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# BASIC OBJECTS FOR AN ALGEBRAIC HOMOTOPY THEORY

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The purposes of this paper are:

(A) To show (§§ 1, 3, 5) that some of the usual notions of homotopy theory (sums, quotients, suspensions, loop functors) exist in the category $k/\mathscr{G}$ of affine $k$-schemes where the affine rings are countably generated.

(B) By example to demonstrate some of the more geometric relations between two objects of $k/\mathscr{G}$ and their quotient or to study the algebraic suspension of one of them. See §§ 2.1, 2.2, 2.3, 3.

(C) To prove (§4) that the algebraic suspension (in $\mathbf{R}/\mathscr{G}$) of the $n$-sphere is homeomorphic to the $n + 1$ sphere for the usual topologies.

(D) To show that the algebraic loop functor is right adjoint to the algebraic suspension functor (§5).

These results can be viewed as a precursor of definitions for an algebraic homotopy theory from a "geometric" point of view (rather than a more algebraic standpoint employing Galois theory [**5**]).

**1. Basic categorical properties of $k$-schemes corresponding to countably generated $k$-algebras.** Let $\mathscr{C}$ be the category of countably generated $k$-algebras, where $k$ is a field and 0 is not an object of $\mathscr{C}$. The comma category $\mathscr{C}/k$ can be formed. This is the category whose objects are $e: A \to k$ (evaluation maps in $C$) and morphisms.

$$(A \xrightarrow{e} k) \xrightarrow{f} (B \xrightarrow{e'} k)$$

are maps $f : A \to B$ such that $e' \circ f = e$.

(A) $\mathscr{C}/k$ has a zero object

$$k \xrightarrow{id} k,$$

as every $k$-algebra map $k \to k$ is the identity.

LEMMA 1. *Every sub-$k$-algebra of a countably generated $k$-algebra is countably generated.*

*Proof.* If $A$ is a countably generated $k$-algebra, it has a countable base and so does any subalgebra. This subalgebra must, then, be countably generated.

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(B) $\mathscr{C}/k$ has equalizers. If

$$(A \xrightarrow{a} k) \underset{g}{\overset{f}{\rightrightarrows}} (B \xrightarrow{b} k)$$

are two maps in $\mathscr{C}/k$, one checks that

$$\text{Equalizer } (f, g) = A' \rightarrow k,$$

where $A' = \{x \in A \mid f(x) = g(x)\}$ and

$$A' \rightarrow k = A' \rightarrow A \xrightarrow{a} k.$$

$A'$ is a countably generated $k$-algebra, by Lemma 1.

(C) $\mathscr{C}/k$ has coequalizers. If

$$(A \xrightarrow{a} k) \underset{g}{\overset{f}{\rightrightarrows}} (B \xrightarrow{b} K)$$

are two maps,

$$\text{Coequalizer } (f, g) = B' \rightarrow k,$$

where $B' = B/(f(x) - g(x))((f(x) - g(x))$ is the ideal in $B$ generated by all $f(x) - g(x)$, $x \in A$) and $B' \rightarrow k$ is induced from

$$B \xrightarrow{b} k$$

as $b \circ f = a = b \circ g$. $B'$ is countably generated as the image of a countably generated $k$-algebra.

(D) $\mathscr{C}/k$ has products. Let

$$A \xrightarrow{a} k, \ B \xrightarrow{b} k$$

be in $\mathscr{C}/k$. One sees that

$$A \times B \xrightarrow{a \times b} k \times k$$

is in $\mathscr{C}$ ($A \times B$ is generated by $A \times 0$ and $0 \times B$). Let

$$A \pi B = \text{Equalizer } (a \times b, s_1 \circ p_1 \circ (a \times b)),$$

where $p_1 : k \times k \rightarrow k$, $s_1 : k \rightarrow k \times k$ are defined by $p_1(x, y) = x$, $s_1(x) = (x, x)$. Then,

$$A \pi B \subset A \times B \xrightarrow{a \times b} k \times k \xrightarrow{p_1} k$$

is the product of $a$ and $b$ in $\mathscr{C}/k$.

(E) $\mathscr{C}/k$ has sums. Let

$$A \xrightarrow{a} k, B \xrightarrow{b} k$$

be in $\mathscr{C}/k$. These induce a map $A \otimes_k B \rightarrow k$ which is the sum of $a$ and $b$.

