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RESEARCH ARTICLE

Limits of nodal surfaces and applications

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Abstract
Let $\mathcal{X} \to \mathbb{D}$ be a flat family of projective complex 3-folds over a disc $\mathbb{D}$ with smooth total space $\mathcal{X}$ and smooth general fibre $\mathcal{X}_t$, and whose special fiber $\mathcal{X}_0$ has double normal crossing singularities, in particular, $\mathcal{X}_0 = A \cup B$, with A, B smooth threefolds intersecting transversally along a smooth surface $R = A \cap B$. In this paper, we first study the limit singularities of a δ-nodal surface in the general fibre $S_t \subset \mathcal{X}_t$, when S_t tends to the central fibre in such a way its δ nodes tend to distinct points in R. The result is that the limit surface S_0 is in general the union $S_0 = S_A \cup S_B$, with $S_A \subset A$, $S_B \subset B$ smooth surfaces, intersecting on R along a δ-nodal curve $C = S_A \cap R = S_B \cap B$. Then we prove that, under suitable conditions, a surface $S_0 = S_A \cup S_B$ as above indeed deforms to a δ-nodal surface in the general fibre of $\mathcal{X} \to \mathbb{D}$. As applications, we prove that there are regular irreducible components of the Severi variety of degree d surfaces with δ nodes in $\mathbb{P}^3$, for every $\delta \leqslant \binom{d-1}{2}$ and of the Severi variety of complete intersection δ-nodal surfaces of type (d, h), with $d \geqslant h - 1$ in $\mathbb{P}^4$, for every $\delta \leqslant \binom{d+3}{3} - \binom{d-h+1}{3} - 1$.

Contents

1. Introduction

The main object of study in this article is Severi varieties of nodal surfaces on smooth, projective, complex threefolds. Severi varieties of nodal hypersurfaces on a smooth variety are a well-known object of study in algebraic geometry that goes back to well more than a century ago. Its importance is underlined by the relationships with other themes in the area. For example, the recent papers [7, 16] explore the relation of Severi varieties with the Hodge conjecture.

Our approach to the subject is via degenerations. Degenerations of smooth complex varieties to complex varieties with simple normal crossings is also a classical object of study. In particular, it has been widely used by several authors for studying Severi varieties of nodal curves on surfaces. The method is powerful and enables one to obtain sharp results on the non-emptiness of some Severi varieties of curves (see, for instance, [3, 5, 6, 10], etc.).

One of the basic ideas in these papers is the well-known and classical fact that the limit of a curve C_t with a node p_t on a smooth surface $\mathcal{X}_t$, when $\mathcal{X}_t$ degenerates to a reducible surface $\mathcal{X}_0 = A \cup B$, with A and B smooth and meeting transversally along a smooth curve $R = A \cap B$, and p_t going to a point $p_0 \in R$, is a curve $C_0 \subset \mathcal{X}_0$ with a tacnode in p_0, which appears scheme theoretically with multiplicity 2. This result is an easy consequence of the study of the versal deformation space of a tacnode, and its proof is in [2, 17]. This result has been proved also using limit linear systems techniques; see [9]. The present article intends to extend this result on the limit of a nodal curve to the case of nodal surfaces in threefolds, and we will take the point of view of [9]. In the sequel, a *node* of a surface will be an A_1-singularity.

Let $\mathcal{X} \to \mathbb{D}$ be a flat family of projective complex 3-folds over a disc $\mathbb{D}$ with smooth total space $\mathcal{X}$ and smooth general fibre, and whose special fiber $\mathcal{X}_0$ has double normal crossing singularities, in particular, $\mathcal{X}_0 = A \cup B$, with A, B smooth threefolds intersecting transversally along a smooth surface $R = A \cap B$.

First of all, we will study in Section 2 the limit singularities of a δ-nodal surface in the general fibre $S_t \subset \mathcal{X}_t$, when S_t tends to the central fibre in such a way that its δ nodes tend to distinct points $p_1, \dots, p_\delta$ in R. The result (see Theorem 2.2) is that the limit surface S_0 is in general the union $S_0 = S_A \cup S_B$, with $S_A \subset A$, $S_B \subset B$ smooth surfaces, that cut out on R the same curve C having nodes at $p_1, \dots, p_\delta$ and no further singularities. In this case, we say that S_0 presents a singularity of type T_1 at every point p_i, $i = 1, \dots, \delta$. The equations of a T_1 singularity are given in (2.9). Finally, in 2.4, we provide the local equation of (an example of) a local deformation of a singularity of type T_1 to a node on the general fibre.

The central part of our paper is Section 3. First of all, we prove in Lemma 3.2 that the only singularity of a surface $S_t \subset \mathcal{X}_t$ to which a singularity of type T_1 of a surface $\mathcal{S}_0 \subset \mathcal{X}_0$ may be deformed is a node. In §3.2.2, we describe the first-order locally trivial deformations in $\mathcal{X}_0$ of surfaces $S_0 = S_A \cup S_B$ with T_1 singularities on R and at most nodes elsewhere. In particular, we find sufficient conditions for smoothness of the equisingular deformation locus of S_0 in the relative Hilbert scheme of $\mathcal{X}$. If these conditions are verified, then the T_1 singularities of S_0 and its nodes can be smoothed independently inside $\mathcal{X}_0$. Next, in §3.2.3, we consider deformations of a surface $S_0 \subset \mathcal{X}_0$, with T_1 singularities on R and at most nodes elsewhere, off the central fibre. We prove, in Theorem 3.13, that under suitable conditions, one can deform S_0 off the central fibre $\mathcal{X}_0$ to a surface S_t in the general fibre $\mathcal{X}_t$, with only nodes that are the deformations of the nodes of S_0 and of the T_1 singularites of S_0, and that the space of this deformation is generically smooth of the expected dimension. Again, generic smoothness means that the nodes of the general surface S_t can be independently smoothed.

In Section 4, we give a couple of applications of our general result. Essentially, we consider the following problem (see Problem 4.3). Let X be a smooth irreducible projective complex threefold. Let L be a line bundle on X such that the general surface in the linear system $|L|$ is smooth and irreducible. Let $V_\delta^{X,|L|}$ be the Severi variety of surfaces S in $|L|$ which are reduced with only δ nodes as singularities. The question we consider is as follows: Given X and L as above, which is the maximal value of δ such that $V_\delta^{X,|L|}$ has a generically smooth component of the expected codimension δ in $|L|$? We give contributions to this problem in two cases. The first one is for $X = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$ (see Theorem 4.6); the second one is when X is a general hypersurface of degree $h \geqslant 2$ in $\mathbb{P}^4$ and $L = \mathcal{O}_X(d)$ with $d \geqslant h-1$ (see Theorem 4.9).

To finish this introduction, it is worth mentioning that the basic idea of a singularity of type T_1 being a limit of a node is already contained, although in a rather obscure form, in B. Segre's paper [18]. In this paper, Segre considers, even more generally, the case of higher dimension. As a matter of fact, we believe that there should be no obstruction in extending our results in higher dimension too. However, we did not dwell on this here because we thought that the surface in threefold case already shows the complexity of the situation. We plan to come back on this in the future.

Notation: In what follows, we use standard notation in algebraic geometry. In particular, we will denote by $\sim$ the linear equivalence.

2. Limit singularity of a node of a surface in a threefold

2.1. The problem

Let $\mathcal{X} \longrightarrow \mathbb{D}$ be a flat family of projective complex 3-folds over a disc $\mathbb{D}$ with smooth total space $\mathcal{X}$ and smooth general fiber $\mathcal{X}_t$, with $t \in \mathbb{D} \setminus \{0\}$, and whose special fiber $\mathcal{X}_0$ has double normal crossing singularities; in particular, $\mathcal{X}_0 = A \cup B$ has two smooth irreducible components A and B, intersecting transversally along a smooth surface $R = A \cap B$.

Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. For each $t \in \mathbb{D}$, we set $\mathcal{L}_t = \mathcal{L}_{|\mathcal{X}_t}$. We consider the following question. Roughly speaking, assume that for $t \in \mathbb{D}$ general, we have a surface $S_t \in |\mathcal{L}_t|$ having a double point p_t. Assume that S_t tends to a surface S_0 in $\mathcal{X}_0$ with p_t tending to a point $p_0 \in R$. The question is: What is the singularity that S_0 has at p_0? Let us make this setting more precise.

2.2. Set up

Let us fix $p = p_0 \in R$, which is a double point for the central fibre $\mathcal{X}_0$, whereas $\mathcal{X}$ is smooth at p. Hence, there are no sections of $\mathcal{X} \longrightarrow \mathbb{D}$ passing through p. So let us consider a smooth bisection γ' of $\mathcal{X} \longrightarrow \mathbb{D}$ passing through p.

Step 0. Let us look at the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{D} & \xrightarrow{\nu_2} & \mathbb{D}, \end{array}$$

where the rightmost square is cartesian and $\nu_2 : u \in \mathbb{D} \longrightarrow u^2 \in \mathbb{D}$. Then $\mathcal{X}'$ is singular along the counterimage of R (that by abuse of notation, we still denote by R), which is a locus of double points for $\mathcal{X}'$, with tangent cone a quadric cone of rank 3. The morphism $\mathcal{Y} \longrightarrow \mathcal{X}'$ is the desingularization of $\mathcal{X}'$ obtained by blowing up $\mathcal{X}'$ along R.

The induced morphism $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$ is $2:1$ outside the central fibre of $\mathcal{Y}$. In particular, for every $t \neq 0$, there are exactly two fibres $\mathcal{Y}_{u_1}$ and $\mathcal{Y}_{u_2}$ of $\mathcal{Y} \longrightarrow \mathbb{D}$ isomorphic to the fibre $\mathcal{X}_t$ of $\mathcal{X} \longrightarrow \mathbb{D}$ via π, where $\{u_1, u_2\} = \nu_2^{-1}(t)$. The family $\mathcal{Y} \longrightarrow \mathbb{D}$ has central fibre $\mathcal{Y}_0 = A \cup \mathcal{E} \cup B$, where, by abusing notation, A and B denote the proper transforms of A and B and $\mathcal{E} \longrightarrow R$ is a $\mathbb{P}^1$-bundle on R. The morphism π is totally ramified along A and B, and it contracts $\mathcal{E}$ to R in $\mathcal{X}$. In particular, $A \cap \mathcal{E}$ and $B \cap \mathcal{E}$ are two non-intersecting sections of $\mathcal{E}$ both isomorphic to R. Denote by F the fibre of $\mathcal{E} \longrightarrow R$ over the point $p \in R \subset \mathcal{X}_0$. One has $F \cong \mathbb{P}^1$. Now the counterimage of γ' on $\mathcal{Y}$ is the union of two sections of $\mathcal{Y} \longrightarrow \mathbb{D}$, each intersecting $\mathcal{Y}_0$ at a smooth point on F. We let γ be one of these two sections and q be the intersection point of γ and F.

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Assume there exists an effective divisor $\mathcal{S} \subset \mathcal{Y}$, with $\mathcal{S} \sim \pi^*(\mathcal{L})$, having double points along γ. Let S be the image of $\mathcal{S}$ in $\mathcal{X}$ via the morphism π. Note that S has points of multiplicity 2 along the bisection γ'. For every $t \neq 0$, if $\mathcal{Y}_{u_1}$ and $\mathcal{Y}_{u_2}$, with $u_1^2 = u_2^2 = t$, are the two fibres of $\mathcal{Y} \to \mathbb{D}$ isomorphic to $\mathcal{X}_t$ via π, we have

$$S_t = S \cap \mathcal{X}_t = S_{u_1} \cup S_{u_2},$$

where $S_{u_i} = \pi(\mathcal{S}_{u_i})$ and $\mathcal{S}_{u_i} = \mathcal{S} \cap \mathcal{Y}_{u_i}$, for $i = 1, 2$. If $t = 0$, we have that

$$S \cap \mathcal{X}_0 = 2S_0 = 2(S_A \cup S_B),$$

where $S_A = \pi(\mathcal{S} \cap A) \subset A$ and $S_B = \pi(\mathcal{S} \cap B) \subset B$.

We want to understand $S|_{\mathcal{X}_0}$. To do this, we will first understand $\mathcal{S}|_{\mathcal{Y}_0}$.

Step 1. Let $\pi_1 : \mathcal{Y}^1 \to \mathcal{Y}$ be the blowing-up of $\mathcal{Y}$ along γ with exceptional divisor Γ. We have a new family $\mathcal{Y}^1 \to \mathbb{D}$ with general fibre the blow up of $\mathcal{Y}_u \cong \mathcal{X}_{\nu_2(u)}$ at its intersection point with γ (that is also the point of multiplicity 2 of the surface $\mathcal{S}_u$), and central fibre $\mathcal{Y}^1_0 = A \cup \mathcal{E}' \cup B$, where $\mathcal{E}'$ is the blow-up of $\mathcal{E}$ at q. Still denoting by F the proper transform of F in $\mathcal{Y}^1$, we have that the proper transform $\mathcal{S}^1$ of $\mathcal{S}$ in $\mathcal{Y}^1$ satisfies

$$\mathcal{S}^1 \sim \pi_1^*(\mathcal{S}) - 2\Gamma. \tag{2.1}$$

We deduce that $\mathcal{S}^1 \cdot F = -2$ and hence $F \subset \mathcal{S}^1$.

Step 2. Let now $\pi_2 : \mathcal{Y}^2 \to \mathcal{Y}^1$ be the blow-up of $\mathcal{Y}^1$ along F with new exceptional divisor Θ. We have the new family $\mathcal{Y}^2 \to \mathbb{D}$, whose general fibre is the same as the general fibre of $\mathcal{Y}^1 \to \mathbb{D}$, and new central fibre $\mathcal{Y}^2_0 = A' \cup \mathcal{E}'' \cup \Theta \cup B'$, where A', $\mathcal{E}''$ and B' are the blow-ups of A, $\mathcal{E}'$ and B at $F \cap A$, $F \subset \mathcal{E}'$ and $B \cap F$, respectively. Notice that $\Theta \to F$ is a $\mathbb{P}^2$-bundle on F, intersecting A' (resp. B') along a surface isomorphic to $\mathbb{P}^2$, which is a fibre of $\Theta \to F$, and at the same time is the exceptional divisor of the blow-up $A' \to A$ at $F \cap A$ (resp. of the blow-up $B' \to B$ at $F \cap B$). Moreover, the surface $E := \Theta \cap \mathcal{E}''$ has a $\mathbb{P}^1$-bundle structure $E \to F$, and it is the exceptional divisor of $\mathcal{E}''$, arising from the blowing-up of F in $\mathcal{E}'$.

We claim that $E \simeq \mathbb{F}_0$. Indeed, since $F \simeq \mathbb{P}^1$ and F is a fibre of $\mathcal{E} \to R$, we have that $\mathcal{N}_{F|\mathcal{E}} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. This implies that $\mathcal{N}_{F|\mathcal{E}'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and hence, $E = \mathbb{P}(\mathcal{N}_{F|\mathcal{E}'}) = \mathbb{F}_0$.

If $\mathcal{S}^2$ is the proper transform of $\mathcal{S}^1$ in $\mathcal{Y}^2$, by (2.1), we deduce that

$$\begin{aligned}
\mathcal{S}^2|_\Theta &\sim \pi_2^*(\mathcal{S}^1)|_\Theta - m_F \Theta|_\Theta \\
&\sim -2f_\Theta + m_F(A' + B' + \mathcal{E}'')|_\Theta \\
&\sim -2f_\Theta + m_F(2f_\Theta + \mathcal{E}''|_\Theta) \\
&\sim (2m_F - 2)f_\Theta + m_F(\mathcal{E}''|_\Theta) \\
&\sim (2m_F - 2)f_\Theta + m_F E,
\end{aligned} \tag{2.2}$$

where f_Θ denotes the linear equivalence class of a fibre of $\Theta \to F$ and m_F is the multiplicity of $\mathcal{S}^1$ along F. Notice that $\mathcal{S}^2|_\Theta$ must be an effective divisor because it is the restriction to Θ of an effective divisor that does not contain Θ. This implies the minimum value of m_F making $\mathcal{S}^2|_\Theta$ effective is $m_F = 1$.

