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ON THE CYCLIC COVERINGS OF THE KNOT 5_2

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We construct a family of hyperbolic 3-manifolds whose fundamental groups admit a cyclic presentation. We prove that all these manifolds are cyclic branched coverings of $\mathbf{S}^3$ over the knot 5_2 and we compute their homology groups. Moreover, we show that the cyclic presentations correspond to spines of the manifolds.

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1. Definitions and main results

In this paper we shall study a countable class of closed, connected, orientable 3-manifolds M_n, whose fundamental groups are cyclically presented groups. We recall the notion of cyclic presentation of a group. Let F_n be the free group on n free generators $x_0, x_1, \cdots, x_{n-1}$, and let θ be the automorphism of F_n defined by $\theta(x_i) = x_{i+1}$, for $i = 0, 1, \cdots, n-1$ (subscripts mod n). For any reduced word w in F_n define $G_n(w) = F_n/R$, where R is the normal closure in F_n of the set $\{w, \theta(w), \cdots, \theta^{n-1}(w)\}$. Then G is said to be *cyclically presented* if G is isomorphic to $G_n(w)$ for some n and w. Some connections between cyclic presentations of groups and cyclic coverings of $\mathbf{S}^3$, branched over knots or links, have been studied in [**2**], [**8**] and [**15**].

A group presentation $< X \mid R >$ is called *geometric* if there is a closed 3-manifold M^3 which admits a Heegaard diagram inducing $< X \mid R >$ as a presentation of $\pi_1(M^3)$. Equivalently, M^3 admits a spine homeomorphic to the canonical complex associated to $< X \mid R >$ (see [**17**]). As is well known, the canonical complex associated to a group presentation is a 2-dimensional cell complex consisting of a unique vertex, a 1-cell for each generator and a 2-cell for each relator, whose boundary is glued to the 1-skeleton according to the corresponding relator. Some relations between cyclic presentation of groups and spines of 3-manifolds are shown in [**4**].

Our main results show that the manifold M_n is the n-fold cyclic covering of the 3-sphere $\mathbf{S}^3$, branched over the knot 5_2 (Rolfsen Notation [**18**]), and that M_n is spherical for $n = 1, 2$ and hyperbolic for $n \geq 3$. Moreover, we find a cyclic presentation

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for $\pi_1(M_n)$ and prove that such a presentation is geometric. Finally, since 5_2 has the property of being a genus one knot, the homology characters of the manifolds can easily be computed as shown in Section 6.

2. Construction of a family of 3-manifolds M_n

The manifolds M_n ($n \geq 1$) is defined by pairwise identification of the 2-faces of a polyhedron P_n, which is homeomorphic to a 3-ball, whose boundary complex provides a tessellation of the 2-sphere as depicted in Figure 1. The tessellation consists of $4n$ quadrilaterals, $8n$ edges and $4n+2$ vertices. The n quadrilaterals around the North Pole N are labelled by $Q_1, Q_2, \ldots, Q_n$. The n quadrilaterals around the South Pole S – which is the point at infinity in Figure 1 – are labelled by $R_1, R_2, \ldots, R_n$, and the other $2n$ quadrilaterals are labelled by $Q_1', R_1', Q_2', R_2', \ldots, Q_n', R_n'$, as indicated in Figure 1. To obtain M_n, we glue Q_i with Q_i' (resp. R_i with R_i'), for each $i = 1, 2, \ldots, n$, by an orientation reversing identification which matches N with A_i (resp. S with B_i). Via this glueing we get, for each $i = 1, 2, \ldots, n$, the following identifications on the edges: $NC_i \equiv A_iC_{i-1} \equiv A_{i-1}B_{i-2}$ (which we shall call x_i), $SA_{i+1} \equiv B_iA_{i+2} \equiv B_{i-1}D_i \equiv C_{i+1}D_{i+1}$ (which we shall call y_i), and $C_1D_2 \equiv C_2D_3 \equiv \cdots \equiv C_nD_1$ (which we shall call z). As a consequence the vertices match as follows: $N \equiv A_i \equiv D_i$ and $S \equiv B_i \equiv C_i$, for each $i = 1, 2, \ldots, n$. Observe that, here and in the following, subscripts are considered mod n. Thus, we obtain a 3-dimensional cellular complex K_n, having one 3-cell, $2n$ quadrilaterals, $2n+1$ edges and two vertices. Since its Euler characteristic is