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Let Spec: $\mathscr{C}/k \to \operatorname{Spec} k/\mathscr{G}$ be the anti-equivalence of categories which associates to each $k$-algebra $A$ and evaluation $e$: $A \to k$ a scheme Spec $A$ and base point $P = \operatorname{Spec} k \subset \operatorname{Spec} A$.

We write $\operatorname{Spec} k/\mathscr{G} = k/\mathscr{G}$. From the above relations we obtain:

PROPOSITION 1. *$k/\mathscr{G}$, the category of "countable" affine $k$-schemes, has*

(a) *a zero object,*

(b) *equalizers and coequalizers,*

(c) *products and sums,*

(d) *kernels and cokernels,*

(e) *pullbacks and pushouts.*

(d) and (e) follow from (a), (b) and (c).

PROPOSITION 2. *$k/\mathscr{G}$ is normal for closed immersions but not conormal.*

*Proof.* $\mathscr{C}$ is surjectively conormal. If $A \xrightarrow{f} B$ is a surjection, $f$ is coequalizer of maps $e, i : k + \ker f \to A$, where $i$ is the inclusion and $e(\ker f) = 0$. $\mathscr{C}/k$ is not normal. Otherwise, $\mathscr{C}/k$ is abelian and, hence, sums equal products, which is impossible. A more illuminating proof is as follows. Let

$$(A \xrightarrow{a} k) \xrightarrow{f} (B \xrightarrow{b} k)$$

be a monomorphism in $\mathscr{C}/k$. If $\mathscr{C}/k$ were normal (and, hence, $\mathscr{C}/k$ abelian), then $f = \ker(\operatorname{cok} f)$. In terms of $\mathscr{C}$, $\operatorname{cok} f = B/(f(x) - a(x))$.

$$\ker(\operatorname{cok} f) = \ker(B \to B/(f(x) - a(x))),$$

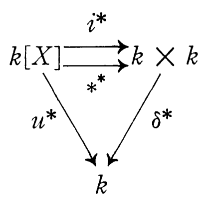
which is not necessarily $A$.

COROLLARY. *If $f$ is a closed immerison in $k/\mathscr{G}$,* $\ker(\operatorname{cok} f) = f$.

**2.1. Examples of categorical constructions in $k/\mathscr{G}$. Quotients of the line by 2 points.** Let $X$ be the $X$-axis, and $1, 0 \in X$. The rings of $X, 0, 1$ are $k[X]$, and $k$, $k$, respectively. Suppose that 0 is the basepoint of $X$ and basepoint of the reducible algebraic variety $\{0, 1\}$. Obviously, $X/\{0\} \cong X$. To determine $X/\{0, 1\}$, we find the coequalizer of

$$\{0, 1\} \underset{*}{\overset{i}{\rightrightarrows}} X,$$

where one map is an inclusion and the other assigns the basepoint of $X$. $i$ and $*$ correspond to $i^*$, $*^*$ in



If $s_1^*: k \to k \times k$ is defined by $s_1^*(x) = (x, x)$, $** = s_1^* \circ u^*$.

$$k[X] \xrightarrow{i^*} k \times k = k[X] \to k[X]/(X(X-1)) \cong k \times k.$$

The last relation follows by the Chinese remainder theorem.

Suppose that $i^*(f) = **(f)$. Then $f(0) \times f(1) = f(0) \times f(0)$.

Thus, $\mathrm{Eq}(i^*, **) = k + (X(X-1))$ (suppressing the evaluation map). To obtain the ideal of $\mathrm{Eq}(i^*, **)$, we define a map

$$k[X_1, \ldots, X_m, \ldots] \to k + (X(X-1))$$

via

$$X_1 \to X(X-1),$$
$$X_2 \to X^2(X-1),$$
$$X_3 \to X(X-1)^2,$$
$$X_4 \to X^3(X-1), \text{ etc.}$$

One obtains equations $X_3X_2 = X_1^3$, $X_4 = X_2 + X_1^2$, $X_5 = X_3 + X_1^2$, $X_1 = (X_2 - X_3)$, etc. If the cardinality of $k$ is bigger than the cardinality of the integers $\mathbf{Z}$, then the zeroes of these equations, which we denote $V_k(\mathrm{Eq}(i^*, **))$, correspond to the closed points (maximal ideals) of $\mathrm{Spec}(k + (X(X-1)))$, when $k$ is algebraically closed. See [**3**].