2.3. *Description of $\mathcal{S}_{|\mathcal{Y}_0}$ and of $S_{|\mathcal{X}_0}$*

We assume now $m_F = 1$. To better understand $\mathcal{S}^2|_\Theta \sim E$, we restrict $\mathcal{S}^2|_\Theta$ to $\mathcal{E}''$. Let σ and f, with $\sigma^2 = f^2 = 0$, be the generators of the Picard group of $E = \mathcal{E}'' \cap \Theta \cong \mathbb{F}_0$. By restricting (2.2) to E, one gets

$$\mathcal{S}^2|_E = \mathcal{S}^2|_{\Theta\cap\mathcal{E}''} \sim \mathcal{E}''_{|E}. \tag{2.3}$$

To compute $\mathcal{E}''_{|E}$, we use the obvious relation $(A' + B' + \Theta + \mathcal{E}'')_{|E} = 0$, which implies the following identity on E:

$$2f + \Theta_{|E} + \mathcal{E}''_{|E} = 0. \tag{2.4}$$

Since $E = \mathcal{E}'' \cap \Theta$, then $\Theta_{|E}$ is the class of $\Theta^2 \cdot \mathcal{E}''$ which is clearly the class of the normal bundle $\mathcal{N}_{E|\mathcal{E}''}$ of E in $\mathcal{E}''$. Similarly, $\mathcal{E}''_{|E}$ is the class of $\Theta \cdot \mathcal{E}''^2 = c_1(\mathcal{N}_{E|\Theta})$. Since $E = \mathbb{P}(\mathcal{N}_{F|\mathcal{E}'})$, denoting by $\pi_E : E \longrightarrow F$ the natural projection morphism, whose fiber is f, and by

$$e := \Theta|_E = \Theta^2 \cdot \mathcal{E}'' = c_1(\mathcal{N}_{E|\mathcal{E}''}),$$

we have that $\mathcal{N}_{E|\mathcal{E}''} \subset \pi_E^*(\mathcal{N}_{F|\mathcal{E}'})$ is the tautological fibre bundle of $E = \mathbb{P}(\mathcal{N}_{F|\mathcal{E}'})$. So we get that

$$f \cdot e = -1 \tag{2.5}$$

and

$$e^2 - c_1(\pi_E^*(\mathcal{N}_{F|\mathcal{E}'})) \cdot e + c_2(\pi_E^*(\mathcal{N}_{F|\mathcal{E}'})) = 0,$$

(see [12, p. 606]). Now

$$c_2(\pi_E^*(\mathcal{N}_{F|\mathcal{E}'})) = \pi_E^*(c_2(\mathcal{N}_{F|\mathcal{E}'})) = 0,$$

since $\mathcal{N}_{F|\mathcal{E}'}$ is a vector bundle on F and $\dim(F) = 1$. So

$$e^2 - c_1(\pi_E^*(\mathcal{N}_{F|\mathcal{E}'})) \cdot e = e^2 - \pi_E^*(c_1(\mathcal{N}_{F|\mathcal{E}'})) \cdot e = e^2 - c_1(\mathcal{N}_{F|\mathcal{E}'})f \cdot e = e^2 + c_1(\mathcal{N}_{F|\mathcal{E}'}) = 0.$$

Thus,

$$e^2 = -c_1(\mathcal{N}_{F|\mathcal{E}'}) = 2. \tag{2.6}$$

Set $e = a\sigma + bf$. By (2.5) and (2.6), one gets $a = -1$ and

$$-2b = (-\sigma + bf)^2 = e^2 = 2, \quad \text{hence} \;\; b = -1.$$

Thus, we have

$$\Theta_{|E} = c_1(N_{E|\mathcal{E}''}) = -\sigma - f.$$

Hence, by (2.4), we get

$$\mathcal{E}''_{|E} = c_1(N_{E|\Theta}) = \sigma - f. \tag{2.7}$$

Remark 2.1. From (2.7), it follows that the divisor E (which does not move on $\mathcal{E}''$ being there an exceptional divisor) does not move in Θ either, since $\mathcal{N}_{E|\Theta}$ is non-effective. Hence, by (2.2) and $m_F = 1$, we have $\mathcal{S}^2|_E = E$.

We are now also able to describe the divisor $\mathcal{S}^2|_{A'\cap\mathcal{E}''} \cong \mathcal{S}^2|_{B'\cap\mathcal{E}''}$. Indeed,

$$\mathcal{S}^2|_{A'\cap\mathcal{E}''} \sim (\pi_2^*\pi_1^*(\mathcal{S}) - 2\pi_2^*(\Gamma) - \Theta)|_{A'\cap\mathcal{E}''} \sim \pi_2^*\pi_1^*(\mathcal{S})|_{A'\cap\mathcal{E}''} - \Theta \cap A' \cap \mathcal{E}'',$$

and, since $\mathcal{S}^2|_\Theta = E$ by Remark 2.1, it contains the (-1)-curve $\Theta \cap A' \cap \mathcal{E}'' = E \cap A'$ in its base locus with multiplicity 1. Thus, $\mathcal{S}^2|_{A' \cap \mathcal{E}''} = \mathcal{D}_A \cup (\Theta \cap A' \cap \mathcal{E}'')$, where

$$\mathcal{D}_A \sim \pi_2^*(\pi_1^*(\mathcal{S}))|_{A' \cap \mathcal{E}''} - 2\Theta \cap A' \cap \mathcal{E}'',$$

and similarly for $\mathcal{S}^2|_{B' \cap \mathcal{E}''}$.

This analysis implies the following:

Theorem 2.2. *Let $\mathcal{S} \subset \mathcal{Y}$ be an effective Cartier divisor as in* **Step 0**. *Then the surface $\mathcal{S}_{|\mathcal{Y}_0}$ is the union of three surfaces $\mathcal{S}_A = \mathcal{S} \cap A, \mathcal{S}_B = \mathcal{S} \cap B$ and $\mathcal{S}_\mathcal{E} = \mathcal{S} \cap \mathcal{E}$, where $\mathcal{S}_A$ (resp. $\mathcal{S}_B$) intersects $A \cap \mathcal{E}$ (resp. $B \cap \mathcal{E}$) along a curve which has a double point at the point $F \cap A$ (resp. $F \cap B$), these two curves are isomorphic, and $\mathcal{S}_\mathcal{E}$ is a $\mathbb{P}^1$-bundle over any one of them.*

Accordingly, $S_{|\mathcal{X}_0} = 2S_0$, with $S_0 \in |\mathcal{L}_0|$ and S_0 is the union of two surfaces S_A, S_B, respectively isomorphic to $\mathcal{S}_A, \mathcal{S}_B$, intersecting along a curve in R that has a double point at p (see Figure 1).

2.3.1. Local equations of S_0

We may assume that $\mathcal{X}$ locally around $p \in \mathcal{X}_0$ is embedded in $\mathbb{A}^5$ with coordinates (x, y, z, u, t) with p corresponding to the origin. We may suppose that $\mathcal{X}$ is defined by the equation $xy = t$ and the map $\mathcal{X} \to \mathbb{D}$ is given by $(x, y, z, u, t) \mapsto t$. So we will assume that A is defined by the equations $x = t = 0$ and B by the equations $y = t = 0$, so that R is defined by $x = y = t = 0$.

The above analysis proves that the surfaces $\mathcal{S}|_A$ and $\mathcal{S}|_B$ belong to the restriction linear systems of $\mathcal{L}$ to A and B, respectively, and moreover are tangent to R at the point p. Thus, $S_0 = S_A \cup S_B$ belongs to the linear system $\mathcal{L}_0(2, p) \subset |\mathcal{L}_0|$ of surfaces with local equations at p given by

$$\begin{cases} (a_1 x + b_1 y) + f_2(x, y, z, u) = 0 \\ xy = 0, \end{cases} \tag{2.8}$$

with $f_2(x, y, z, u)$ an analytic function with terms of degree at least 2.

Definition 2.2.1. Let $S_0 = S_A \cup S_B$ be a surface that is the union of two irreducible components S_A, S_B intersecting along a curve C. Let $p \in C$. We will say that S_0 has at p a *singularity of type* T_1 if S_A and S_B are smooth at p and C has at p a node.

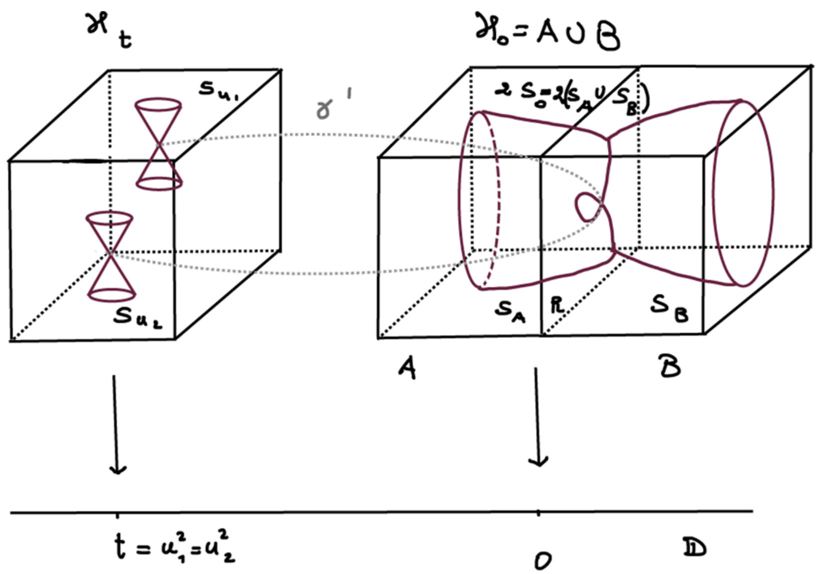

Figure 1. *A T_1-singularity of a surface $S_0 = S_A \cup S_B$ in $\mathcal{X}_0 = A \cup B$.*

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Remark 2.3. If in (2.8), $a_1 b_1 \neq 0$, then S_0 has a T_1 singularity at the origin p, and, up to a linear change of coordinates, the local equations are given by

$$\begin{cases} x + y + f_2(x, y, z, u) = 0, \text{ with } f_2(0, 0, z, u) = 0 \text{ having a node at } \underline{0}, \\ xy = 0. \end{cases} \tag{2.9}$$

In the sequel, we will also refer to $\mathcal{L}_0(2, p)$ as the sublinear system of $|\mathcal{L}_0|$ of surfaces with at least a T_1 singularity at p.

Remark 2.4. We have that $\mathcal{L}_0(2, p) \subset |\mathcal{L}_0|$ has dimension

$$\dim(\mathcal{L}_0(2, p)) \geqslant \dim |\mathcal{L}_0| - 3.$$

2.4. *Local deformation of a singularity of type T_1 to a node*

In §2.3.1, we saw that a singularity of type T_1 appears as a *generic* limit of a double point of a surface. In this section, we will show that locally the converse happens, i.e., that locally a singularity of type T_1 can be deformed to a node.

In local coordinates (x, y, z, u, t), we consider as before the family of 3-folds $\mathcal{X}_t : xy = t$. We further consider the one parameter family of 3-folds in $\mathbb{A}^4$ of local equation at $\underline{0}$ given by

$$\mathcal{S}_\alpha : x - y - \alpha(t) - z^2 - u^2 = 0,$$

where $\alpha(t)$ is a suitable function of $t \in \mathbb{A}^1$ to be determined, such that $\alpha(0) = 0$. We will set $S_t = \mathcal{S}_{\alpha(t)} \cap \mathcal{X}_t$ for any $t \in \mathbb{A}^1$. The surface S_0 has a T_1 singularity at $\underline{0}$ and $\mathcal{S}_\alpha$ is smooth. Our requirement on the function $\alpha(t)$ is that for any $t \neq 0$, there exists a singular point $q(t) = (x(t), y(t), z(t), u(t)) \in S_t$, that is, such that

$$T_{q(t)}(\mathcal{S}_\alpha) = T_{q(t)}(\mathcal{X}_t).$$

This is equivalent to asking if there exists $q(t) = (x(t), y(t), z(t), u(t)) \in \mathbb{A}^4$ satisfying

$$x(t) - y(t) - \alpha(t) - z(t)^2 - u(t)^2 = x(t)y(t) - t = 0$$

and

$$(x - x(t)) - (y - y(t)) - 2z(t)(z - z(t)) - 2u(t)(u - u(t)) = c(t)\Big(y(t)(x - x(t)) + x(t)(y - y(t))\Big),$$

for a non-zero $c(t)$. This implies

$$z(t) = u(t) = 0, x(t) = -y(t), \alpha(t) = 2x(t) \text{ and } t = -x(t)^2 = -\frac{\alpha(t)^2}{4}.$$

Thus, for every $t \neq 0$, there exist exactly two divisors $\mathcal{S}_{\alpha_i}$, with $i = 1, 2$ and $\alpha_i(t)^2 = -4t$ so that

$$S_{\alpha_i(t)} = \mathcal{S}_{\alpha_i} \cap \mathcal{Y}_t : \begin{cases} x = y + \alpha_i(t) + z^2 + u^2 \\ y(y + \alpha_i(t) + z^2 + u^2) = t \end{cases}$$

is a one-nodal surface, with tangent cone at $q_i(t) = (\frac{\alpha_i(t)}{2}, -\frac{\alpha_i(t)}{2}, 0, 0)$ given by

$$TC_{q_i(t)}(S_{\alpha_i(t)}) : x - y - \alpha_i(t) = 2\Big(y + \frac{\alpha_i(t)}{2}\Big)^2 - \alpha_i(t)z^2 - \alpha_i(t)u^2 = 0.$$

Notice that, for every $i = 1, 2$, we have that $\alpha_i(t)$ is a well-defined continuous function on $\mathbb{D}^o_\epsilon = \mathbb{D}(\underline{0}, \epsilon) \setminus \{a + i0 \,|\, 0 < a < \epsilon\}$ (the disk cut along a radius), vanishing at 0 and holomorphic on

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$\mathbb{D}_\epsilon^o \setminus \underline{0}$. Each family $\mathcal{S}_{\alpha_i} \to \mathbb{D}_\epsilon^o$, for $i = 1, 2$, is not algebraic, while the complete intersection family of surfaces

$$\mathcal{D} : \begin{cases} (x - y - u^2 - z^2)^2 = -4t \\ xy = t. \end{cases} \tag{2.10}$$

is algebraic. As usual, we set $D_t = \mathcal{D} \cap \mathcal{X}_t$. One has $D_t = S_{\alpha_1(t)} \cup S_{\alpha_2(t)}$ for $t \neq 0$ and non-reduced fibre $D_0 = 2S_0$ for $t = 0$.

The locus $x^2 + t = x + y = z = u = 0$, whose general point is singular for D_t, is a bisection of $\mathcal{X} \to \mathbb{A}^1$ passing through $(\underline{0}, 0)$.

3. Deformations of surfaces with T_1 singularities and nodes

Throughout this section, we will consider $\mathcal{X} \to \mathbb{D}$ a family of projective complex 3-folds over a disc $\mathbb{D}$ as in the previous section, and we let $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ be its relative Hilbert scheme, whose fiber over $t \in \mathbb{D}$ is the Hilbert scheme of $\mathcal{X}_t$ and it is denoted by $\mathcal{H}^{\mathcal{X}_t}$. Moreover, we will consider $S_0 = S_A \cup S_B \subset \mathcal{X}_0$, with $S_A \subset A$ and $S_B \subset B$ an effective reduced Cartier divisor.