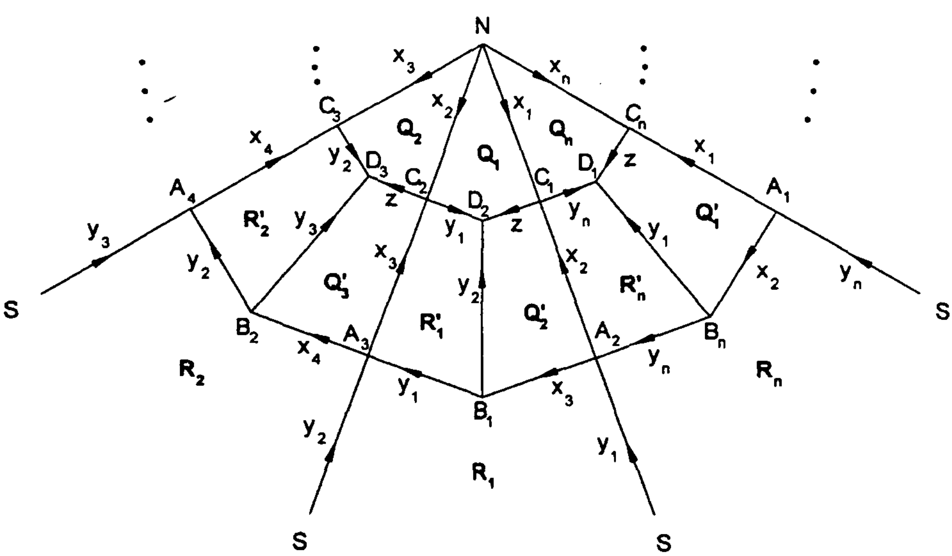

Figure 1

$\chi(K_n) = 2 - (2n+1) + 2n - 1 = 0$, the space $M_n = |K_n|$ is a genuine closed, connected, orientable 3-manifold according to the Seifert-Threlfall criterion (see [**20**, p. 216]).

3. M_n as branched cyclic covering of the 3-sphere

Let θ_n be the clockwise rotation of $2\pi/n$ radians around the polar axis of the 3-ball P_n. It is easy to see that all the above defined identifications are invariant with respect to this rotation; therefore θ_n induces an orientation preserving homeomorphism g_n on M_n. The set $\mathrm{Fix}(g_n)$ consists of the points of the polar diameter NS and the points of the edge z. Let G_n be the cyclic group of homeomorphisms of M_n generated by g_n. Of course, G_n has order n and $\mathrm{Fix}(g_n^k) = \mathrm{Fix}(g_n)$ for each $k = 1, 2, \ldots, n-1$. The quotient space M_n/G_n is homeomorphic to M_1 and the canonical quotient map

$$p_n : M_n/G_n \to M_1$$

is an n-fold branched cyclic covering, whose branching set is the 1-subcomplex of M_1 composed of NS and z (see Figure 2, where the branching set is shown by a thick line and each of the boundary quadrilaterals Q, Q', R, R' is subdivided into four triangles).

Figures 2–7 depict, in detail, the identifications performed on the 2-sphere of Figure 2 to obtain M_1, showing the development of the branching set. More precisely, we have successively performed the identifications between the following regions: q_1 and q_4 with q_1' and q_4' (Fig. 2 → Fig. 3), q_2 and q_3 with q_2' and q_3' (Fig 3 → Fig. 4), r_1 with r_1' (Fig. 4 → Fig. 5), r_2 with r_2' (Fig. 5 → Fig. 6). Notice that the complex is a three-ball at each of these stages. As a final step we identify r_3 and r_4 with r_3' and r_4' obtaining a three-sphere, where the branching set is a knot embedded as in Figure 7.[1]

Hence, M_1 is homeomorphic to a 3-sphere and the branching set is the two-bridge knot $\mathbf{b}(7,3)$, according to Schubert's notation (see [**1**. p. 181]), which is the knot 5_2 of the Alexander, Briggs, Reidemeister table ([**1**, p. 312]).