Proposition 1. (a) $\mathrm{Eq}(i^*, **)$ *has a function field* $K$.

(b) *A model for* $K$ *is* $Y^2 = X^2(X+1)$. *This is the cubic with one node* ($k$ *algebraically closed*).

(c) *The node is the basepoint.*

*Proof.* (a) $\mathrm{Eq}(i^*, **) \subset k(X)$, which is a field.

(b) One projects onto $X_1$, $X_3$ coordinates which satisfy:

$$X_2X_3 = X_1^3,$$
$$X_2 = X_3 + X_1,$$
$$X_3^2 + X_1X_3 = X_1^3.$$

The tangents at the singular point $(0, 0)$ are $X_3 = 0$, $X_3 + X_1 = 0$. Therefore, $X_3^2 + X_1X_3 = X_1^3$ is the singular cubic with one node, and is projectively equivalent with $Y^2 = X^2(X+1)$. See [**6**].

(c) The basepoint of $\mathrm{Eq}(i^*, **)$ corresponds to $X = 0$; i.e., $X_1 = 0$, $X_2 = 0$, $X_3 = 0, \ldots$ . But $(X_1, X_3) = (0, 0)$ is the node of $X_3^2 + X_1X_3 = X_1^3$.

One can easily see that if $k = \mathbf{R}$, the equation $X_3^2 + X_1X_3 = X_1^3$ defines the usual picture of a singular cubic with one node.

It is possible to show that the quotients of a non-singular irreducible algebraic curve by finite collections of a finite number of points is again

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algebraic. Serre [7] contains this result, and also a local statement of the above result. The advantage of the above construction is that it is more explicit.

**2.2. Quotients in $k/\mathcal{G}$.** Let $k$ be algebraically closed and let card $k >$ card **Z**. Suppose that

$$(V, P) \xrightarrow{f} (W, Q)$$

are two based maps in $k/\mathcal{G}$ where $P \in V$, $Q \in W$ and $*(V) = Q$. Then,

$$\operatorname{cok} f = \operatorname{Coeq}(f, *) = \operatorname{Spec}(\operatorname{Eq}(f^*, *^*)),$$

$$f^*\colon (k[W] \xrightarrow{w^*} k) \to (k[V] \xrightarrow{v^*} k),$$

$$*^*\colon (k[W] \xrightarrow{w^*} k) \to (k[V] \xrightarrow{v^*} k).$$

Here, $k[W]$, $k[V]$ are the elements of $\mathcal{C}$ corresponding to $W$ and $V$, and $*^* = i^* \circ w^*$, where $i^*$ includes $k$ in $k[V]$. Thus,

$$\operatorname{Eq}(f^*, *^*) = \{x \in k[W] \mid f^*(x) = w^*(x)\}.$$

LEMMA 1. $\operatorname{Eq}(f^*, *^*) = k + \ker f^*$.

*Proof.* Let $x \in k + \ker f^*$. Then $x = a + b$, where $a \in k$, $b \in \ker f^*$. We have $f^*(x) = a$ and $w^*(x) = v^*f^*(x) = v^*(a) = a$. So, $x \in \operatorname{Eq}(f^*, *^*)$. Let $x \in \operatorname{Eq}(f^*, *^*)$. Then $x = a + b$, where $a \in k$, $b \in \ker(w^*)$. We have $f^*(x) = f^*(a) + f^*(b) = a + f^*(b)$ and $w^*(x) = a$. As $x$ equalizes $f^*$, $w^*$, $f^*(b) = 0$, $b \in \ker f^*$.

PROPOSITION 1. (a) *Let* $(V, P) \xrightarrow{f} (W, Q)$ *be a map in* $k/\mathcal{G}$. $\operatorname{Cok}(f) = (\operatorname{Spec}(k + \ker f^*), \ker f^*)$

(b) *If $f$ is an inclusion*, $(W/V, V) = \operatorname{Cok} f = (\operatorname{Spec}(k + I_v), I_v)$, *where $I_v$ is the ideal of $V$.*

*Proof.* (a) follows from Lemma 1 and remarks on equalizers in $\mathcal{C}$ in §1.
(b) One has a ring map

$$k[W] \xrightarrow{f^*} k[W]/I_v.$$

**2.3. Example of categorical constructions in $k/G$. The quotient of the line by the divisor $2 0$. $0$ the basepoint of the line, etc.**

$$k/2 0 \cong \operatorname{Spec}(k + (X^2)).$$

Map
$$X_1 \to X^2,$$
$$X_2 \to X^3,$$
$$X_3 \to X^4, \text{ etc.}$$

A model of the function field of $k + (X^2)$ is given by $X_1^3 = X_2^2$. This is the non-singular cubic with one cusp.