3.1. Deformations and a smoothness criterion

Definition 3.0.1. Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$, with $S_A \subset A$ and $S_B \subset B$ being an effective reduced Cartier divisor and let

$$\mathcal{H}^{\mathcal{X}|\mathbb{D}}_{[S_0]} \to \mathbb{D}$$

be an irreducible component of the relative Hilbert scheme of $\mathcal{X}$ containing $[S_0]$. *A deformation of* S_0 *in* $\mathcal{X}$ *not in* $\mathcal{X}_0$ is the total space $S \subset \mathcal{X}$ of an irreducible local r-multisection γ of $\mathcal{H}^{\mathcal{X}|\mathbb{D}}_{[S_0]}$ passing through $[S_0]$. Equivalently, a deformation of S_0 is an effective divisor

$$\begin{array}{ccc} S & \hookrightarrow & \mathcal{X} \\ & \searrow & \downarrow \\ & & \mathbb{D} \end{array}$$

dominating $\mathbb{D}$, whose central fibre is $S \cap \mathcal{X}_0 = rS_0$ (i.e., the surface S_0 counted with multiplicity r) and whose general fibre is a reduced surface with r irreducible components $S \cap \mathcal{X}_t = S_t^1 \cup \cdots \cup S_t^r$, with $[S_t^i] \in \mathcal{H}^{\mathcal{X}|\mathbb{D}}_{[S_0]}$, for every $i = 1, \ldots, t$. We will also say that every irreducible component S_t^i of $S \cap \mathcal{X}_t$ is a deformation of S_0 or that S_0 is a limit of S_t^i. Let $\mathcal{Y}$ be the smooth family of threefolds obtained from $\mathcal{X} \to \mathbb{D}$ after a base change

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{D} & \xrightarrow{\nu_r} & \mathbb{D} \end{array}$$

of order r and after minimally desingularizing the total space of the obtained family. $\mathcal{Y}$ has central fibre $\mathcal{Y}_0 = A \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{r-1} \cup B$ with normal crossing singularities of multiplicity two, where every $\mathcal{E}_i$ is a $\mathbb{P}^1$-bundle over $\mathcal{E}_{i-1} \cap \mathcal{E}_i$, intersecting $\mathcal{E}_{i-1}$ and $\mathcal{E}_{i+1}$, with $\mathcal{E}_0 = A$ and $\mathcal{E}_r = B$. We denote by $\pi : \mathcal{Y} \to \mathcal{X}$ the induced morphism. Then the pullback divisor $\pi^*(S) = \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^r$ has r irreducible distinct components $\mathcal{S}^1, \ldots, \mathcal{S}^r$, where now every $\mathcal{S}^i$ has irreducible general fibre and has central fibre given by $\mathcal{S}_0^i = \mathcal{S}^i \cap \mathcal{X}_0 \cong S_0$.

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Proposition 3.1. *Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$, with $S_A \subset A$ and $S_B \subset B$, be a reduced effective Cartier divisor as above. Let p be a point of the intersection curve $C = S_A \cap S_B \subset R$ where S_A and S_B intersect transversally (i.e., such that S_A and S_B are smooth at p and $T_p(S_A) \neq T_p(S_B)$). Then for every deformation $S \subset \mathcal{X}$ of S_0 not in $\mathcal{X}_0$, we have that p is limit only of smooth points of the irreducible components of the general fibre of S (i.e., in a sufficiently small analytic neighborhood of p in $\mathcal{X}$, all irreducible components of the general fibre of S are smooth). In particular, if S_A and S_B intersect transversally along C, then S_0 is limit only of smooth surfaces.*

Proof. Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ and $p \in R = S_A \cap S_B$ as in the statement. Then, there exists an analytic coordinate system (x, y, z, u, t) of $\mathcal{X}$ at $p = \underline{0}$ and such that the local equation of S_0 at p is given by $xy = t = z + f_2(x, y, z, u) = 0$, where $f_2(x, y, z, u) \in (x, y, z, u)^2$.

Assume that the assertion is not true. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be the morphism defined in Definition 3.0.1, from which we keep the notation. Then the chain of fibres $F_p^1 \cup \cdots \cup F_p^{r-1}$ of $\pi^{-1}(S_0)$ contracted to p by π intersects the singular locus of every divisor $\mathcal{S}^i$. In particular, there exists an analytic s-multisection γ of $\mathcal{X}$ (with $s \geqslant r$) passing through p, whose general point is a singular point of an irreducible component of $S \cap \mathcal{X}_t$, with t general. Every analytic s-multisection of $\mathcal{X} \to \mathbb{D}$ at p gives rise to s distinct continuous sections $\gamma^1, \ldots, \gamma^s$ over $\mathbb{D}^o = \mathbb{D} \setminus \{a + i0 \mid 0 < a < 1\}$, which are holomorphic on $\mathbb{D}^o \setminus \underline{0}$. If t varies in $\mathbb{D}^o$, then there exists a one-parameter analytic family of irreducible surfaces $\mathcal{Z} \subset S$, singular along γ^1, whose fibre $\mathcal{Z}_t$ over $t \neq 0$ is an irreducible component of $S \cap \mathcal{X}_t$ and whose fibre over 0 is $\mathcal{Z}_0 = S_0$. The equation of $\mathcal{Z}_t$ in $\mathbb{A}^4$ with coordinates (x, y, z, u) is given by

$$\begin{cases} p(x, y, z, u; t) = 0 \\ xy = t, \end{cases}$$

where $p(x, y, z, u; t) = 0$ is an analytic function in (x, y, z, u), whose coefficients are continuous functions in the variable $t \in \mathbb{D}^o$ which are holomorphic on $\mathbb{D}^o \setminus \underline{0}$. If

$$\gamma^1(t) = (x(t), y(t), z(t), u(t)),$$

then by the hypothesis that the general fibre of $\mathcal{Z}$ is singular along γ, we have that

$$p(x, y, z, u; t) = c(t)(y(t)(x - x(t)) + x(t)(y - y(t))) + g_2(x - x(t), y - y(t), z - z(t), u - u(t)),$$

where $g_2(x - x(t), y - y(t), z - z(t), u - u(t)) \in (x - x(t), y - y(t), z - z(t), u - u(t))^2$.

We moreover have that $p(x, y, z, u; t)$ specializes to $p(x, y, z, u; 0) = z + f_2(x, y, z, u)$ $= z - z(0) + f_2(x - x(0), y - y(0), z - z(0), u - u(0))$ as t goes to 0. This is not possible. Thus, every irreducible component of the general fibre of a deformation $S \subset \mathcal{X}$ of S_0 is smooth in a neighborhood of p. □

3.2. *Deformations of T_1 singularities*

3.2.1. Deformations not in $\mathcal{X}_0$ of surfaces with T_1 singularities

In this section, we prove that the only singularity of a surface in $\mathcal{X}_t$, with $t \neq 0$, to which a T_1 singularity of a surface $S_0 \subset \mathcal{X}_0$ may be deformed is a node.

Lemma 3.2. *Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ be a reduced effective Cartier divisor, with $S_A \subset A$ and $S_B \subset B$ as above. Let p be a point of the intersection curve $C = S_A \cap S_B \subset R$, where S_0 has a T_1 singularity. Let $S \subset \mathcal{X}$ be a deformation of S_0 not in $\mathcal{X}_0$. Then there exists a sufficiently small analytic neighborhood of p in $\mathcal{X}$ such that all irreducible components of the general fibre of S in that neighborhood are smooth or are 1-nodal.*

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Proof. By Proposition 3.1, if $S \subset \mathcal{X}$ is any deformation of S_0 not in $\mathcal{X}_0$, then all irreducible components of the general fibre of S have, in a sufficiently small neighborhood of p, only isolated singularities. We want to prove that if the T_1 singularity of S_0 at p is limit of an isolated singularity, then this is a node. We argue as in the proof of Proposition 3.1.

Let $\mathbb{D}_\epsilon = \mathbb{D}(\underline{0}, \epsilon) \subset \mathbb{A}^1$ be the open disc with center at the origin and radius ϵ and let $\mathbb{D}_\epsilon^o = \mathbb{D}(\underline{0}, \epsilon) \setminus \{a + i0 \,|\, 0 < a < \epsilon\}$. We denote by $t = a + ib$ the coordinate on $\mathbb{D}_\epsilon$ and by (x, y, z, u) the coordinates in $\mathbb{A}^4$. In $\mathbb{A}^4 \times \mathbb{D}_\epsilon^o$, we consider a one parameter family of 3-folds

$$\mathcal{S}_t : p(x, y, z, u; t) = 0, \;\; t \in \mathbb{D}_\epsilon^o,$$

where $p(x, y, z, u; t)$ is a polynomial in x, y, z, u whose coefficients are holomorphic functions on $\mathbb{D}_\epsilon^o \setminus \underline{0}$, continuous in $\underline{0}$, and the one parameter family of 3-folds

$$\mathcal{X}_t : xy = t, \;\; t \in \mathbb{D}_\epsilon^o.$$

Assume that the surface

$$S_0 = \mathcal{S}_0 \cap \mathcal{X}_0 : \begin{cases} p(x, y, z, u; 0) = x + y + p_2(x, y, z, u) + o(3) = 0 \\ xy = 0 \end{cases} \tag{3.1}$$

has a T_1 singularity at $\underline{0}$, where $p_2(x, y, z, u)$ is the homogeneous part of degree 2 of $p(x, y, z, u; 0)$, where $o(3)$ is the sum of terms of degree at least 3 in $p(x, y, z, u; 0)$, and where, by assumption, $p_2(0, 0, z, u)$ has nonzero discriminant.

Assume that, for $t \neq 0$, there exists $q(t) = (x(t), y(t), z(t), w(t); t) \in S_t = \mathcal{S}_t \cap \mathcal{X}_t$ specializing to $\underline{0}$, as t goes to $\underline{0}$ and such that S_t has a singular point at $q(t)$. Thus, $\mathcal{S}_t$ is smooth at $q(t)$ since $\mathcal{S}_0$ is smooth at $q(0) = \underline{0}$, and we have that

$$T_{q(t)}(\mathcal{S}_t) = T_{q(t)}(\mathcal{X}_t).$$

In particular, there exists a function $c(t)$, which is nonzero if $t \neq 0$, such that

$$\begin{aligned} y(t)(x - x(t)) + x(t)(y - y(t)) = c(t)\frac{\partial p}{\partial x}|_{q(t)}(x - x(t)) + c(t)\frac{\partial p}{\partial y}|_{q(t)}(y - y(t)) \\ + c(t)\frac{\partial p}{\partial z}|_{q(t)}(z - z(t)) + c(t)\frac{\partial p}{\partial u}|_{q(t)}(u - u(t)), \end{aligned}$$

from which we deduce that

$$y(t) = c(t)\frac{\partial p}{\partial x}|_{q(t)}, \;\; x(t) = c(t)\frac{\partial p}{\partial y}|_{q(t)}, \tag{3.2}$$

$$\frac{\partial p}{\partial z}|_{q(t)} = 0 \text{ and } \frac{\partial p}{\partial u}|_{q(t)} = 0. \tag{3.3}$$

As t goes to $\underline{0}$, $c(0) = 0$, since $x(t) \neq 0 \neq y(t)$ if $t \neq 0$ but $x(0) = y(0) = 0$ and $\frac{\partial p}{\partial x}|_{q(t)} \neq 0 \neq \frac{\partial p}{\partial y}|_{q(t)}$ for any t in a neighborhood of $\underline{0}$. We now write down the local equations

$$\mathcal{X}_t : y(t)(x - x(t)) + x(t)(y - y(t)) + (x - x(t))(y - y(t)) = 0 \tag{3.4}$$

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of $\mathcal{X}_t$ at $q(t)$, and the local equation

$$\mathcal{S}_t : \frac{\partial p}{\partial x}|_{q(t)}(x - x(t)) + \frac{\partial p}{\partial y}|_{q(t)}(y - y(t)) \tag{3.5}$$
$$+ \frac{\partial p}{\partial x \partial y}|_{q(t)}(x - x(t))(y - y(t)) + \frac{\partial p}{\partial x \partial z}|_{q(t)}(x - x(t))(z - z(t))$$
$$+ \frac{\partial p}{\partial x \partial u}|_{q(t)}(x - x(t))(u - u(t)) + \frac{\partial p}{\partial y \partial z}|_{q(t)}(y - y(t))(z - z(t))$$
$$+ \frac{\partial p}{\partial y \partial u}|_{q(t)}(y - y(t))(u - u(t)) + \frac{\partial p}{\partial u \partial z}|_{q(t)}(u - u(t))(z - z(t))$$
$$+ \frac{1}{2}\frac{\partial p}{\partial x^2}|_{q(t)}(x - x(t))^2 + \frac{1}{2}\frac{\partial p}{\partial y^2}|_{q(t)}(y - y(t))^2 + \frac{1}{2}\frac{\partial p}{\partial u^2}|_{q(t)}(u - u(t))^2$$
$$+ \frac{1}{2}\frac{\partial p}{\partial z^2}|_{q(t)}(z - z(t))^2 + o(3) = 0$$

of $\mathcal{S}_t$ at $q(t)$, where $o(3) \in (x - x(t), y - y(t), z - z(t), u - u(t))^3$. By (3.4), one may write

$$x - x(t) = -\frac{x(t)(y - y(t))}{y - y(t) + y(t)}. \tag{3.6}$$

Let d be the maximum degree of $x - x(t)$ in (3.5). By substituting in (3.5), by multiplying by $y^d = (y - y(t) + y(t))^d$, and by using that $T_{q(t)}(\mathcal{S}_t) = T_{q(t)}(\mathcal{X}_t)$ (i.e., $y(t)\frac{\partial p}{\partial y}|_{q(t)} = x(t)\frac{\partial p}{\partial x}|_{q(t)}$) for any $t \neq 0$, we find that the local equation of $S_t = \mathcal{S}_t \cap \mathcal{X}_t$ is given by

$$S_t : \frac{\partial p}{\partial x}|_{q(t)}\Big(-x(t)(y - y(t))\Big)\Big(y - y(t) + y(t)\Big)^{d-1}$$
$$+ \Big(\frac{\partial p}{\partial x \partial y}|_{q(t)}(y - y(t))\Big)\Big(-x(t)(y - y(t))\Big)\Big(y - y(t) + y(t)\Big)^{d-1}$$
$$+ \Big(\frac{\partial p}{\partial x \partial z}|_{q(t)}(z - z(t))\Big)\Big(-x(t)(y - y(t))\Big)\Big(y - y(t) + y(t)\Big)^{d-1}$$
$$+ \Big(\frac{\partial p}{\partial x \partial u}|_{q(t)}(u - u(t))\Big)\Big(-x(t)(y - y(t))\Big)\Big(y - y(t) + y(t)\Big)^{d-1}$$
$$+ \frac{1}{2}\frac{\partial p}{\partial x^2}|_{q(t)}\Big(-x(t)(y - y(t))\Big)^2\Big(y - y(t) + y(t)\Big)^{d-2}$$
$$+ \frac{\partial p}{\partial y}|_{q(t)}\Big(y - y(t)\Big)\Big(y - y(t) + y(t)\Big)^{d}$$
$$+ \Big(\frac{\partial p}{\partial y \partial z}|_{q(t)}(z - z(t)) + \frac{\partial p}{\partial y \partial u}|_{q(t)}(u - u(t))\Big)\Big(y - y(t)\Big)\Big(y - y(t) + y(t)\Big)^{d}$$
$$+ \Big(\frac{1}{2}\frac{\partial p}{\partial y^2}|_{q(t)}(y - y(t))^2 + \frac{\partial p}{\partial u \partial z}|_{q(t)}(u - u(t))(z - z(t))\Big)\Big(y - y(t) + y(t)\Big)^{d}$$
$$+ \Big(\frac{1}{2}\frac{\partial p}{\partial u^2}|_{q(t)}(u - u(t))^2 + \frac{1}{2}\frac{\partial p}{\partial z^2}|_{q(t)}(z - z(t))^2\Big)\Big(y - y(t) + y(t)\Big)^{d} + o(3)$$
$$= y(t)^{d-2}\Big(\frac{\partial p}{\partial y}|_{q(t)}y(t) - \frac{\partial p}{\partial x \partial y}|_{q(t)}x(t)y(t) + \frac{1}{2}\frac{\partial p}{\partial x^2}|_{q(t)}x(t)^2 + \frac{1}{2}\frac{\partial p}{\partial y^2}|_{q(t)}y(t)^2\Big)\Big(y - y(t)\Big)^2$$
$$+ y(t)^{d-1}\Big(y(t)\frac{\partial p}{\partial y \partial z}|_{q(t)} - x(t)\frac{\partial p}{\partial x \partial z}|_{q(t)}\Big)\Big(y - y(t)\Big)\Big(z - z(t)\Big)$$

