$-\Delta$ IS POSITIVE DEFINITE ON A "SPINY URCHIN"

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In a recent note [3] in this department C. Clark has shown that Rellich's theorem on the compactness of the imbedding $H_0^1(G) \to L^2(G)$ is valid if G is the "spiny urchin" domain obtained by removing from the plane the union of the sets S_k $(k = 1, 2, \ldots)$ defined in polar coordinates by

$$S_k = \{(r, \theta) : r \geq k, \quad \theta = n\pi/2k, \quad n = 1, 2, \ldots, 4k\} .$$

It follows that the eigenvalue problem

$$(1) \qquad \begin{cases} -\Delta u = \lambda u & \text{in } G \\ u = 0 & \text{on bdry } G \end{cases}$$

has a discrete spectrum. We can show further that

$$-\int_G \Delta\phi\, \overline{\phi}\, dx \geq C\|\phi\|_{1,G}^2 , \qquad C > 0 ,$$

where $\|\phi\|_{1,G}^2 = \|\phi\|_{0,G}^2 + |\phi|_{1,G}^2$, $\|\cdot\|_{0,G}$ being the norm in $L^2(G)$, and $|\phi|_{1,G}^2 = \|\partial\phi/\partial x\|_{0,G}^2 + \|\partial\phi/\partial y\|_{0,G}^2$. In particular, the generalized Dirichlet problem for $-\Delta$ has a unique solution and all the eigenvalues of (1) are positive. Since $-\int_G \Delta\phi\,\overline{\phi}\, dx = |\phi|_{1,G}^2$ the above result is an immediate consequence of the

THEOREM. *The norms* $|\cdot|_{1,G}$ *and* $\|\cdot\|_{1,G}$ *are equivalent in* $H_0^1(G)$.

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Proof. We must show that $\|\phi\|_{0,G} \leq$ const. $|\phi|_{1,G}$ (Poincaré's inequality) for all $\phi \in C_0^\infty(G)$. If $G_R = \{x \in G : |x| \geq R\}$, Clark shows in [3] that if $R \geq 1$ then

$$(2) \qquad \|\phi\|_{0,G_R} \leq C(R) \|\phi\|_{1,G} .$$

We require (2) only for $R = 1$. To obtain the remaining part of Poincaré's inequality, namely

$$(3) \qquad \|\phi\|_{0,B} \leq \text{const.} |\phi|_{1,G}, \quad B = \{x \in G : |x| \leq 1\},$$

let us take a new origin at the point $(2, 0)$ and use polar coordinates (r, θ) with respect to this point. Then B is contained in the annulus $A = \{(r, \theta) : 1 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$. Since ϕ vanishes on $\theta = 0$, $1 \leq r \leq 3$ we have

$$\phi(r, \theta) = \int_0^\theta \frac{\partial}{\partial \gamma} (r, \gamma)\, d\gamma .$$

Hence by Schwarz's inequality

$$\int_B |\phi(x)|^2 dx \leq \int_1^3 r\, dr \int_0^{2\pi} |\phi(r, \theta)|^2 d\theta$$

$$\leq \int_1^3 r\, dr \cdot 2\pi \int_0^{2\pi} d\theta \int_0^{2\pi} \left|\frac{\partial}{\partial \gamma} \phi(r, \gamma)\right|^2 d\gamma$$

$$\leq 4\pi^2 \int_A \left|\frac{\partial}{\partial \gamma} \phi\right|^2 dx .$$

Since $\left|\frac{\partial}{\partial \gamma} \phi(r, \gamma)\right|^2 \leq \text{const. } r^2 \left(\left|\frac{\partial \phi}{\partial x}\right|^2 + \left|\frac{\partial \phi}{\partial y}\right|^2\right)$ (3) follows at once.

One clearly does not require in this proof the fact that G is quasi-bounded. The proof remains the same if G is replaced by the less prickly set obtained by removing from the plane the union of the sets T_k $(k = 1, 2, \ldots)$ defined in polar coordinates by

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$$T_k = \{(r, \theta) : r \geq k^2, \ \theta = n\pi/2k, \ n = 1, 2, \ldots, 4k\},$$

though the eigenvalue problem (1) need not have a discrete spectrum in this case.

The method used above can be extended to yield theorems on the equivalence of norms in the Sobolev space $W_o^{m,p}(G)$ for a large class of domains G for which dist $(x, \text{bdry } G)$ remains bounded as x tends to infinity in G. For such spaces some preliminary inequalities similar to (2) may be found in [1] and [2] where they are used to prove compactness theorems for imbeddings of $W_o^{m,p}(G)$.

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