Similarly,

$$k/20 + P \cong \operatorname{Spec}(k + (X + 1)(X^2))$$

has a model with equation $X_1^4 = X_2^2 X_3$, $X_3 - X_2 = X_1$.

Often the quotient of algebraic varieties is not an algebraic variety.

*Definition* 1. Let $K$ be the function field of an irreducible reduced $k$-scheme in $\mathcal{G}$. Then $K$ is a countably generated $k$-algebra, and each subalgebra is a countably generated $k$-algebra. Let $V(k)$ denote the $k$-points of a closed affine $k$-scheme corresponding to such a subalgebra and let $\{V(k)\}_K$, or simply $\{V\}_K$, denote the collection of all $V(k)$. We write $V > V'$ if there is a birational projection $V \to V'$. Minimal models are elements of $\{V\}_K$ not bigger than any other element of $\{V\}_K$ by $>$.

COROLLARY. *The examples of* §§ 2.1, 2.3 *are minimal models.*

As a consequence of these heuristics, one can propose:

PROPOSITION 1. *Let $k$ be an algebraically closed field and $F$ an algebraic curve in $k/\mathcal{G}$.*

(a) *$F$ appears at the end of a string of algebraic curves*

$$H_1 \xrightarrow{\alpha_1} H_2 \xrightarrow{p_1} H_2 \xrightarrow{\alpha_2} H_3 \to \ldots . \to H_n \xrightarrow{\alpha_n} F,$$

*where the $\alpha_i$ are quotients, the $p_i$ are constant maps, and $H_1$ is non-singular, the normalization of $F$.*

(b) *If $F$ has one singular point, $n = 1$.*

(c) *$F$ is the coequalizer of maps*

$$\{P_1\}, \ldots \ldots, \{P_n\} \to H_1.$$

(d) *If $F$ is of genus zero, and a plane curve,*

$$(\deg F - 1)(\deg F - 2) = \textstyle\sum_i r_i(r_i - 1),$$

*where the $r_i$ are the multiplicities of the singularities of $F$.*

The proofs of (a), (b), (c) are to be essentially found in [**7**]; (d) is a formula in [**8**].

## 2.4. Examples of categorical constructions in $k/\mathcal{G}$. Higher dimensional quotients.

PROPOSITION 1. *Let 0 be the basepoint of the plane $P$ and the $X$-axis in the plane. Then the quotient of $(P, 0)$ by $(X$-axis$, 0)$ has minimal model with equation $XZ = Y^2$.*

*Proof.* The ring of the quotient is $k + (X)$. One defines a map

$$k[X_1, \ldots, X_n, \ldots] \to k + (X)$$

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by

$$X_1 \to X,$$
$$X_2 \to XY,$$
$$X_3 \to XY^2,$$
$$X_4 \to XY^3, \text{ etc.}$$

Then $X_1X_3 = X_2^2$.

PROPOSITION 2. *The quotient of* $(P, 0)$ *by* $(X\text{-axis} \cup Y\text{-axis}, 0)$ *has minimal model with equation* $YZ = X^3$.

*Proof.* The map

$$k[X_1, \ldots, X_n, \ldots] \to k + (XY)$$

is

$$X_1 \to XY,$$
$$X_2 \to X^2Y,$$
$$X_3 \to XY^2,$$
$$X_4 \to X^3Y, \text{ etc.}$$

Then $X_2X_3 = X_1^3$.

PROPOSITION 3. *Let S be the cylinder in* 3*-space defined by* $X^2 + Y^2 = 1$ *with basepoint* $(1, 0, 0)$ *and let L be the line* $X = 1$, $Y = 0$ *with basepoint* $(1, 0, 0)$. *Then,* $S/L$ *has a minimal model with equation* $Z^2 + X^3(X + 2) = 0$ (*with basepoint* $(0, 0, 0)$).