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$$+\, y(t)^{d-1}\Big(y(t)\frac{\partial p}{\partial y\partial u}|_{q(t)} - x(t)\frac{\partial p}{\partial x\partial u}|_{q(t)}\Big)\Big(y-y(t)\Big)\Big(u-u(t)\Big)$$
$$+\, y(t)^{d}\Big(\frac{1}{2}\frac{\partial p}{\partial z^2}|_{q(t)}(z-z(t))^2 + \frac{1}{2}\frac{\partial p}{\partial u^2}|_{q(t)}(u-u(t))^2\Big)$$
$$+\, y(t)^{d}\Big(\frac{\partial p}{\partial u\partial z}|_{q(t)}(u-u(t))(z-z(t))\Big) + o(3) = 0.$$

Up to the irrelevant factor $y(t)^{d-2}$, the Hessian matrix at $q(t)$ of the above polynomial is

$$H_{q(t)} = \begin{pmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{12}(t) & \frac{y(t)^2}{2}\frac{\partial p}{\partial z^2}|_{q(t)} & \frac{y(t)^2}{2}\frac{\partial p}{\partial z\partial u}|_{q(t)} \\ A_{13}(t) & \frac{y(t)^2}{2}\frac{\partial p}{\partial z\partial u}|_{q(t)} & \frac{y(t)^2}{2}\frac{\partial p}{\partial u^2}|_{q(t)} \end{pmatrix}, \tag{3.7}$$

where

$$A_{11}(t) = \frac{\partial p}{\partial y}|_{q(t)}y(t) - \frac{\partial p}{\partial x\partial y}|_{q(t)}x(t)y(t) + \frac{1}{2}\frac{\partial p}{\partial x^2}|_{q(t)}x(t)^2 + \frac{1}{2}\frac{\partial p}{\partial y^2}|_{q(t)}y(t)^2,$$
$$A_{12}(t) = \frac{y(t)}{2}\Big(y(t)\frac{\partial p}{\partial y\partial z}|_{q(t)} - x(t)\frac{\partial p}{\partial x\partial z}|_{q(t)}\Big),$$
$$A_{13}(t) = \frac{y(t)}{2}\Big(y(t)\frac{\partial p}{\partial y\partial u}|_{q(t)} - x(t)\frac{\partial p}{\partial x\partial u}|_{q(t)}\Big).$$

Now S_t has a node at $q(t)$ if and only if $\det(H_{q(t)}) \neq 0$. If we substitute the equalities (3.2) in $H_{q(t)}$, we see that this matrix has the first column divisible by $c(t)$ and the second and third columns divisible by $c(t)^2$. Let $B_{q(t)}$ be the matrix obtained by $H_{q(t)}$ by dividing the first column by $c(t)$ and the second and third columns by $c(t)^2$. We have that

$$B_{q(t)} = \begin{pmatrix} B_{11}(t) & B_{12}(t) & B_{13}(t) \\ B_{21}(t) & \frac{1}{2}(\frac{\partial p}{\partial x}|_{q(t)})^2\frac{\partial p}{\partial z^2}|_{q(t)} & \frac{1}{2}(\frac{\partial p}{\partial x}|_{q(t)})^2\frac{\partial p}{\partial z\partial u}|_{q(t)} \\ B_{31}(t) & \frac{1}{2}(\frac{\partial p}{\partial x}|_{q(t)})^2\frac{\partial p}{\partial z\partial u}|_{q(t)} & \frac{1}{2}(\frac{\partial p}{\partial x}|_{q(t)})^2\frac{\partial p}{\partial u^2}|_{q(t)} \end{pmatrix}, \tag{3.8}$$

where

$$B_{11}(t) = \frac{\partial p}{\partial y}|_{q(t)}\frac{\partial p}{\partial x}|_{q(t)} + c(t)\Big(-\frac{\partial p}{\partial x\partial y}|_{q(t)}\frac{\partial p}{\partial x}|_{q(t)}\frac{\partial p}{\partial y}|_{q(t)}$$
$$+\frac{1}{2}\frac{\partial p}{\partial x^2}|_{q(t)}\Big(\frac{\partial p}{\partial y}|_{q(t)}\Big)^2 + \frac{1}{2}\frac{\partial p}{\partial y^2}|_{q(t)}\Big(\frac{\partial p}{\partial x}|_{q(t)}\Big)^2\Big),$$
$$B_{21}(t) = \frac{1}{2}\frac{\partial p}{\partial x}|_{q(t)}c(t)\Big(\frac{\partial p}{\partial x}|_{q(t)}\frac{\partial p}{\partial y\partial z}|_{q(t)} - \frac{\partial p}{\partial y}|_{q(t)}\frac{\partial p}{\partial x\partial z}|_{q(t)}\Big),$$
$$B_{31}(t) = \frac{1}{2}\frac{\partial p}{\partial x}|_{q(t)}c(t)\Big(\frac{\partial p}{\partial x}|_{q(t)}\frac{\partial p}{\partial y\partial u}|_{q(t)} - \frac{\partial p}{\partial y}|_{q(t)}\frac{\partial p}{\partial x\partial u}|_{q(t)}\Big).$$

As t goes to 0, $c(t)$ goes to 0 and the matrix $B_{q(t)}$ specializes to the matrix

$$B_{\underline{0}} = \begin{pmatrix} \frac{\partial p}{\partial y}|_{\underline{0}}\frac{\partial p}{\partial x}|_{\underline{0}} & B_{12}(0) & B_{13}(0) \\ 0 & \frac{1}{2}(\frac{\partial p}{\partial x}|_{\underline{0}})^2\frac{\partial p}{\partial z^2}|_{\underline{0}} & \frac{1}{2}(\frac{\partial p}{\partial x}|_{\underline{0}})^2\frac{\partial p}{\partial z\partial u}|_{\underline{0}} \\ 0 & \frac{1}{2}(\frac{\partial p}{\partial x}|_{\underline{0}})^2\frac{\partial p}{\partial z\partial u}|_{\underline{0}} & \frac{1}{2}(\frac{\partial p}{\partial x}|_{\underline{0}})^2\frac{\partial p}{\partial u^2}|_{\underline{0}} \end{pmatrix}. \tag{3.9}$$

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Using that $\frac{\partial p}{\partial x}|_{\underline{0}} = 1 = \frac{\partial p}{\partial y}|_{\underline{0}}$, we see that $\det(B_{\underline{0}})$ coincides with the discriminant of the degree 2 homogeneous polynomial $p_2(0,0,z,u)$, which is nonzero by the hypothesis that S_0 has a T_1 singularity at $\underline{0}$. We finally deduce that $\det(B_{q(t)}) \neq 0 \neq \det(H_{q(t)})$ and thus the surface S_t has a node at $q(t)$ for $t \neq 0$. □

3.2.2. Equisingular deformations of surfaces with T_1 singularities

We go on considering the setting we introduced at the beginning of Section 3. Assume that $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ is a surface with S_A and S_B smooth, intersecting transversally along $R = A \cap B$, except for δ distinct points $p_1, \dots p_\delta \in S_A \cap S_B$, where S_0 has a singularity of type T_1. We recall the standard exact sequence

$$0 \longrightarrow \Theta_{S_0} \longrightarrow \Theta_{\mathcal{X}}|_{S_0} \xrightarrow{\alpha} \mathcal{N}_{S_0|\mathcal{X}} \xrightarrow{\beta} T^1_{S_0} \longrightarrow 0, \tag{3.10}$$

where $\Theta_{S_0} = \mathfrak{hom}(\Omega^1_{S_0}, \mathcal{O}_{S_0})$ is the tangent sheaf of S_0, $\Theta_{\mathcal{X}}|_{S_0}$ is the tangent sheaf of $\mathcal{X}$ restricted to S_0, $\mathcal{N}_{S_0|\mathcal{X}}$ is the normal bundle of S_0 in $\mathcal{X}$, and $T^1_{S_0}$ is the first cotangent sheaf of S_0 [20, Section 1.1.3]. The latter is supported on the singular locus $\mathrm{Sing}(S_0) = S_A \cap S_B$. The kernel $\mathcal{N}'_{S_0|\mathcal{X}}$ of β is the so-called *equisingular normal sheaf to* S_0 *in* $\mathcal{X}$, whose global sections are the first-order locally trivial deformations of S_0 in $\mathcal{X}$.

In the sequel, an equisingular (first-order) deformation of S_0 in $\mathcal{X}$ will be a (first-order) locally trivial deformation of S_0 in $\mathcal{X}$.

We will denote by $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]} \subseteq \mathcal{H}^{\mathcal{X}_0} \subset \mathcal{H}^{\mathcal{X}|\mathbb{D}}$ the locally closed set of equisingular deformations of S_0 in $\mathcal{X}_0$. Similarly, if $p \in S_0$ is a point, we will denote by $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p} \subseteq \mathcal{H}^{\mathcal{X}_0} \subset \mathcal{H}^{\mathcal{X}|\mathbb{D}}$ the locally closed set of deformations of S_0 in $\mathcal{X}_0$, which are equisingular at p.

Lemma 3.3. *If* $\mathcal{ES}^{\mathcal{X}|\mathbb{D}}_{[S_0]} \subseteq \mathcal{H}^{\mathcal{X}|\mathbb{D}}$ *is the locally closed set of equisingular deformations of* S_0 *in* $\mathcal{X}$*, then* $\mathcal{ES}^{\mathcal{X}|\mathbb{D}}_{[S_0]}$ *coincides set-theoretically with* $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]} \subseteq \mathcal{H}^{\mathcal{X}_0}$.

Proof. This is a straightforward consequence of Lemma 3.2. □

If $\mathbb{T}_\delta \subset \mathcal{H}^{\mathcal{X}_0}$ is the Zariski closure of the family of surfaces in $\mathcal{X}_0$ with δ singularities of type T_1, then every irreducible component of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ is a Zariski open set in an irreducible component of $\mathbb{T}_\delta$.

Consider the rational map

$$\varphi : \mathcal{H}^{\mathcal{X}_0} \dashrightarrow \mathcal{H}^R,$$

where $\mathcal{H}^R$ is the Hilbert scheme of R, and φ maps the general subscheme of $\mathcal{X}_0$ to its intersection with R.

Lemma 3.4. *Let* $[S_0] \in \mathbb{T}_\delta$ *be a point corresponding to a surface* $S_0 = S_A \cup S_B$ *as above and suppose that* $[S_0]$ *is a smooth point of the Hilbert scheme* $\mathcal{H}^{\mathcal{X}_0}$ *so that there is a unique component* $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ *of* $\mathcal{H}^{\mathcal{X}_0}$ *containing* S_0*. Let* C *be the curve cut out by* S_0 *on* R*. Assume that* $h^1(\mathcal{N}_{C|R} \otimes \mathcal{I}_{\{p_1,\dots,p_\delta\}|R}) = 0$*, where* $p_1, \dots, p_\delta$ *are the nodes of* C*, which implies that* $\mathcal{H}^R$ *is smooth at the point* $[C]$ *and that the Severi variety of curves on* R *with* δ *nodes is smooth at the point* $[C]$ *of codimension* δ *in the unique irreducible component* $\mathcal{H}^R_C$ *of the Hilbert scheme* $\mathcal{H}^R$ *containing* $[C]$*. Suppose moreover that the map*

$$\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}} : \mathcal{H}^{\mathcal{X}_0}_{S_0} \dashrightarrow \mathcal{H}^R_C \tag{3.11}$$

is dominant.

Then there is an irreducible component $\mathbb{T}$ *of* $\mathbb{T}_\delta$ *containing* S_0 *that has codimension at most* δ *in* $\mathcal{H}^{\mathcal{X}_0}_{S_0}$.

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Proof. Let $V \subset \mathcal{H}^R_C$ be the unique irreducible component of the locally closed set of curves on R with δ nodes that contains the point $[C]$. Since $[C]$ sits in the image of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$, V intersects the image of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$. Since $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ is dominant, the intersection of V with the image of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ is an open dense subset of V; hence, there is an irreducible component $\mathbb{T}$ of $\mathbb{T}_\delta$ containing S_0 such that the map

$$\varphi_{|\mathbb{T}} : \mathbb{T} \dashrightarrow V$$

is dominant. Let a be the dimension of the general fibre of $\varphi_{|\mathbb{T}}$ and let b be the dimension of the general fibre of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$. Of course, $a \geqslant b$. We have

$$\dim(\mathbb{T}) = \dim(V) + a, \text{ and } \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) = \dim(\mathcal{H}^R_C) + b.$$

Hence,

$$\dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - \dim(\mathbb{T}) = \dim(\mathcal{H}^R_C) - \dim(V) + b - a \leqslant \delta,$$

and the assertion follows. □

Remark 3.5. Note that in the previous lemma, one has that $\mathbb{T}$ has exactly codimension δ in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ if and only if $a = b$.

Lemma 3.6. *Assume that $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ is a surface with S_A and S_B smooth, intersecting transversally along $R = A \cap B$, except for δ points $p_1, \dots, p_\delta \in S_A \cap S_B$, where S_0 has a singularity of type T_1. Then the equisingular first-order infinitesimal deformations of S_0 in $\mathcal{X}$ coincide with the equisingular first-order infinitesimal deformations of S_0 in $\mathcal{X}_0$. More precisely, we have that*

$$H^0(S_0, \mathcal{N}'_{S_0|\mathcal{X}}) \subseteq H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}), \tag{3.12}$$

where $I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}$ is the ideal sheaf of $\{p_1, \dots, p_\delta\}$ in $\mathcal{X}_0$.

Proof. Let $p = p_i$, for $i = 1, \dots, \delta$, be a point where S_0 has a T_1 singularity. Consider the localized exact sequence

$$0 \longrightarrow \mathcal{N}'_{S_0|\mathcal{X},\,p} \longrightarrow \mathcal{N}_{S_0|\mathcal{X},\,p} \longrightarrow T^1_{S_0,p} \longrightarrow 0. \tag{3.13}$$

Let (x, y, z, u, t) be an analytic coordinate system of $\mathcal{X}$ centered at p such that $\mathcal{X}$ is given by $xy = t$ and such that we have the following identifications:

- the local ring $\mathcal{O}_{S_0,\,p} = \mathcal{O}_{\mathcal{X},p}/\mathcal{I}_{S_0|\mathcal{X},p}$ of S_0 at p is identified with $\mathbb{C}[x, y, z, u]/(h_1, h_2)$, localized at the origin, where $h_1(x, y, z, u) = x + y + h_{12}(x, y, z, u)$, $h_{12}(x, y, z, u) \in (x, y, z, u)^2$ and $h_{12}(0, 0, z, u) = 0$ having a node at $\underline{0} = p$, and $h_2(x, y, z, u) = xy$;
- the $\mathcal{O}_{S_0,p}$-module $\mathcal{N}_{S_0|\mathcal{X},\,p}$ is identified with the free $\mathcal{O}_{\mathcal{X},\,p}$-module $\mathfrak{hom}_{\mathcal{O}_{\mathcal{X},\,p}}(\mathcal{I}_{S_0|\mathcal{X},\,p}, \mathcal{O}_{S_0,p})$, generated by the morphisms h_1^* and h_2^*, defined by

$$h_i^*(s_1(x, y, z, u)h_1(x, y, z, u) + s_2(x, y, z, u)h_2(x, y, z, u)) = s_i(x, y, z, u), \text{ for } i = 1,\ 2$$

and, finally,

- the $\mathcal{O}_{S_0,p}$-module

$$\begin{aligned}(\Theta_{\mathcal{X}}|_{S_0})_p &\simeq \Theta_{\mathcal{X},p} \otimes \mathcal{O}_{S_0,p} \\ &\simeq \langle \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial u, \partial/\partial t\rangle_{\mathcal{O}_{S_0,\,p}} / \langle \partial/\partial t - x\partial/\partial y - y\partial/\partial x\rangle\end{aligned}$$

  is identified with the free $\mathcal{O}_{\mathcal{X},\,p}$-module generated by the derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial u$.

