi \colon \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I \to \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega$ is a complete quotient mapping. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_I & \longrightarrow & \mathrm{MAX}\,\ell_1^n \check{\otimes} \mathcal{M}_\omega \\ \big\| & & \big\| \\ CB(\mathrm{MIN}\,\ell_\infty^n, \mathcal{M}_I) & \longrightarrow & CB(\mathrm{MIN}\,\ell_\infty^n, \mathcal{M}_\omega), \end{array}$$

where the columns are complete isometries and the top row is a complete quotient mapping. It follows that the bottom row is a complete quotient mapping, and thus given $\varepsilon > 0$, any $\varphi \in CB(\mathrm{MIN}\,\ell_\infty^n, \mathcal{M}_\omega)$ has a lifting ψ with $\|\psi\|_{cb} < \|\varphi\|_{cb} + \varepsilon$, which is impossible for $n > 2$ see [11, lemma 14.5.3]. ■

References

[1] R. J. Archbold and C. J. Batty, *C*-tensor norms and slice maps.* J. London Math. Soc. **22**(1980), no. 1, 127–138. http://dx.doi.org/10.1112/jlms/s2-22.1.127

[2] D. P. Blecher, *Tensor products of operator spaces. II.* Canad. J. Math. **44**(1992), no. 1, 75–90. http://dx.doi.org/10.4153/CJM-1992-004-5

[3] D. P. Blecher and V. I. Paulsen, *Tensor products of operator spaces.* J. Funct. Anal. **99**(1991), 262–292. http://dx.doi.org/10.1016/0022-1236(91)90042-4

[4] Z. Dong, *The OLLP and $\mathcal{T}$-local reflexivity of operator spaces.* Illinois J. Math. **51**(2006), 1103–1122.

[5] E. G. Effros and U. Haagerup, *Lifting problems and local reflexivity for C*-algebras.* Duke Math. J. **52**(1985), 103–128. http://dx.doi.org/10.1215/S0012-7094-85-05207-X

[6] E. G. Effros, M. Junge, and Z.-J. Ruan, *Integral mapping and the principle of local reflexive for non-commutative L^1 spaces.* Ann. of Math. **151**(2000), no. 1, 59–92. http://dx.doi.org/10.2307/121112

[7] E. G. Effros and Z.-J. Ruan, *On approximation properties for operator spaces.* Internat. J. Math. **1**(1990), 163–181. http://dx.doi.org/10.1142/S0129167X90000113

[8] ———, *Mapping spaces and liftings for operator spaces.* Proc. London Math. Soc. **69**(1994), 171–197. http://dx.doi.org/10.1112/plms/s3-69.1.171

[9] ———, *The Grothedieck-Pietsch and Dvoretzky-Rogers theory for operator spaces.* J. Funct. Anal. **122**(1994), 428–450. http://dx.doi.org/10.1006/jfan.1994.1075

[10] ———, *On the analogues of integral mappings and local reflexivity for operator spaces.* Indiana Univ. Math. J. **46**(1997), 1289–1310. http://dx.doi.org/10.1512/iumj.1997.46.1429

[11] ———, *Operator spaces.* London Mathematical Society Monographs. New Series, 23, The Clarendon Press, Oxford University Press, New York, 2000.

[12] K. H. Han, *An operator space approach to condition C.* J. Math. Anal. Appl. **336**(2007), 569–576. http://dx.doi.org/10.1016/j.jmaa.2007.02.074

[13] E. Kirchberg, *The Fubini theory for exact C*-algebras.* J. Operator Theory **10**(1983), 3–8.

[14] ———, *On non-semisplit extensions, tensor products and exactness of group C*-algebras.* Invent. Math. **112**(1993), 449–489. http://dx.doi.org/10.1007/BF01232444

[15] ———, *On subalgebras of the CAR-algebra.* J. Funct. Anal. **129**(1995), 35–63. http://dx.doi.org/10.1006/jfan.1995.1041

[16] N. Ozawa, *On the lifting property for universal C*-algebras of operator spaces.* J. Operator Theory **46**(2001), no. 3, 579–591.

[17] G. Pisier, *Exact operator spaces. Recent advances in operator algebras (Orléans, 1992).* Astérisque **233**(1995), 159–186.

https://doi.org/10.4153/CMB-2016-064-3 Published online by Cambridge University Press

[18] ———, *Introduction to operator space theory.* London Mathematical Society Lecture Note Series, 294, Cambridge University Press, Cambridge, 2003. http://dx.doi.org/10.1017/CBO9781107360235

[19] Z.-J. Ruan, *Subspaces of C*-algebras.* J. Funct. Anal. **76**(1988), 217–230. http://dx.doi.org/10.1016/0022-1236(88)90057-2

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