*Proof.* The map

$$k[X_1, \ldots, X_n, \ldots] \to k + (X - 1) \subset k(X, Y, Z]$$

is defined by

$$X_1 \to X - 1,$$
$$X_2 \to Z(X - 1),$$
$$X_3 \to Y(X - 1),$$
$$X_4 \to Z^2(X - 1), \text{ etc.}$$

Then $X_3^2 + X_1^3(X_1 + 2) = 0$.

Note that in these examples $Y$ is not integral over the ring of the quotient, and the quotient can not be shown to be an algebraic variety by the method of Serre [7].

**3. Algebraic Suspensions.** One of the main reasons to study quotients in $k/\mathcal{G}$ is to define suspensions. Let $(X, P)$ be in $k/\mathcal{G}$, and let $S_1$ be the circle in $\mathcal{G}$, defined by $X^2 + Y^2 = 1$ with basepoint $(0, 1)$. Then:

*Definition* 1. The algebraic suspension of $X$, written $S(X)$, is the quotient of $(X, P) \times (S_1, (0, 1))$ by $(P \times S_1 \cup X \times (0, 1), P \times (0, 1))$.

This is the same, categorically, as the definition in the topological case. Thus, one expects:

PROPOSITION 1. *The algebraic suspension of two points is the circle.*

*Proof.* The ring of the suspension is the equalizer $E$ of

$$k[S_1] \otimes_k (k \times k) \overset{f^*}{\underset{*^*}{\rightrightarrows}} k[S_1] \times k \times k,$$

where $f^*$ $(g, a, b,) = (ag, g(0, 1)\, a, g(0, 1)b)$.

This is seen to be the ring generated by

$$\{g \otimes (a, b) | (ag, g(0, 1)a, g(0, 1)b) = (f(0, 1)a, f(0, 1)a, f(0, 1)a)\}.$$

If $a \neq 0$, elements $f(1, 1) \otimes (a, a)$ are in $E$. If $a \neq 0$, $b \neq 0$, elements of $M_{(0,1)} \otimes_k k$ are in $E$. Thus, as $M_{(0,1)}$ is the maximal ideal in $k[S_1]$ at $(0, 1)$, $E = k + M_{(0,1)} = k[S_1]$.

PROPOSITION 2. *The algebraic suspension of the line has a minimal model with equation*

$$Z = Z,$$
$$X^6 + Y^4 = -2Y^3X.$$

A graph of this surface can be found in Frost [**1**, plate IV, fig. 3].

*Proof.* One defines a map

$$k[X_1, \ldots, X_n, \ldots] \rightarrow k + (X - 1)(Y)$$

via $(X' = (X - 1))$

$$X_1 \rightarrow X'Y,$$
$$X_2 \rightarrow X'^2Y,$$
$$X_3 \rightarrow X'Y^2,$$
$$X_4 \rightarrow Z(X'Y)X,$$
$$X_5 \rightarrow Z^2(X'Y), \text{ etc.}$$

Then $X_1X_5 = X_4^2$, $X_2X_3 = X_1^3$, and $X_2^2 + X_3^2 = -2X_2X_1$, the last relation following from $X'^2 + Y^2 = -2X'$. Projecting onto $(X_1, X_2)$ plane and substituting $X_3 = X_1^3/X_2$, one obtains $X_2^4 + X_1^6 = -2X_2^3X_1$. $X_4$ is free.

## 4. Algebraic suspensions of $S^n$.

THEOREM. *Let $S^n$ be the $n$-*sphere *defined over $k$.*

(a) *The equations of a minimal model of the algebraic suspension $S(S^n)$ of $S^n$ are*

$$X_3X_4 = X_1^3,$$
$$X_2 + X_3^2 = -2X_3X_1,$$
$$X_4^2 + \ldots + X_{n+4}^2 = -2X_4X_1,$$

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*in* $n + 4$ *space. The basepoint is* $(0, 0, \ldots, 0)$ *and this is clearly a singular point.* $S^1 = S_1$.