So we have proved the following:

Theorem 1. *The manifold M_n is the n-fold cyclic covering of $\mathbf{S}^3$, branched over the two-bridge knot* $\mathbf{b}(7,3)$.

As already known, the 2-fold branched coverings of the two-bridge knot or link $\mathbf{b}(p,q)$ is the lens space $L(p,q)$ (see [**19**]). Therefore, we immediately have:

Corollary 2. *The manifold M_2 is the lens space* $L(7,3)$.

Remark 1. From a result of [**16**], each M_n turns out to be an element of a certain class of manifolds $S(b,l,t,c)$, depending on four integer parameters, introduced in [**13**].

[1] For the reader's convenience, the starred reference points in the figures underline the above identifications.

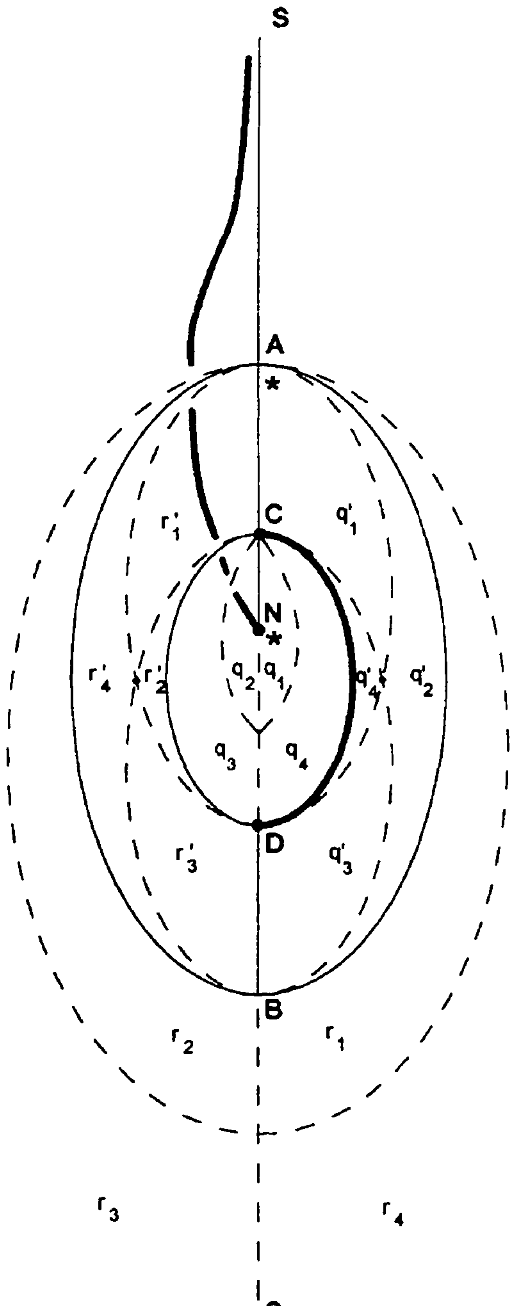

Figure 2

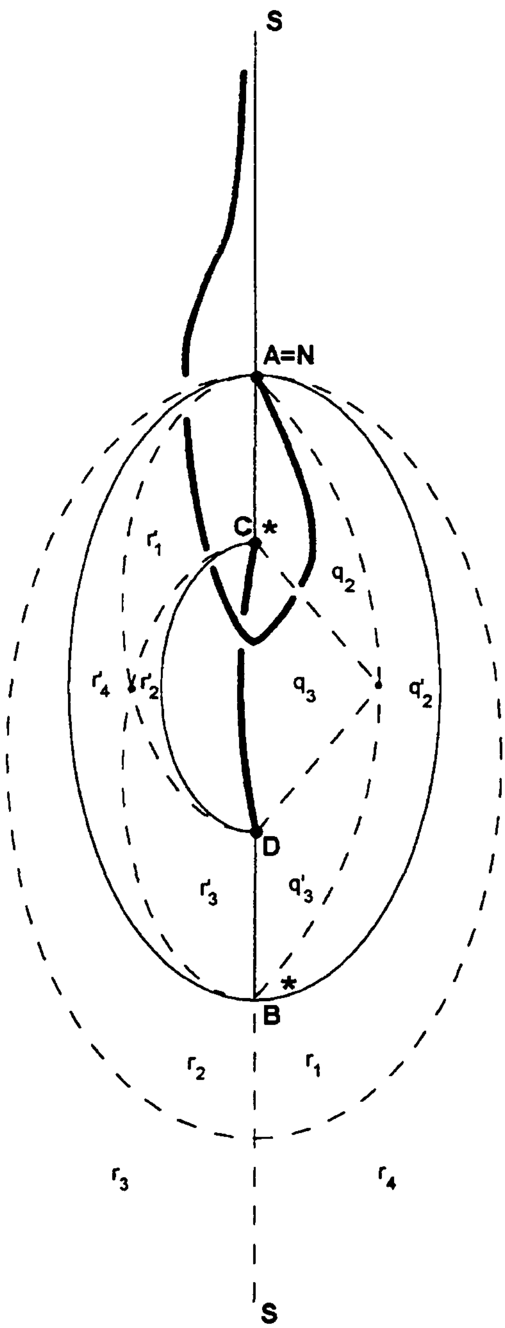

Figure 3

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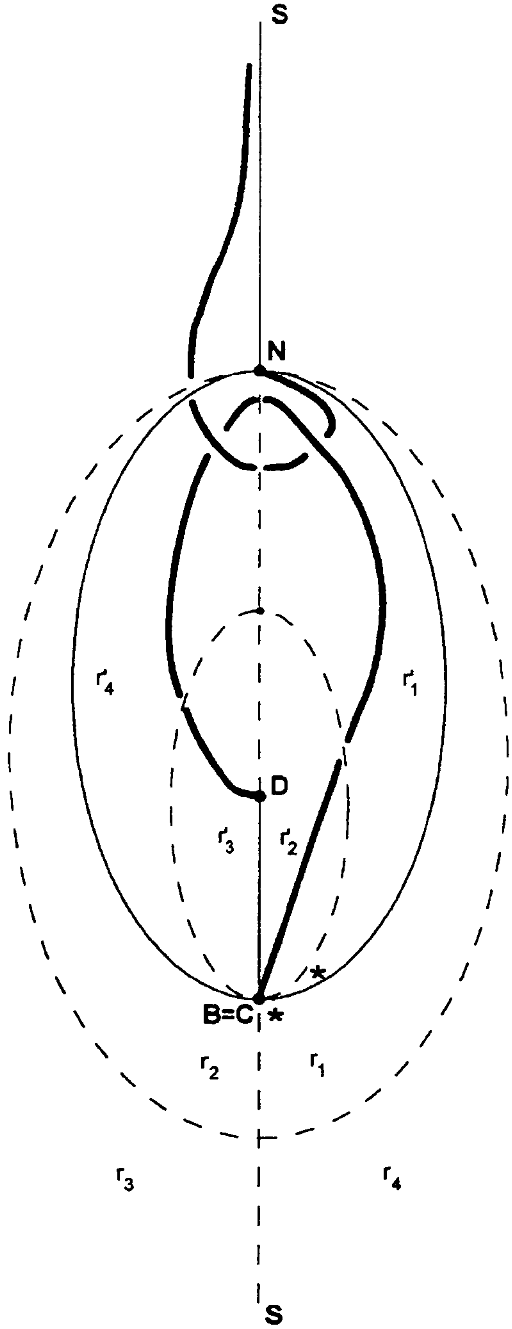