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With these identifications, the localization $\alpha_p : (\Theta_{\mathcal{X}}|_{S_0})_p \to \mathcal{N}_{S_0|\mathcal{X},\,p}$ of the sheaf map α from (3.10) is defined by

$$\begin{aligned}\alpha_p(\partial/\partial x) &= \Big(s = s_1h_1 + s_2h_2 \rightarrowtail \partial s/\partial x =_{\mathcal{O}_{S_0,p}} s_1\partial h_1/\partial x + s_2\partial h_2/\partial x\Big)\\ &= (1+\partial h_{12}/\partial x)h_1^* + yh_2^*,\\ \alpha_p(\partial/\partial y) &= (1+\partial h_{12}/\partial y)h_1^* + xh_2^*,\\ \alpha_p(\partial/\partial z) &= (\partial h_{12}/\partial z)h_1^* \text{ and}\\ \alpha_p(\partial/\partial u) &= (\partial h_{12}/\partial u)h_1^*.\end{aligned}$$

By definition of $\mathcal{N}'_{S_0|\mathcal{X}}$, a local section s of $\mathcal{N}'_{S_0|\mathcal{X},\,p}$ is such that there exists a local section v of $\Theta_{\mathcal{X}}|_{S_0\,p}$ with

$$v = v_x(x,y,z,u)\partial/\partial x + v_y(x,y,z,u)\partial/\partial y + v_z(x,y,z,u)\partial/\partial z + v_u(x,y,z,u)\partial/\partial u,$$

such that $s = \alpha_p(v)$. Hence, locally at p, first-order equisingular deformations of S_0 in $\mathcal{X}$ have equations

$$\begin{cases} x+y+h_{12}(x,y,z,u) & + \;\epsilon\Big(v_x(1+\partial h_{12}/\partial x) + v_y(1+\partial h_{12}/\partial y)\\ & + \quad v_z(\partial h_{12}/\partial z) + v_u(\partial h_{12}/\partial u)\Big) = 0\\ xy + \epsilon(yv_x + xv_y) & = 0. \end{cases} \tag{3.14}$$

The first equation above gives a first-order infinitesimal deformation of the Cartier divisor cutting S_0 on $\mathcal{X}_0$, while the second equation gives a first-order infinitesimal deformation of $\mathcal{X}_0$ in $\mathcal{X}$. More precisely, by the exact sequence

$$0 \longrightarrow \Theta_{\mathcal{X}_0} \longrightarrow \Theta_{\mathcal{X}}|_{\mathcal{X}_0} \longrightarrow \mathcal{N}_{\mathcal{X}_0|\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}_0} \longrightarrow T^1_{\mathcal{X}_0} \cong \mathcal{O}_R \longrightarrow 0,$$

one sees that $xy + \epsilon(yv_x + xv_y) = 0$ is the local equation at p of a first-order equisingular deformation of $\mathcal{X}_0$ in $\mathcal{X}$. But $H^0(\mathcal{X}_0, \mathcal{N}'_{\mathcal{X}_0|\mathcal{X}}) = H^0(\mathcal{X}_0, \mathcal{I}_{R|\mathcal{X}_0}) = 0$. It follows that the polynomial $yv_x(x,y,z,u) + xv_y(x,y,z,u)$ in the second equation of (3.14) must be identically zero, proving the first assertion of the lemma. In particular, by expanding v_x and v_y in Taylor series, we see that

$$v_x(\underline{0}) = v_y(\underline{0}) = 0.$$

Looking at the first equation of (3.14), we have that $\frac{\partial h_{12}}{\partial z}(\underline{0}) = \frac{\partial h_{12}}{\partial u}(\underline{0}) = 0$ since $h_{12}(x,y,z,u) \in (x,y,z,u)^2$. This shows the inclusion (3.12). □

Remark 3.7. The argument in the proof of Lemma 3.6 proves more than stated. In fact, it proves that if S_0 is any surface in $\mathcal{X}_0$ with δ singularities of type T_1 at $p_1, \ldots, p_\delta$ (and may be other singularities which we do not care about), the first-order infinitesimal deformations of S_0 in $\mathcal{X}$ which are equisingular at $p_1, \ldots, p_\delta$ are a linear subspace of $H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\ldots,p_\delta\}|\mathcal{X}_0})$.

Corollary 3.7.1. *Same hypotheses as in Lemma 3.4. Assume moreover that*

$$H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\ldots,p_\delta\}|\mathcal{X}_0}) = 0 \tag{3.15}$$

or, equivalently, that

$$H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0 \text{ and } h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\ldots,p_\delta\}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta \tag{3.16}$$

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(assuring that the Hilbert scheme $\mathcal{H}^{\mathcal{X}_0}$ is smooth at $[S_0]$). Then the schemes $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ and $\mathbb{T}_\delta$ are smooth at $[S_0]$ of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}))$, with tangent space $T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}) \simeq T_{[S_0]}(\mathbb{T}_\delta) \simeq H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0})) \simeq H^0(S_0, \mathcal{N}'_{S_0|\mathcal{X}})$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0} \longrightarrow \mathcal{N}_{S_0|\mathcal{X}_0} \longrightarrow \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{O}_{\{p_1,\dots,p_\delta\}} \longrightarrow 0, \tag{3.17}$$

from which one deduces that $h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}) = 0$ if and only if $h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0$ and

$$h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta.$$

Assume that $h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}) = 0$. Thus, $[S_0]$ is a smooth point of $\mathcal{H}^{\mathcal{X}_0}$. In particular, there exists a unique component $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ of $\mathcal{H}^{\mathcal{X}_0}$ containing $[S_0]$ and having dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0})$ at $[S_0]$. Now, by Lemma 3.4, one has that

$$h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta = \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - \delta \leqslant \dim_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]})) \leqslant \dim(T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]})).$$

However, by (3.12), one has that

$$\dim(T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]})) \leqslant h^0(S_0, \mathcal{N}'_{S_0|\mathcal{X}}) \leqslant h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta.$$

The corollary follows. □

We note the following:

Lemma 3.8. *Let $[S_0] \in \mathcal{H}^{\mathcal{X}|\mathbb{D}}$ be any point corresponding to a reduced effective Cartier divisor $S_0 = S_A \cup S_B \subset \mathcal{X}_0$. Assume that $H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0$. Then the space of first-order infinitesimal deformations of S_0 in $\mathcal{X}$ is given by*

$$H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}}) \simeq H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) \oplus H^0(S_0, \mathcal{O}_{S_0}),$$

and

$$H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}}) \simeq H^1(S_0, \mathcal{O}_{S_0})$$

is an obstruction space for $\mathcal{O}_{\mathcal{H}^{\mathcal{X}|\mathbb{D}},[S_0]}$.

Proof. By the hypothesis, we have $\operatorname{Ext}^1(\mathcal{O}_{S_0}, \mathcal{N}_{S_0|\mathcal{X}_0}) \simeq H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0$, and by the exact sequence

$$0 \longrightarrow \mathcal{N}_{S_0|\mathcal{X}_0} \longrightarrow \mathcal{N}_{S_0|\mathcal{X}} \longrightarrow \mathcal{N}_{\mathcal{X}_0|\mathcal{X}}|_{S_0} = \mathcal{O}_{S_0} \longrightarrow 0, \tag{3.18}$$

we have that

$$\mathcal{N}_{S_0|\mathcal{X}} \simeq \mathcal{N}_{S_0|\mathcal{X}_0} \oplus \mathcal{O}_{S_0}.$$

The statement then follows by standard deformation theory. □

Corollary 3.8.1. *Let $[S_0] \in \mathcal{H}^{\mathcal{X}|\mathbb{D}}$ be a point corresponding to a reduced effective Cartier divisor $S_0 = S_A \cup S_B \subset \mathcal{X}_0$. Assume that $[S_0]$ belongs to an irreducible component $\mathcal{H}$ of $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ that dominates $\mathbb{D}$. Suppose that $H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0$. Then $[S_0]$ is a smooth point for $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ and* $\dim(\mathcal{H}) = \dim(\mathcal{H}^{\mathcal{X}_0}_{[S_0]}) + 1$.

Proof. One has $\dim(\mathcal{H}) \geqslant \dim(\mathcal{H}^{\mathcal{X}_0}_{[S_0]}) + 1 = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) + 1$. However, $\dim(\mathcal{H}) \leqslant h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) + 1$ by Lemma 3.8. The assertion follows. □

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Corollary 3.8.2. *In the same setting as in Lemma 3.6 and same hypotheses as in Lemma 3.4, suppose that (3.15) (or equivalently (3.16)) holds, assuring that $\mathbb{T}_\delta$ is smooth at $[S_0]$ of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_\delta\}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta$. Then for every positive integer $r < \delta$, the variety $\mathbb{T}_r$ is nonempty and $[S_0] \in \mathbb{T}_r$. More precisely, in an analytic neighborhood of $[S_0]$, $\mathbb{T}_r$ consists of $\binom{\delta}{r}$ smooth analytic branches each of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - r$ that intersect at $[S_0]$ along a smooth analytic branch of $\mathbb{T}_\delta$.*

Informally speaking, this is saying that the δ singularities of type T_1 of S_0 can be independently smoothed inside $\mathcal{X}_0$.

Proof. We first prove the assertion for $r = 1$. Let $\mathbb{T}$ be the closure of the subset of the Hilbert scheme $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ consisting of all surfaces S'_0 such that the intersection curve of S'_0 with $R = A \cap B$ is singular with at most nodes. Observe that $[S_0] \in \mathbb{T}$.

We claim that any irreducible component of $\mathbb{T}$ that contains $[S_0]$ has exactly codimension 1 in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$. Indeed, let $\mathbb{T}'$ be such a component. Consider the dominant map $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ as in (3.11), which is defined at a general point of $\mathbb{T}'$.

By our hypotheses, the general element in $\mathcal{H}^R_C$ is a smooth curve; hence, $\mathbb{T}'$ has codimension at least 1 in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$. Let $\mathcal{T}$ be the image of the restriction of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ to $\mathbb{T}'$. Then $\mathcal{T}$ has codimension 1 in $\mathcal{H}^R_C$.

Let α be the dimension of the general fibre of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ and let β be the dimension of the general fibre of the restriction of $\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}$ to $\mathbb{T}'$. One has $\alpha \leqslant \beta$. Then

$$\dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) = \dim(\mathcal{H}^R_C) + \alpha$$

and

$$\dim(\mathbb{T}') = \dim(\mathcal{T}) + \beta = \dim(\mathcal{H}^R_C) - 1 + \beta \geqslant \dim(\mathcal{H}^R_C) - 1 + \alpha = \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - 1.$$

Since $\dim(\mathbb{T}') < \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0})$, we have $\dim(\mathbb{T}') = \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - 1$ and $\alpha = \beta$, as claimed.

Now, we consider a suitably small analytic open neighborhood U of $[S_0]$ in $\mathbb{T}$. Every surface S'_0 such that $[S'_0] \in U$ has at most δ singularities of type T_1.

Consider the variety $I \subset U \times R$ consisting of all pairs $([S'_0], q)$ with q a T_1 singularity of S'_0. Let $\pi_1 : I \longrightarrow U$ and $\pi_2 : I \longrightarrow R$ be the two projections. The former one has finite fibres, implying that every irreducible component of I has dimension $\dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - 1$. As for the latter, it is dominant because we assume the hypotheses of Lemma 3.4. Moreover, if $q \in R$ is a point, the fibre $\pi_2^{-1}(q)$ is the locally closed set of surfaces S'_0 in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ having a T_1 singularity in q.

Let V_i be a sufficiently small analytic neighborhood of p_i in R for $i = 1, \dots, \delta$. By the above considerations, $\pi_1(\pi_2^{-1}(V_i)) \subseteq U$ is an analytic open set that parametrizes deformations of S_0 which are analytically equisingular at p_i. Hence, by Remark 3.7, the tangent space to $\pi_1(\pi_2^{-1}(V_i))$ at $[S_0]$ is contained in $H^0(S, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{p_i|\mathcal{X}_0})$ and

$$\dim_{[S_0]}(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - 1 = \dim_{[S_0]}(\pi_1(\pi_2^{-1}(V_i))) \leqslant h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{p_i|\mathcal{X}_0}).$$

If (3.15) holds, then $h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{p_i|\mathcal{X}_0}) = 0$ for all $i = 1, \dots, \delta$, and one has

$$h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{p_i|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - 1 = \dim_{[S_0]}(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - 1.$$

Thus, $\pi_1(\pi_2^{-1}(V_i))$, that is an open analytic subset of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_i}$, is an analytic branch of U of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - 1$, smooth at $[S_0]$.

Next, we prove that the general element $[S'_0]$ in $\pi_1(\pi_2^{-1}(V_i))$ has a unique T_1 singularity. We argue for the case $i = 1$, and the proof is analogous in the other cases. Suppose this is not the case, and that S'_0 has

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s singularities $q_1, \dots, q_s$ of type T_1 with $s > 1$. When S_0' specializes to S_0, $q_1, \dots, q_s$ specialize, say, to $p_1, \dots, p_s$. By the same argument as above, the tangent space to $\pi_1(\pi_2^{-1}(V_1))$ at $[S_0']$ is contained in $H^0(S_0', \mathcal{N}_{S_0'|\mathcal{X}_0} \otimes I_{\{q_1,\dots,q_s\}|\mathcal{X}_0})$ and, under the hypothesis (3.15), one has

$$h^0(S_0', \mathcal{N}_{S_0'|\mathcal{X}_0} \otimes I_{\{q_1,\dots,q_s\}|\mathcal{X}_0}) \leqslant h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_1,\dots,p_s\}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - s < h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - 1$$

and this is a contradiction. This proves the assertion for $r = 1$.

Consider now the case $\delta > r > 1$. Fix $p_{i_1}, \dots, p_{i_r}$ distinct points among $p_1, \dots, p_\delta$. The intersection

$$\mathfrak{T}_{i_1,\dots,i_r} := \bigcap_{j=1}^{r} \pi_1(\pi_2^{-1}(V_{i_j})),$$

that is an analytic open subset of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{i_1},\dots,p_{i_r}}$, is an analytic variety in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ parametrizing deformations of S_0 that are analytic equisingular at the points $p_{i_1}, \dots, p_{i_r}$. With the same argument as above, one sees that, under the hypothesis (3.15), $\mathfrak{T}_{i_1,\dots,i_r}$ is smooth of codimension r in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$, with tangent space at $[S_0]$ given by $H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\{p_{i_1},\dots,p_{i_r}\}|\mathcal{X}_0})$.

Moreover, again by the same argument as above, the general element S_0' in $\mathfrak{T}_{i_1,\dots,i_r}$ has exactly r singularities of type T_1 at points specializing to $p_{i_1}, \dots, p_{i_r}$ when S_0' specializes to S_0. So $\mathfrak{T}_{i_1,\dots,i_r}$ is a smooth analytic branch of $\mathbb{T}_r$ containing $[S_0]$, and this ends the proof of the corollary. □

Let now $\mathbb{T}_{\delta_A,\delta_B,\delta_R} \subseteq \mathcal{H}^{\mathcal{X}_0}$ be the Zariski closure of the family of surfaces $S_0 = S_A \cup S_B$ in $\mathcal{X}_0$ with δ_A nodes on A and δ_B nodes on B off R and δ_R singularities of type T_1 on R.