(b) *Let* $k = \mathbf{R}$. *Then* $S(S^n)$ *is homeomorphic to* $S^{n+1}$, *for the usual topologies.*

*Proof.* (a) Let $k[x, y]$ and $k[x_1, \ldots, x_{n+1}]$ be the affine rings of $S^1$ and $S^n$, respectively. The ideal of

$$(1, 0) \times S_1 \cup S_1 \times (1, 0, \ldots 0)$$

in

$$k[x, y] \otimes k[x_1, \ldots, x_{n+1}] = k[x, y, x_1, \ldots, x_{n+1}]$$

is $((x - 1)(x_1 - 1))$. Therefore, the ring of the suspension is

$$k + ((x - 1)(x_1 - 1)) \subset k[x, y, x_1, \ldots, x_{n+1}].$$

Let $\bar{x} = x - 1$ and $\bar{x}_1 = x_1 - 1$.
One defines a map

$$k[X_1, \ldots, X_m, \ldots] \rightarrow k + (\bar{x}\bar{x}_1)$$

via

$$X_1 \rightarrow \bar{x}\bar{x}_1.$$
$$X_2 \rightarrow y\bar{x}\bar{x}_1,$$
$$X_3 \rightarrow \bar{x}\bar{x}\bar{x}_1,$$
$$X_4 \rightarrow \bar{x}_1\bar{x}\bar{x}_1,$$
$$X_5 \rightarrow x_2\bar{x}\bar{x}_1,$$
$$\vdots$$
$$X_{n+4} \rightarrow x_{n+1}\bar{x}\bar{x}_1, \text{ etc.}$$

One sees that $X_3X_4 = X_1^3$, $X_2^2 + X_3^2 = -2X_3X_1$, and

$$X_4^2 + \ldots + X_{n+4}^2 = -2X_4X_1,$$

by inspection. The basepoint of $S^n \times S_1$ corresponds to $\bar{x}, \bar{x}_1 = 0$. Therefore, the basepoint of these equations is $(0, 0, \ldots)$.

(b) We prove the case $n = 1$ first, where intuition is clearer. In this situation the equations are

$$X_3X_4 = X_1^3,$$
$$X_2^2 + X_3^2 = -2X_3X_1,$$
$$X_4^2 + X_5^2 = -2X_4X_1.$$

For a solution to exist (from the last two equations), $X_3, X_1$ must have opposite signs and $X_4, X_1$ must have opposite signs. But then, from the first equation, $X_1$ must be positive. Thus, using the last two equations, we obtain the conditions

$$X_1 \geqq 0,$$
$$X_3, X_4 \geqq 0.$$

Substituting $X_4 = X_1^3/X_3$ into the last equation we have

$$(\alpha) \qquad X_1^4(X_1^2 + 2X_3) = - X_5^2X_3^2.$$

The second equation can be written

$$(\beta) \qquad X_2^2 = - X_3(X_3 + 2X_1).$$

These two equations yield

$$X_1^2 + 2X_3 \leqq 0,$$
$$X_3 + 2X_1 \geqq 0.$$

Graphing ($\alpha$), subject to the above conditions, one obtains a graph homeomorphic to a disc with $X_3 + 2X_1 = 0$ on the boundary. Then the graph of $S(S^1)$ in 4-space, subject to condition ($\beta$), will be two discs joined around their boundary where $X_3 + 2X_1 = 0$. A 2-sphere is thus obtained.

For arbitrary $n$, one obtains

$$(\alpha) \qquad X_1^4(X_1^2 + 2X_3) = - (X_3^2X_5^2 + \ldots + X_3^2X_{n+4}^2)$$

and

$$(\beta) \qquad X_2^2 = - X_3(X_3 + 2X_3),$$

together with conditions

$$X_1 \geqq 0,$$
$$X_3, X_4 \leqq 0,$$
$$X_1^2 + 2X_3 \leqq 0,$$
$$X_3 + 2X_1 \geqq 0.$$

The projection of ($\alpha$), subject to $X_3 + 2X_1 = 0$, is the interior of the region bounded by

$$X_1^3(4 - X_1) = 4(X_5^2 + \ldots + X_{n+4}^2).$$

This is seen to be an $n + 1$ disc, topologically. Therefore, using equation ($\beta$), one has $S(S^n)$ is homeomorphic to the $n + 1$ sphere.

We note that the projection of $S(S^n)$ into the first $n + 4$ coordinates is a homeomorphism.