Figure 4

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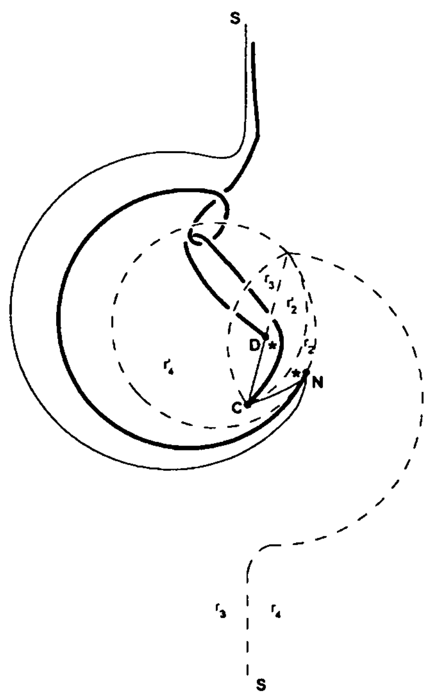

Figure 5

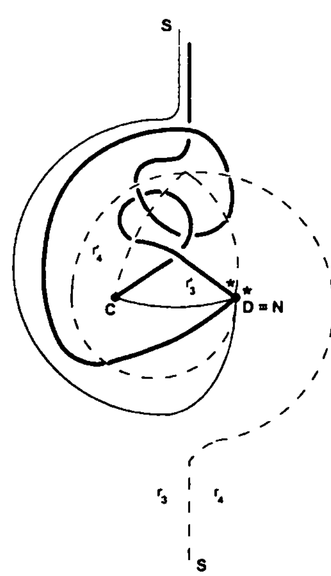

Figure 6

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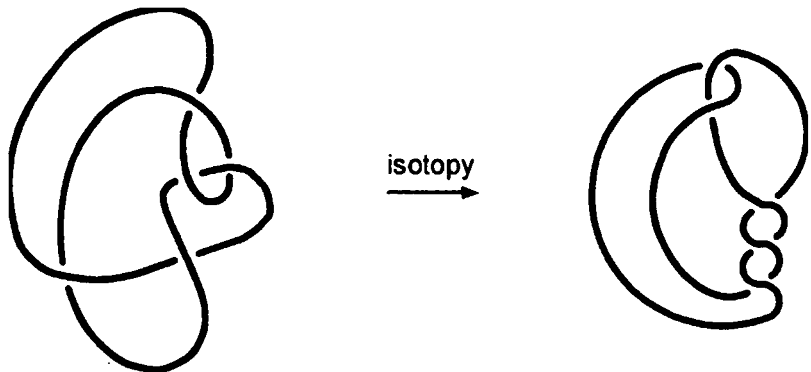

Figure 7

In particular, M_n is homeomorphic to the Lins-Mandel space $S(n, 7, 3, n-1) = S(n, 7, 4, 1)$.

4. Geometric structure on M_n

The topological properties of M_n established by Theorem 1 easily allow us to derive its geometric structure:

Proposition 3. *The manifold M_n has a spherical structure for $n = 1, 2$ and a hyperbolic structure for $n > 2$.*

Proof. As is well known, the n-fold cyclic covering of $\mathbf{S}^3$ branched over a knot K has the same geometric structure of the orbifold (K, n), which has $\mathbf{S}^3$ as its underlying space and K as its singular set with a cyclic isotropy group of order n (for example, see [**3**, p. 69]). Since the orbifold $(\mathbf{b}(7, 3), n)$ is hyperbolic for $n > 2$ and spherical for $n = 1, 2$ ([**9**, Theorem 3.1]), the statement is proved. □

Remark 2. Observe that M_3 is the Fomenko-Matveev-Weeks manifold Q_1, which is the hyperbolic 3-manifold with the smallest known volume (vol. = 0.9427...). For more details, see [**11**], [**14**], [**21**] and Chapter 2 of [**12**].