Corollary 3.8.3. *Let $S_0 = S_A \cup S_B$ be a reduced effective Cartier divisor such that S_A and S_B have, respectively, δ_A and δ_B nodes $p_{A,1}, \dots, p_{A,\delta_A}$ and $p_{B,1}, \dots, p_{B,\delta_B}$ off R, are elsewhere smooth and intersect transversally along a curve $C = S_A \cap S_B$, except for δ_R distinct points $p_{R,1}, \dots, p_{R,\delta_R} \in C \subset R$ where S_0 has singularities of type T_1. Let $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ be the locally closed set of equisingular deformations of S_0 in $\mathcal{X}_0$. Consider the ideal sheaf $I_{\mathfrak{Z}|\mathcal{X}_0}$ in $\mathcal{X}_0$ of the 0-dimensional reduced scheme $\mathfrak{Z}$ of lenght $\delta = \delta_A + \delta_B + \delta_R$ given by*

$$\mathfrak{Z} = \sum_{i=1}^{\delta_A} p_{A,i} + \sum_{i=1}^{\delta_B} p_{B,i} + \sum_{i=1}^{\delta_R} p_{R,i}.$$

Then

$$T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}) \subseteq H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}|\mathcal{X}_0}). \tag{3.19}$$

If

$$h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}|\mathcal{X}_0}) = h^1(C, \mathcal{N}_{C|R} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|R}) = 0 \tag{3.20}$$

and the map

$$\varphi_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}} : \mathcal{H}^{\mathcal{X}_0}_{S_0} \dashrightarrow \mathcal{H}^R_C$$

defined as in Lemma 3.4 is dominant, then the equality holds in (3.19), and the locally closed set $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ of locally trivial deformations of S_0 in $\mathcal{X}_0$ is smooth at $[S_0]$ of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\mathfrak{Z}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta$. In particular, there exists only one irreducible component $\mathbb{T} \subset \mathbb{T}_{\delta_A,\delta_B,\delta_R}$ containing the point $[S_0]$ (which is smooth at $[S_0]$ and contains $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ as a Zariski open set). Moreover, under these hypotheses, the singularities of S_0 may be smoothed independently in $\mathcal{X}_0$. More precisely, for

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every $\delta'_A \leqslant \delta_A$, $\delta'_B \leqslant \delta_B$ and $\delta'_R \leqslant \delta_R$, we have that $\mathbb{T}_{\delta'_A,\delta'_B,\delta'_R}$ is nonempty and $[S_0] \in \mathbb{T}_{\delta'_A,\delta'_B,\delta'_R}$. In an analytic neighborhood of $[S_0]$, $\mathbb{T}_{\delta'_A,\delta'_B,\delta'_R}$ consists of

$$\binom{\delta_R}{\delta'_R}\binom{\delta_A}{\delta'_A}\binom{\delta_B}{\delta'_B}$$

smooth analytic branches of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta'$, where $\delta' = \delta'_A + \delta'_B + \delta'_R$, that intersect at $[S_0]$ along a smooth analytic branch of $\mathbb{T} \subset \mathbb{T}_{\delta_A,\delta_B,\delta_R}$, corresponding to deformations of $[S_0]$ preserving δ'_R points of type T_1 and δ'_A nodes on A and δ'_B nodes on B.

Proof. Let S_0 be a surface as in the statement. The inclusion (3.19) follows from Lemma 3.6, Remark 3.7 and well-known deformation theory of nodal surfaces (see [11, §2.3]). It can be proved by using (3.10). In particular, if one localizes (3.10) at a node p of S_0, then $H^0(S_0, T^1_{S_0,p}) \cong \mathbb{C}$ can be identified with the tangent space to the versal deformation space of a node.

Now we want to prove that, under the hypotheses of the corollary, the locally closed set $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ is smooth at $[S_0]$ of codimension δ in the Hilbert scheme $\mathcal{H}^{\mathcal{X}_0}_{S_0}$.

Let $C \subset R$ be the δ_R-nodal curve cut out by S_0 on R. By the hypothesis (3.20), one has that $h^1(C, \mathcal{N}_{C|R} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|R}) = 0$. This implies that $[C]$ is a smooth point of the locally closed Severi variety $\mathcal{V}_\delta$ of δ-nodal curves in $\mathcal{H}^R_C$. Let V be the unique irreducible component of $\mathcal{V}_\delta$ containing $[C]$. Now, as we saw in Lemma 3.4, $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$ has at least one irreducible component of codimension at most δ_R in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$. However, by Lemma 3.6, we have that

$$T_{[S_0]}(\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)) \subseteq H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|\mathcal{X}_0}),$$

and by (3.20), we have $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|\mathcal{X}_0}) = \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) - \delta_R$. Thus, $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$ is smooth at $[S_0]$ of codimension δ_R. We observe that $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$ is an analytic open set of the variety $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{R,1},\dots,p_{R,\delta_R}}$ of deformations of S_0 in $\mathcal{X}_0$ that are locally trivial at every T_1 singularity $p_{R,i}$, and we just proved that

$$T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{R,1},\dots,p_{R,\delta_R}}) = T_{[S_0]}(\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)) = H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|\mathcal{X}_0}).$$

We morever observe that the general element $[S'_0]$ of $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$ corresponds to a surface $S'_0 = S'_A \cup S'_B$, where S'_A and S'_B intersect transversally along a curve C' on R, except for δ_R points $p'_{R,1}, \dots, p'_{R,\delta_R} \in C'$, which are singularities of type T_1 of S'_0, and specialize to $p_{R,1}, \dots, p_{R,\delta_R}$ as S'_0 specializes to S_0.

We claim that S'_A and S'_B are smooth outside R. Indeed, since $[S_0]$ belongs to $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$, the surface S'_0 may have at most $\delta'_A \leqslant \delta_A$ nodes $p_{A,1}, \dots, p_{A,\delta'_A}$ on A and $\delta'_B \leqslant \delta_B$ nodes $p_{B,1}, \dots, p_{B,\delta'_B}$ on B, deformations of δ'_A nodes of S_0 on A and δ'_B nodes of S_0 on B. If this happens, denoting by $\mathfrak{Z}'$ the scheme of singular points of S'_0, then $T_{[S'_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{R,1},\dots,p_{R,\delta_R}}) = T_{[S'_0]}\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V) \subseteq H^0(S'_0, \mathcal{N}_{S'_0|\mathcal{X}_0} \otimes I_{\mathfrak{Z}'|\mathcal{X}_0})$. But, once again by (3.20) and by semicontinuity, one has that $h^0(S'_0, \mathcal{N}_{S'_0|\mathcal{X}_0} \otimes I_{\mathfrak{Z}'|\mathcal{X}_0}) = H^0(S'_0, \mathcal{N}_{S'_0|\mathcal{X}_0}) - \delta_R - \delta'_A - \delta'_B$. It follows that $\delta'_A = \delta'_B = 0$ (i.e., S'_A and S'_B are smooth off R and $\varphi^{-1}_{|\mathcal{H}^{\mathcal{X}_0}_{S_0}}(V)$ is a locally closed set in one irreducible component $\mathbb{T} \subset \mathbb{T}_{\delta_R}$, smooth at $[S_0]$). We just proved that, under our hypotheses, one may deform S_0 in $\mathcal{X}_0$ by smoothing all nodes of S_0 and by preserving all T_1 singularities.

Let now $\mathfrak{Z}_A$ and $\mathfrak{Z}_B$ be, respectively, the scheme of nodes of S_0 on A and B. Let $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ be the scheme of deformations of S_0 which are locally trivial at every node of S_0. By standard deformation

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theory of nodal surfaces, one has that, under the hypothesis $h^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}_A \cup \mathfrak{Z}_B|\mathcal{X}_0}) = 0$ (that holds by (3.20)), $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ is smooth of codimension $\delta_A + \delta_B$ in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ at $[S_0]$, and moreover,

$$T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}) = H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}_A \cup \mathfrak{Z}_B|\mathcal{X}_0}).$$

With a similar argument as above, one sees that the general element $[\tilde{S}_0]$ of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ corresponds to a surface $\tilde{S}_0 = \tilde{S}_A \cup \tilde{S}_B$, with $\tilde{S}_A$ and $\tilde{S}_B$ intersecting transversally along a smooth curve $\tilde{C} \subset R$ and having, respectively, δ_A and δ_B nodes as singularities. In particular, $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ is a locally closed set in an irreducible component $\tilde{\mathbb{T}}$ of $\mathbb{T}_{\delta_A,\delta_B,0}$, of which $[S_0]$ is a smooth point.

Now the equisingular deformation locus $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ of S_0 in $\mathcal{X}_0$ is the intersection of the loci $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{R,1},\dots,p_{R,\delta_R}}$ and $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$. Hence, $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ has codimension at most δ in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$ because $[S_0]$ is a smooth point of $\mathcal{H}^{\mathcal{X}_0}_{S_0}$.

However, one has

$$\begin{aligned} T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}) &= T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p_{R,1},\dots,p_{R,\delta_R}}) \cap T_{[S_0]}(\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}) \\ &= H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\{p_{R,1},\dots,p_{R,\delta_R}\}|\mathcal{X}_0}) \cap H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}_A \cup \mathfrak{Z}_B|\mathcal{X}_0}) \\ &= H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}|\mathcal{X}_0}). \end{aligned}$$

By (3.20), we have

$$h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{Z}|\mathcal{X}_0}) = h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - \delta.$$

This proves that $\mathcal{ES}^{\mathcal{X}_0}_{[S_0]}$ is smooth at $[S_0]$ of codimension exactly δ in $\mathcal{H}^{\mathcal{X}_0}_{S_0}$, as wanted. This proves the first part of the corollary.

The second part is proved with analogous arguments as the ones used in the proof of Corollary 3.8.2. □

Proposition 3.9. *Let $S_0 = S_A \cup S_B$ be a reduced effective Cartier divisor as in the statement of Corollary 3.8.3. Assume that $[S_0]$ belongs to an irreducible component $\mathcal{H}$ of $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ that dominates $\mathbb{D}$ and that (3.20) holds.*

Let $\mathcal{ES}^{\mathcal{X}}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ be the locus in $\mathcal{H}^{\mathcal{X}}$ of deformations of S_0 which are equisingular at every node of S_0. Then $\mathcal{ES}^{\mathcal{X}}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ is generically smooth of codimension $\delta_A + \delta_B$ in $\mathcal{H}$, and it contains $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{Z}_A,\mathfrak{Z}_B}$ as a subscheme of codimension 1.

In simple words, $[S_0]$ can be deformed out of $\mathcal{X}_0$ preserving the $\delta_A + \delta_B$ nodes.

Proof. From the hypotheses, it follows that $H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) = 0$. Then, by Corollary 3.8.1, $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ is smooth at $[S_0]$ with dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) + 1 = \dim(\mathcal{H}^{\mathcal{X}_0}_{S_0}) + 1$. By standard deformation theory, there is an analytic neighborhood U of $[S_0]$ in $\mathcal{H}$ and a versal morphism

$$f : U \longrightarrow \prod_{p \in \mathfrak{Z}_A + \mathfrak{Z}_B} \Delta_p,$$

where Δ_p is the versal deformation space of a node, and therefore, it has dimension 1. Let $U' = U \cap \mathcal{H}^{\mathcal{X}_0}_{S_0}$. Then f restricts to

$$g : U' \longrightarrow \prod_{p \in \mathfrak{Z}_A + \mathfrak{Z}_B} \Delta_p.$$

The differential of g at $[S_0]$ is

$$H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) \to \prod_{p \in \mathfrak{z}_A + \mathfrak{z}_B} T^1_{S_0,p} \cong \mathbb{C}^{\delta_A+\delta_B},$$

and this map is surjective since its kernel is $H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes \mathcal{I}_{\mathfrak{z}_A \cup \mathfrak{z}_B|\mathcal{X}_0})$, which has codimension $\delta_A+\delta_B$ in $H^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0})$ by (3.20). Hence, g has maximal rank at $[S_0]$ and therefore also f is of maximal rank at $[S_0]$. Hence, $f^{-1}(0)$ and $g^{-1}(0)$ are analytic subvarieties of U and U', respectively, smooth at $[S_0]$ and of codimension $\delta_A + \delta_B$ in U and U', respectively. By versality, $f^{-1}(0)$ (resp. $g^{-1}(0)$) coincides with $\mathcal{ES}^{\mathcal{X}}_{[S_0],\mathfrak{z}_A,\mathfrak{z}_B}$ (resp. $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],\mathfrak{z}_A,\mathfrak{z}_B}$). The statement follows. □

3.2.3. Global deformations of surfaces with T_1 singularities to nodal surfaces

In this section, we will assume the following setup. We have the family $\pi : \mathcal{X} \to \mathbb{D}$ as usual with its relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$, whose fibre over $t \in \mathbb{D}$ is the Hilbert scheme of $\mathcal{H}^{\mathcal{X}_t}$ of $\mathcal{X}_t$.

Let $\mathcal{V}^{\mathcal{X}|\mathbb{D}}_\delta$ be the Zariski closure in $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ of the relative Severi variety $\mathcal{W}^{\mathcal{X}\setminus\mathcal{X}_0|\mathbb{D}\setminus 0}_\delta \subset \mathcal{H}^{\mathcal{X}\setminus\mathcal{X}_0|\mathbb{D}\setminus 0}$ of δ-nodal surfaces. We want to provide sufficient conditions for $\mathcal{V}^{\mathcal{X}|\mathbb{D}}_\delta$ to be nonempty.

We will suppose that we have a line bundle $\mathcal{L}$ on $\mathcal{X}$ with the following properties:

(1) $h^0(\mathcal{X}_t, \mathcal{L}_t)$ is a constant $r+1$ in t and greater or equal than 4. In particular, every surface S_0 in $|\mathcal{L}_0|$ belongs to an irreducible component $\mathcal{H}$ of the relative Hilbert scheme $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$ that dominates $\mathbb{D}$;
(2) $|\mathcal{L}_0|$ is base point free, so that we can assume that $|\mathcal{L}_t|$ is base point free for all $t \in \mathbb{D}$;
(3) if $p_t \in \mathcal{X}_t$ is a general point, then the general surface in $|\mathcal{L}_t|$ with a singular point at p_t is singular only at finitely many points, for the general $t \in \mathbb{D}$.

In this setting, we can consider the rank r projective bundle $\bar{\pi} : \mathbb{P}(\pi_*(\mathcal{L})) \to \mathbb{D}$. A point in $\mathbb{P} := \mathbb{P}(\pi_*(\mathcal{L}))$ that maps to $t \in \mathbb{D}$ is a nonzero section of $H^0(\mathcal{X}_t, \mathcal{L}_t)$ up to a constant. In particular, if $t \neq 0$, a point in $\mathbb{P}$ corresponds to a surface in $|\mathcal{L}_t|$. Consider the open Zariski subset $\mathbb{P}' := \bar{\pi}^{-1}(\mathbb{D} \setminus \{0\})$, which, by the above considerations, can be regarded as a subvariety of the relative Hilbert scheme of surfaces in $\mathcal{X}$. By a standard parameter count, one sees that there is a subscheme Z of pure codimension 1 in $\mathbb{P}'$ whose points correspond to sections vanishing along singular surfaces. We will denote by $\bar{Z}$ the closure of Z in $\mathbb{P}$ that has also codimension 1.