**5. Algebraic loop functors.** Suppose that the suspension functor $S$ (a functor because of its categorical construction) has a right adjoint $\Omega$ in $k/\mathscr{G}$. Then, suppose that

$$\text{Hom}(SX, Y) \cong \text{Hom}(X, \Omega(Y)),$$

as bifunctors in $X$, $Y$. Let $X$ be an element of $k/\mathscr{G}$ consisting of 2 points. Then, as sets,

$$\Omega(Y) \cong \text{Hom}(X, \Omega(Y) \cong \text{Hom}(S^1, Y)$$

(suppressing basepoints). The algebraic loop functor should bave the same form as the topological loop functor. For the above to make sense, however, $\mathrm{Hom}(S^1, Y)$ must be given the structure of an element in $k/\mathscr{G}$.

Let $X$, $Y$ be elements in $k/\mathscr{G}$ whose geometric $k$ points $X(k)$, $Y(k) \neq \emptyset$. We show that $\mathrm{Hom}(X, Y)$ can be given the structure of an object in $k/\mathscr{G}$. Each element $f \in \mathrm{Hom}(X, Y)$ defines a morphism

$$f(a_i) = (g^j(b_k^j, a_i)),$$

where the $b_k^j$ are the coefficients of the polynomial in $a_i$ in the $j$ coordinate of $f(a_i)$. Viewing the $a_i$ as indeterminants and substituting $(g^j(b_k^j, a_i))$ into the equations of $Y$, one has

$$\mathrm{Hom}(k^{\mathbf{Z}}, Y) \cong \{(b_k^j) | F(g^j(b_k^j, a_i)) = 0, \text{ for all } F \text{ in the ideal of } Y\}$$

is an element of $k/\mathscr{G}$ with basepoint the map $e: k^{\mathbf{Z}} \to Y$ which maps $k^{\mathbf{Z}}$ to the basepoint of $Y$. Let

$$Z = \{(b_k^j) \in \mathrm{Hom}(k^{\mathbf{Z}}, k^{\mathbf{Z}}) | (g^j(b_k^j, a)) = P \text{ for all } (a_i) \in X(k)\}.$$

$Z$ is an element of $k/\mathscr{G}$ with basepoint $e$. $\mathrm{Hom}(X, k^{\mathbf{Z}})$ is then the quotient of $\mathrm{Hom}(k^{\mathbf{Z}}, k^{\mathbf{Z}})$ by $Z$. $\mathrm{Hom}(X, Y)$ is the image in $k/\mathscr{G}$ of $\mathrm{Hom}(k^{\mathbf{Z}}, Y)$ in

$$\mathrm{Hom}(X, k^{\mathbf{Z}}).$$

THEOREM 1. *In $k/\mathscr{G}$, $S$ is the left adjoint to $\Omega$.*

*Proof.* As $k/\mathscr{G}$ is normal for closed immersions, one has an exact sequence

$$0 \to S_1 \times R \cup Q \times X \to S_1 \times X \to S(X) \to 0.$$

Define a map

$$\alpha: \mathrm{Hom}(S(X), Y) \to \mathrm{Hom}(X, \mathrm{Hom}(S_1, Y))$$

via

$$\alpha(f)(x)(s) = f(s, x),$$

where $f \in \mathrm{Hom}(S(X), Y)$ and $(s, x)$ is a representative for an element of $S(X)$. Then,

$$\alpha(f)(X)(Q) = f(Q, X) = P,$$
$$\alpha(f)(R)(S_1) = f(S_1, R) = P,$$

and as $e$ (as defined above) is the basepoint of $\mathrm{Hom}(S_1, Y)$, $\alpha$ behaves properly with respect to basepoints. Define a map

$$\beta': \mathrm{Hom}(X, \mathrm{Hom}(S_1, Y)) \to \mathrm{Hom}(S_1 \times X, Y)$$

via

$$\beta'(f)(s, x) = f(x)(s).$$

As

$$\beta'(f)(S_1, R) = f(R)(S_1) = P,$$
$$\beta'(f)(Q, X) = f(X)(Q) = P,$$

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$\beta'$ induces a map

$$\beta\colon \mathrm{Hom}(X, \mathrm{Hom}(S_1, Y)) \to \mathrm{Hom}(S(X), Y).$$

The theorem is then complete, as it is clear that $\alpha$ and $\beta$ are inverse natural transformations and that $\Omega(Y) = \mathrm{Hom}(S_1, Y)$.

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