Proposition 3.10. *Set up as above with the following further condition: the subspace of sections of $H^0(\mathcal{X}_0, \mathcal{L}_0)$ that vanish on $R = A \cap B$, with A and B the irreducible components of $\mathcal{X}_0$, has codimension strictly larger than 1 in $H^0(\mathcal{X}_0, \mathcal{L}_0)$.*

Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ be a surface corresponding to a section of $\mathcal{L}_0$. We suppose that:

(a) *S_A and S_B are smooth and intersect transversally along a curve $C = S_A \cap S_B$, except for a point $p = p_1 \in C \subset R$ where S_0 has a singularity of type T_1 and the hypotheses of Lemma 3.4 hold for $\delta = 1$;*
(b) *the sublinear system $\mathcal{L}_0(2, p)$ of $|\mathcal{L}_0|$ of surfaces with at least a T_1 singularity at p has codimension 3 in $|\mathcal{L}_0|$;*
(c) *(3.15) (for $\delta = 1$) holds for S_0.*

Then:

(i) *$\mathbb{T}_1$ is smooth at $[S_0]$ of codimension 1 in $\mathcal{H}_0$;*
(ii) *S_0 can be deformed to a 1-nodal surface $S_t \subset \mathcal{X}_t$;*
(iii) *if $\mathbb{T} \subseteq \mathbb{T}_1$ is the unique irreducible component containing $[S_0]$, then there exists a reduced, irreducible component $\mathcal{V} \subset \mathcal{V}^{\mathcal{X}|\mathbb{D}}_1$ of dimension $\dim(\mathcal{H}) - 1$ whose central fibre $\mathcal{V}_0$ contains $\mathbb{T}$ as an irreducible component.*

Before giving the proof of the proposition, we make a preliminary lemma. For this, we need some notation. Let $I \subset |\mathcal{L}_0| \times R$, with $R = A \cap B$ be the locally closed subset consisting of pairs (S_0, p) such that S_0 cuts out on R a curve singular at p. We will consider the two projections $\pi_1 : I \to |\mathcal{L}_0|$ and $\pi_2 : I \to R$. Note that if $p \in R$, then $\pi_2^{-1}(p)$ can be identified with $\mathcal{L}_0(2, p)$.

Lemma 3.11. (i) *There is at most one irreducible component I' of I such that the restriction of π_2 to I' is dominant to R via π_2.*

(ii) *If I' exists, and if its general element (S_0, p) is such that S_0 cuts out on R a curve with finitely many singular points, then $\mathcal{L}_0(2, p)$ has codimension 3 in $|\mathcal{L}_0|$. Moreover,* $\dim(I') = \dim(|\mathcal{L}_0|) - 1$, *and its image in $|\mathcal{L}_0|$ via π_1 has codimension 1 in $|\mathcal{L}_0|$.*

(iii) *If there is a pair (S_0, p) in I such that S_0 cuts out on R a curve with finitely many singular points and if $\mathcal{L}_0(2, p)$ has codimension 3 in $|\mathcal{L}_0|$, then (S_0, p) belongs to an irreducible component I' of I dominating R via π_2. For this component, one has* $\dim(I') = \dim(|\mathcal{L}_0|) - 1$, *and its image in $|\mathcal{L}_0|$ via π_1 has codimension 1 in $|\mathcal{L}_0|$.*

Proof. (i) Let I' be an irreducible component of I such that I' is dominant to R via π_2. If $p \in R$ is a general point, we know that $\pi_2^{-1}(p)$ can be identified with $\mathcal{L}_0(2, p)$, and $\mathcal{L}_0(2, p)$ is a projective space with dimension s independent on the general point p. This clearly implies that I' is unique.

(ii) Suppose the dominating component I' exists. With the same notation as above, we have $\dim(I') = \dim(R) + s = s + 2$. However, by Remark 2.4, $s \geqslant \dim(|\mathcal{L}_0|) - 3$; hence, $\dim(I') \geqslant \dim(|\mathcal{L}_0|) - 1$. By the hypotheses, the map π_1, restricted to I', is generically finite onto the image, and this image cannot be dense in $|\mathcal{L}_0|$ by Bertini's theorem. Hence, $\dim(I') \leqslant \dim(|\mathcal{L}_0|) - 1$, so the equality holds, and this implies that $s = \dim(|\mathcal{L}_0|) - 3$, as wanted.

(iii) Keep the same notation as above. The dimension of the fibre of π_2 over a general point of R is $r \geqslant \dim(|\mathcal{L}_0|) - 3 = \mathcal{L}_0(2, p) \geqslant 0$. Moreover, there is an open dense subset U of R, containing p, such that for all $q \in U$, one has that $\mathcal{L}_0(2, p)$ has dimension $s = \dim(|\mathcal{L}_0|) - 3$. Hence, there is a component I' of I dominating R via π_2. □

We can now give the following:

Proof of Proposition 3.10. We notice that by Corollary 3.8.1, $[S_0]$ is a smooth point for $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$. Part (i) follows by Corollary 3.7.1.

Let us prove part (ii). For this, we go back to the notation introduced before the statement of Proposition 3.10. Consider then the intersection $\bar{Z}_0$ of $\bar{Z}$ with $\bar{\pi}^{-1}(0) \cong |\mathcal{L}_0|$, such that any of its irreducible components has codimension 1 in $|\mathcal{L}_0|$. By the hypotheses we made, if $\bar{Z}_0'$ is any irreducible component of $\bar{Z}_0$, its general element does not contain R; hence, it is a surface $S_0' \in |\mathcal{L}_0|$ that intersects R along a curve C'. By Proposition 3.1, the curve C' is singular.

Claim 3.12. There is an irreducible component $\bar{Z}_0'$ of $\bar{Z}_0$, such that for $S_0' \in \bar{Z}_0'$ general, S_0' intersects R in a curve C' that is singular at a general point p' of R. Moreover, S_0' is limit of reduced singular surfaces $S_t \in |\mathcal{L}_t|$.

Proof of the Claim 3.12. This will be a consequence of the following fact that we are going to prove: given a general point $p' \in R$, there is some $S_0' \in \bar{Z}_0$ such that the curve C' cut out by S_0' on R is singular at p'. Indeed, given $p' \in R$ general, take a smooth bisection γ' of $\mathcal{X} \to \mathbb{D}$ that passes through p'. As in §2.1, we can consider the family $\mathcal{Y} \to \mathbb{D}$ obtained by desingularising the variety $\mathcal{X}' \to \mathbb{D}$ gotten via 2-fold base change $\nu_2 : \mathbb{D} \to \mathbb{D}$. The variety $\mathcal{Y} \to \mathbb{D}$ has a section γ that is mapped to γ' via the map $\pi : \mathcal{Y} \to \mathcal{X}$. We consider $\pi^*(\mathcal{L})$. Our assumption (3) implies that there are nonzero sections of $\pi^*(\mathcal{L})$, on $\mathcal{Y} \setminus \mathcal{Y}_0$, vanishing with multiplicity at least 2 along γ. The assertion is now a consequence of Theorem 2.2. □

By the hypothesis (b) and by Lemma 3.11(iii), the pair (S_0, p) belongs to the unique irreducible component I' of I dominating R via π_2, and I' has dimension equal to $\dim(|\mathcal{L}_0|) - 1$. Consider now the subset $I'' \subseteq I$ of the pairs (S_0', p') with $S_0' \in \bar{Z}_0'$, where $\bar{Z}_0'$ is as in Claim 3.12. We notice that I'' also dominates R via π_2. So by Lemma 3.11(i), I'' coincides with I'. This implies that $\bar{Z}_0' = \pi_1(I')$; hence, $S_0 \in \bar{Z}_0'$ and therefore, the general surface $S_0' \in \bar{Z}_0'$ has a unique T_1 singularity. By Lemma 3.2, the assertion (ii) follows.

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To prove (iii), we remark first of all that (ii) implies that $\mathcal{V}_1^{\mathcal{X}|\mathbb{D}}$ is nonempty and there is an irreducible component $\mathcal{V}$ of $\mathcal{V}_1^{\mathcal{X}|\mathbb{D}}$ that dominates $\mathbb{D}$ and contains $[S_0]$. The general point in $\mathcal{V}$ corresponds to a surface S_t with $t \neq 0$, with a unique node at a general point $p_t \in \mathcal{X}_t$. Moreover, since $[S_0]$ is a smooth point of $\mathcal{H}$, we have that $[S_t]$ is a smooth point of $\mathcal{H}$, and by the hypothesis (c) and by semicontinuity, we have that $h^1(S_t, \mathcal{N}_{S_t|\mathcal{X}_t} \otimes I_{p_t}) = 0$. This yields that $\mathcal{V} \cap \mathcal{H}_t$ is smooth of dimension $\dim(\mathcal{H}_t) - 1$. Hence, $\mathcal{V}$ has dimension $\dim(\mathcal{H}) - 1$. To prove that $\mathcal{V}$ is reduced, it suffices to prove that $\mathcal{V}$ is smooth at $[S_t]$. To see this, consider the exact sequence

$$0 \to \mathcal{N}'_{S_t|\mathcal{X}} \to \mathcal{N}_{S_t|\mathcal{X}} \to T^1_{S_t} \to 0,$$

where $T^1_{S_t}$ is supported on p_t with stalk $\mathbb{C}$, and $H^0(S_t, \mathcal{N}'_{S_t|\mathcal{X}})$ is the Zariski tangent space to $\mathcal{V}$ at $[S_t]$. The map

$$H^0(S_t, \mathcal{N}_{S_t|\mathcal{X}}) \to T^1_{S_t} = \mathbb{C}$$

is surjective because S_t is smoothable inside $\mathcal{H}$, by the hypothesis (2) at the beginning of this section. Hence, $h^0(S_t, \mathcal{N}'_{S_t|\mathcal{X}}) = h^0(S_t, \mathcal{N}_{S_t|\mathcal{X}}) - 1 = \dim(\mathcal{H}) - 1$, as wanted. □

We can now prove the main result of this section extending Proposition 3.10 to the case $\delta > 1$:

Theorem 3.13. *Set up as in Proposition 3.10. In particular, we have the following condition: the subspace of sections of $H^0(\mathcal{X}_0, \mathcal{L}_0)$ that vanish on $R = A \cap B$, with A and B the irreducible components of $\mathcal{X}_0$, has codimension strictly larger than 1 in $H^0(\mathcal{X}_0, \mathcal{L}_0)$.*

Let $S_0 = S_A \cup S_B \subset \mathcal{X}_0$ be a surface corresponding to a section of $\mathcal{L}_0$. We suppose that:

(a) *S_A and S_B have, respectively, δ_A and δ_B nodes $p_{A,1}, \dots, p_{A,\delta_A}$ and $p_{B,1}, \dots, p_{B,\delta_B}$ off R are elsewhere smooth and intersect transversally along a curve $C = S_A \cap S_B$, except for δ_R distinct points $p_{R,1}, \dots, p_{R,\delta_R} \in C \subset R$, where S_0 has singularities of type T_1 and that the hypotheses of Lemma 3.4 hold;*

(b) *the sublinear system $\mathcal{L}_0(2, p_i)$ of $|\mathcal{L}_0|$ of surfaces with at least a T_1 singularity at p_i has codimension 3 in $|\mathcal{L}_0|$, for every $1 \leqslant i \leqslant \delta$;*

(c) *if $\mathfrak{Z}$ is the 0-dimensional scheme of length $\delta = \delta_A + \delta_B + \delta_R$ given by*

$$\mathfrak{Z} = \sum_{i=1}^{\delta_A} p_{A,i} + \sum_{i=1}^{\delta_B} p_{B,i} + \sum_{i=1}^{\delta_R} p_{R,i},$$

then $H^1(S_0, \mathcal{N}_{S_0|\mathcal{X}_0} \otimes I_{\mathfrak{Z}|\mathcal{X}_0}) = 0$ (where $I_{\mathfrak{Z}|\mathcal{X}_0}$ is the ideal sheaf of the scheme $\mathfrak{Z}$ in $\mathcal{X}_0$).

Then:

(i) *S can be deformed to a δ-nodal surface $S_t \subset \mathcal{X}_t$;*

(ii) *if $\mathbb{T} \subseteq \mathbb{T}_{\delta_A,\delta_B,\delta_R}$ is the unique irreducible component containing $[S]$, then there exists an irreducible component $\mathcal{V} \subset \mathcal{V}_\delta^{\mathcal{X}|\mathbb{D}}$ of dimension $\dim(\mathcal{H}) - \delta$ whose central fibre $\mathcal{V}_0$ contains $\mathbb{T}$ as an irreducible component.*

Proof. Again, by Corollary 3.8.1, $[S_0]$ is a smooth point for $\mathcal{H}^{\mathcal{X}|\mathbb{D}}$. Moreover, by Corollary 3.8.3, $\mathbb{T}_{\delta_A,\delta_B,\delta_R}$ is smooth at $[S_0]$.

We denote by $\mathbb{T}$ the unique irreducible component of $\mathbb{T}_{\delta_A,\delta_B,\delta_R}$ containing $[S_0]$, which is smooth at $[S_0]$. Furthermore, we set $\mathfrak{Z} = \mathfrak{Z}_A + \mathfrak{Z}_B + \mathfrak{Z}_R$, where $\mathfrak{Z}_A$ is the scheme of nodes of S_0 on A, $\mathfrak{Z}_B$ is the scheme of nodes of S_0 on B, and $\mathfrak{Z}_R$ is the scheme of T_1-singularities of S_0.

Again by Corollary 3.8.3, in an analytic neighborhood of $[S]$, $\mathbb{T}$ consists of an analytic branch $\mathcal{T}$ that is the transverse intersection of δ smooth analytic branches of dimension $h^0(S_0, \mathcal{N}_{S_0|\mathcal{X}_0}) - 1$, each

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branch corresponding to the locus of deformations of S_0 that are equisingular at a given point in $\mathfrak{Z}$; that is, we have that

$$\mathcal{T} = \bigcap_{p \in \mathfrak{Z}} \mathcal{ES}^{\mathcal{X}_0}_{[S_0],p}.$$

The general element S_0' of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p}$ is a surface in $\mathcal{X}_0$ that has a unique singularity analytically equivalent to the singularity of S_0 at p (i.e., a node if $p \in \mathfrak{Z}_A + \mathfrak{Z}_B$, a T_1 singularity otherwise).

By Proposition 3.9, for every $p \in \mathfrak{Z}_A + \mathfrak{Z}_B$, $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p}$ is contained in $\mathcal{ES}^{\mathcal{X}}_{[S_0],p}$ as a subvariety of codimension 1, and $\mathcal{ES}^{\mathcal{X}}_{[S_0],p}$ is an analytic branch of the Severi variety $\mathcal{V}_1^{\mathcal{X}|\mathbb{D}}$.

If $p \in \mathfrak{Z}_R$, then by the hypothesis (b), for general element S_0' of $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p}$, the condition (b) of Proposition 3.10 holds. Then, by Proposition 3.10, $\mathcal{ES}^{\mathcal{X}_0}_{[S_0],p}$ is contained, as a subvariety of codimension 1, in an analytic branch $\mathcal{T}_p$ of $\mathcal{V}_1^{\mathcal{X}|\mathbb{D}}$ having codimension 1 in $\mathcal{H}$, which is smooth at the general point corresponding to a 1-nodal surface.

Now, the intersection

$$\mathcal{T}' = \bigcap_{p \in \mathfrak{Z}_R} \mathcal{T}_p \cap \bigcap_{p \in \mathfrak{Z}_A + \mathfrak{Z}_B} \mathcal{ES}^{\mathcal{X}}_{[S_0],p}$$

has codimension at most δ in $\mathcal{H}$, and it contains the smooth analytic branch $\mathcal{T}$ of $\mathbb{T}$, which has codimension $\delta + 1$ in $\mathcal{H}$. The general element of $\mathcal{T}'$ corresponds to a surface $\tilde{S}$, not contained in $\mathcal{X}_0$, with at least δ singularities, precisely δ_A (resp. δ_B) singularities in neighborhoods of the nodes $p \in \mathfrak{Z}_A$ (resp. $p \in \mathfrak{Z}_B$) and δ_R singularities in neighborhoods of the T_1 singularities $p \in \mathfrak{Z}_R$. Taking into account Lemma 3.2, we deduce that $\tilde{S}$ has δ nodes and no further singularities. This proves (i).

If $\tilde{S} \subset \mathcal{X}_t$ and it has nodes at $\tilde{q}_1, \dots, \tilde{q}_\delta$, by semicontinuity, we have that $H^0(\tilde{S}, \mathcal{N}_{\tilde{S}|\mathcal{X}_t} \otimes I_{\{\tilde{q}_1,\dots,\tilde{q}_\delta\}|\mathcal{X}_t})$ has dimension $\dim(\mathcal{H}_t) - \delta$. Thus, $\mathcal{T}'$ is an analytic branch containing the point $[\tilde{S}]$ in an irreducible component $\mathcal{V} \subset \mathcal{V}_\delta^{\mathcal{X}|\mathbb{D}}$ of dimension $\dim(\mathcal{H}) - \delta$ whose central fibre $\mathcal{V}_0$ contains $\mathbb{T}$ as an irreducible component. This proves (ii). □

4. Applications

4.1. Severi varieties

Let X be a smooth irreducible projective complex threefold. Let L be a line bundle on X such that the general surface in the linear system $|L|$ is smooth and irreducible. We denote by $V_\delta^{X,|L|}$ the *Severi variety*, that is the locally closed subscheme in $|L|$ parametrizing surfaces S in $|L|$ which are reduced and with only δ nodes as singularities. If $[S] \in V_\delta^{X,|L|}$, then the Zariski tangent space to $V_\delta^{X,|L|}$ at $[S]$ coincides with

$$T_{[S]}(V_\delta^{X,|L|}) \simeq H^0(S, \mathcal{O}_S(L) \otimes I_{N|S}),$$

where N is the reduced scheme of nodes of S. In particular, $\dim(V_\delta^{X,|L|}) \leqslant h^0(S, \mathcal{O}_S(L) \otimes I_{N|S})$. Moreover, by standard deformation theory, $H^1(S, \mathcal{O}_S(L) \otimes I_{N|S})$ is an obstruction space for $\mathcal{O}_{V_\delta^{X,|L|},[S]}$, and thus,

$$h^0(S, \mathcal{O}_S(L) \otimes I_{N|S}) - h^1(S, \mathcal{O}_S(L) \otimes I_{N|S}) \leqslant \dim(V_\delta^{X,|L|}) \leqslant h^0(S, \mathcal{O}_S(L) \otimes I_{N|S}).$$

If $h^1(S, \mathcal{O}_S(L) \otimes I_{N|S}) = 0$, then $V_\delta^{X,|L|}$ is smooth at $[S]$ of dimension

$$h^0(S, \mathcal{O}_S(L) \otimes I_{N|S}) = \dim(|L|) - \delta.$$

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In this case, one says that $[S]$ is a *regular* point of the $\dim(V_\delta^{X,|L|})$. An irreducible component V of $\dim(V_\delta^{X,|L|})$ is said to be *regular* if it is regular at its general point.

Remark 4.1. Suppose V is a regular irreducible component of $\dim(V_\delta^{X,|L|})$. By standard deformation theory already used in Section 3, the nodes of the surface corresponding to any smooth point in V can be independently smoothed. This implies that there are regular components of $\dim(V_{\delta'}^{X,|L|})$ for any $\delta' < \delta$.

One can consider the following two questions.

Problem 4.2. Given X and L as above, which is the maximal value of δ such that the Severi variety $V_\delta^{X,|L|}$ is nonempty?

Problem 4.3. Given X and L as above, which is the maximal value of δ such that the Severi variety $V_\delta^{X,|L|}$ has a regular component?

As for Problem 4.2, this is a classical and difficult question, for which there are several contributions, too many to be quoted here. Probably the most efficient one is given by the Miyaoka's bound [15, Formulae (2) and (8)]. In particular, the problem has been completely solved for $X = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$ with $d \leqslant 6$ (see, for example, [14] and references therein). However, in this section, we will not consider Problem 4.2, but we will give some contribution to Problem 4.3.

Remark 4.4. One could be tempted to believe that the maximal δ for which the Severi variety is nonempty is bounded above by the dimension of $|L|$. This is not true. In fact, there are classical examples, for $X = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$ for suitable d, for which $V_\delta^{X,|L|}$ is nonempty and δ is greater than the dimension of $|L|$ (cf. [1], [19]). In these cases, every component of the Severi variety is not regular.

Remark 4.5. Referring to Problem 4.3, it is rather natural to conjecture that the δ which answers the question should be bounded below by $\delta_0 = [\frac{\dim(|L|)}{4}]$. The reason for such a conjecture is the following: choose $p_1, \ldots, p_{\delta_0}$ general points on X. Since a double point imposes at most four conditions to $|L|$, certainly there are surfaces which are singular at every p_i. If the general such surface has only nodes at $p_1, \ldots, p_{\delta_0}$ and no other singularities, then it belongs to a regular component of the Severi variety. However, this heuristic argument is very difficult to be made rigorous in general.

4.2. The case of $\mathbb{P}^3$

In this section, we give a contribution to Problem 4.3, in the case $X = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$. More precisely, we will prove the following:

Theorem 4.6. *There is an irreducible, regular component of* $V_\delta^{\mathbb{P}^3,|\mathcal{O}_{\mathbb{P}^3}(d)|}$*, for any* $\delta \leqslant \binom{d-1}{2}$.

Proof. In view of Remark 4.1, it is sufficent to consider only the case $\delta = \binom{d-1}{2}$.

Let $\mathcal{X}' = \mathbb{P}^3 \times \mathbb{D} \to \mathbb{D}$ be a trivial family. Let us consider $\mathcal{X} \to \mathcal{X}'$ the blow-up of a point q in the central fibre $\mathbb{P}^3$ over $0 \in \mathbb{D}$. Let $\mathcal{X} \to \mathbb{D}$ be the new family. The fibre over $t \in \mathbb{D} \setminus \{0\}$ of this family is $\mathcal{X}_t \cong \mathbb{P}^3$. The central fibre $\mathcal{X}_0$ consists of two components $A \cup B$, where $f : A \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at q, whereas $B \cong \mathbb{P}^3$ is the exceptional divisor in $\mathcal{X}$, and $A \cap B = R \cong \mathbb{P}^2$ is the exceptional divisor in A and a plane in B.

On $\mathcal{X}'$, there is a line bundle $\mathcal{L}'$, which is the pull-back via the first projection, of $\mathcal{O}_{\mathbb{P}^3}(d)$. We pull this back to $\mathcal{X}$ and denote it $\mathcal{L}$. Now we consider on $\mathcal{X}$ the line bundle $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B)$. Its restriction to the general fibre $\mathcal{X}_t$ is given by $(\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B))|_{\mathcal{X}_t} \simeq \mathcal{O}_{\mathbb{P}^3}(d)$. As for the restriction of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B)$ to $\mathcal{X}_0$, we observe that $(\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B))|_A \simeq \mathcal{O}_A(d) \otimes \mathcal{O}_A(-(d-1)R)$, where $\mathcal{O}_A(d) \simeq f^*(\mathcal{O}_{\mathbb{P}^3}(d))$, whereas $(\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B))|_B \simeq \mathcal{O}_{\mathbb{P}^3}(d-1)$ and, finally, the restriction of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B)$ to R is $\mathcal{O}_{\mathbb{P}^2}(d-1)$. One easily checks that the line bundle $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}((1-d)B)$ verifies the hypotheses (1), (2) and (3) at the beginning of Section 3.2.3.

We now consider on R a curve C which consists of the union of $d-1$ general lines. It has $\delta = \binom{d-1}{2}$ nodes as singularities. By standard application of Bertini's theorem, there exists a smooth surface S_B

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in B of degree $d-1$ cutting out on R the curve C. Similarly, there exists a smooth surface $S_A \in |\mathcal{O}_A(d)\otimes\mathcal{O}_A(-(d-1)R)|$ restricting to C on R. To see this, let (x,y,z) be an affine coordinates system on $\mathbb{P}^3$ centered at q. If $\phi_1(x,y,z)=0$ is the equation of C in the plane at infinity, and $\phi_2(x,y,z)$ is a general homogeneous polynomial of degree d in (x,y,z), then the projective closure S_B of the degree d affine surface with equation $\phi_1(x,y,z)+\phi_2(x,y,z)=0$ has a point of multiplicity $d-1$ at q and no other singularities, and its minimal resolution obtained by blowing up q is the required surface.

Now $S_0 = S_A \cup S_B$ is a Cartier divisor in $\mathcal{X}_0$ belonging to the linear system $|\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}((1-d)B)|$. Moreover, S_0 verifies all hypotheses of Theorem 3.13. In particular, if $\mathfrak{Z}$ is the reduced scheme of the nodes of C, then $\mathfrak{Z}$ imposes independent conditions to $|\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}((1-d)B)|_R| = |\mathcal{O}_{\mathbb{P}^2}(d-1)|$, and therefore to $|\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}((1-d)B)|$, because the Severi varieties of nodal curves in the plane are well known to be regular. By applying Theorem 3.13, one may deform S_0 to a surface $S_t\subset\mathcal{X}_t$ with δ nodes and no further singularities, which are deformations of the δ singularities of type T_1 of S_0. Finally, the nodes of S_t impose independent conditions to surfaces of degree d on $\mathcal{X}_t\simeq\mathbb{P}^3$. Hence, $[S_t]\in V_\delta^{\mathbb{P}^3,|\mathcal{O}_{\mathbb{P}^3}(d)|}$ belongs to a regular component of the Severi variety. □

Remark 4.7. Taking into account Remark 4.5, we believe that the previous result is far from being sharp, not even asymptotically. Indeed, we may expect that the bound on δ for the existence of regular components of the Severi variety of nodal surfaces of degree d in $\mathbb{P}^3$ could asymptotically go as $\delta\sim\frac{d^3}{24}$. See also [14, Corollary 4.1] and related references for a very large upper bound of the number of nodes δ of a surface in $\mathbb{P}^3$ in a regular component of the Severi variety (if nonempty). Moreover, our results could in principle be improved by imposing to S_0 nodes off R, but we do not dwell on this here.

Remark 4.8. The known results about Problem 4.3 are very few. For example, in [13], one proves that *if* $V_\delta^{\mathbb{P}^3,|\mathcal{O}_{\mathbb{P}^3}(d)|}$ is nonempty, then every component of it is regular for $d\leqslant 7$ and for $d\geqslant 8$ and $\delta\leqslant 4d-5$, and this last bound is sharp (the case $d\leqslant 7$ was already proved in [8]). Nonemptiness results for $d\leqslant 7$ are also well known (see, for example, [14, p. 120]). In particular, our Theorem 4.6 is, at the best of our knowledge, new as soon as $d\geqslant 8$.

4.3. Complete intersections in $\mathbb{P}^4$

In this section, we want to provide a partial answer to Problem 4.3 in the case of complete intersections in $\mathbb{P}^4$.

Let X be a general hypersurface of degree $h\geqslant 2$ in $\mathbb{P}^4$. We consider on X the linear system $|\mathcal{O}_X(d)|$. Our aim is to construct regular components of $V_\delta^{X,|\mathcal{O}_X(d)|}$ with suitable δ.

Theorem 4.9. *Let $d\geqslant h-1$ be an integer. There are regular components of $V_\delta^{X,|\mathcal{O}_X(d)|}$ for*

$$\delta\leqslant\binom{d+3}{3}-\binom{d-h+1}{3}-1.$$

Proof. As usual, to prove the theorem, it suffices to do the case

$$\delta=\binom{d+3}{3}-\binom{d-h+1}{3}-1.$$

Let Y be a general hypersurface of $\mathbb{P}^4$ of degree $h-1$ and H be a general hyperplane, cutting Y along a surface R, which is a general surface of degree $h-1$ in $H\simeq\mathbb{P}^3$. Let X be a general hypersurface of degree h and let us consider the pencil generated by X and $Y\cup H$. Specifically, if X has equation $f=0$, Y has equation $g=0$ and H has equation $\ell=0$, we will consider the hypersurface $\mathcal{X}''$ in $\mathbb{P}^4\times\mathbb{A}^1$, with equation $\{tf+g\ell=0, \text{ with } t\in\mathbb{A}^1\}$. Via the second projection $\mathcal{X}''\to\mathbb{A}^1$, this becomes a flat family of 3-folds, with smooth general fibre $\mathcal{X}''_t$, corresponding to a general hypersurface of degree h in $\mathbb{P}^4$, and whose fibre over 0 is $\mathcal{X}''_0 = Y\cup H\subset\mathbb{P}^4$. We are interested in the singularities of $\mathcal{X}''$ in a neighborhood

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of the central fibre (i.e., we are interested in what happens if t belongs to a disc $\mathbb{D}$, centered at the origin). Thus, we consider the family

$$\mathcal{X}' = \{tf + g\ell = 0, \text{ with } t \in \mathbb{D}\} \to \mathbb{D}.$$

It is immediate to see that the singular locus of $\mathcal{X}'$ coincides with the curve $D : t = f = g = \ell = 0 \subset \mathcal{X}'_0$, which is isomorphic to a smooth complete intersection curve of type $(1, h-1, h)$ in $\mathbb{P}^4$ cut out on $R = Y \cap H$ by X. Moreover, $\mathcal{X}'$ has double points along D with tangent cone a quadric of rank 4. We resolve these singularities by blowing up $\mathcal{X}'$ along D. One obtains a new family $\tilde{\mathcal{X}} \to \mathbb{D}$ with the same general fibre as $\mathcal{X}' \to \mathbb{D}$ and whose central fibre consists of three components $\tilde{Y}$ and $\tilde{H}$, the blow-ups of Y and H along D and the exceptional divisor $\tilde{\Theta}$ that is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over D. Now we can contract $\tilde{\Theta}$ by contracting one of the two rulings of the $\mathbb{P}^1 \times \mathbb{P}^1$ bundle. We choose to do this in the direction of Y. We obtain a new family of 3-folds $\mathcal{X} \to \mathbb{D}$, with $\mathcal{X}$ smooth, with fiber $\mathcal{X}_t = \mathcal{X}'_t$ over $t \neq 0$, and whose central fiber $\mathcal{X}_0 = A \cup B$, where now $B = H \simeq \mathbb{P}^3$ and $A = \mathrm{Bl}_D(Y)$ is the blowing up of Y along D and A and B intersect transversally along a surface isomorphic to R, which we still denote by $R = A \cap B$. The exceptional divisor Θ in $A = \mathrm{Bl}_D(Y)$ is a $\mathbb{P}^1$-bundle on $D \subset R$, intersecting R along D. In particular, $\Theta \simeq \mathbb{P}(\mathcal{N}_{D|Y})$.

Notice that one has a natural morphism $\tilde{\mathcal{X}} \to \mathbb{P}^4$. This factors through a morphism $\phi : \mathcal{X} \to \mathbb{P}^4$. The action of ϕ on $\mathcal{X}_0$ is as follows: it maps B isomorphically to H, and it maps A to Y by contracting the exceptional divisor Θ. Let us now set $\mathcal{L}_d = \phi^*(\mathcal{O}_{\mathbb{P}^4}(d))$ and assume that $d \geqslant h-1$.

Recall that R is a general surface of degree $h-1$ in $\mathbb{P}^3$, with $h \geqslant 2$. By [4], $V_\delta^{R,|\mathcal{O}_R(d)|}$ is nonempty and contains a regular component V for

$$\delta = \dim(|\mathcal{O}_R(d)|) = \binom{d+3}{3} - \binom{d-h+1}{3} - 1.$$

So we can choose a general curve C in V, that is a complete intersection of type $(h-1, d)$ on R with δ nodes. Using Bertini's theorem, we can assume that there is a divisor $S_0 \in \mathcal{L}_{d|\mathcal{X}_0}$ that cuts out C on R and $S_0 = S_A \cup S_B$ (the notation is obvious), with S_A and S_B smooth.

Now S_0 verifies all hypotheses of Theorem 3.13. In particular, if $\mathfrak{Z}$ is the reduced scheme of nodes of C, then $\mathfrak{Z}$ imposes independent conditions to $\mathcal{L}_d|_{\mathcal{X}_0}$ because the component V of the Severi variety is regular. By applying Theorem 3.13, one may deform S_0 to a surface $S_t \subset \mathcal{X}_t$ with δ nodes and no further singularities, which are deformations of the δ singularities of type T_1 of S_0. Finally, the nodes of S_t impose independent conditions to surfaces in $|\mathcal{O}_{X_t}(d)|$. Hence, $[S_t] \in V_\delta^{\mathcal{X}_t,|\mathcal{O}_{\mathcal{X}_t}(d)|}$ belongs to a regular component of the Severi variety, as wanted. □

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