5. A geometric cyclic presentation for $\pi_1(M_n)$

From the 2-skeleton of K_n it is easy to get a presentation of the fundamental group $\pi_1(M_n)$. Orienting the edges of K_n as in Figure 1 and squeezing z to a point, we get $2n$ generators $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ subject to n relations of type $x_{i+1}y_ix_i^{-1} = 1$ (derived from Q_i) and n relations of type $y_ix_{i+2}y_iy_{i+1}^{-1} = 1$ (derived from R_i). Since the

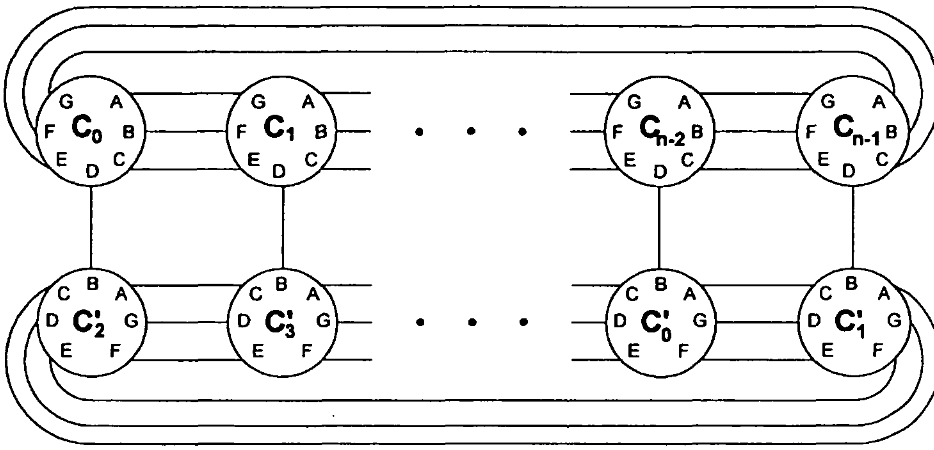

Figure 8

first relations give $y_i = x_{i+1}^{-1}x_i$, the second relations become $x_{i+1}^{-1}x_ix_{i+2}x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2} = 1$. Hence, the fundamental group of M_n admits the following cyclic presentation with n generators:

$$\pi_1(M_n) = < \{x_i\}_{i\in\mathbf{Z}_n} \mid \{x_{i+1}^{-1}x_ix_{i+2}x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2}\}_{i\in\mathbf{Z}_n} > . \tag{1}$$

A family of Heegaard diagrams H_n, $n \geq 1$, corresponding to such cyclic presentations is depicted in Figure 8 (where the circle C_i must be identified with the circle C_i', for each $i = 0, \ldots, n-1$, according to the labelling of their vertices). Note that this family has cyclic symmetry and it is a particular case of the ones studied in [4]. More precisely, it is exactly the $(7, 3, 0, 1, 2, -2)$ class of Table 1 of that paper.

It is easy to see that H_1 is a Heegaard diagram of $\mathbf{S}^3$ and that it is the quotient of H_n via the cyclic action. Therefore, H_n is a Heegaard diagram of a 3-manifold, which is an n-fold cyclic covering of $\mathbf{S}^3$ with branching set independent of n. A simple test[2] shows that H_2 is a Heegaard diagram of the lens space $L(7, 3)$. Since $L(7, 3)$ admits a unique representation as 2-fold branched covering of $\mathbf{S}^3$ (namely, over the two-bridge knot $\mathbf{b}(7, 3)$), we have the following result:

Proposition 4. *The manifold M_n admits a spine which corresponds to the cyclic presentation* (1).

[2] The test has been conducted with the aid of a computer program, written by the first author's research group, which checks topological and algebraic properties of (pseudo-) manifolds of relatively little "complexity" using combinatorial tools. See [5] for a survey of these techniques.

6. Homology characters of M_n

The two-bridge knot $\mathbf{b}(7,3)$ is a genus one knot (see [7, Satz 5.1]), and so the homology characters of M_n can be computed:

Proposition 5. *The first homology group of M_n is*

$$H_1(M_n) \cong \begin{cases} \mathbf{Z}_{7|a_n|} \oplus \mathbf{Z}_{|a_n|} & \textit{if } n \textit{ is even} \\ \mathbf{Z}_{|b_n|} \oplus \mathbf{Z}_{|b_n|} & \textit{if } n \textit{ is odd} \end{cases},$$

where, for each $n > 0$:

$a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} - 2a_n;$

$b_1 = 1, b_2 = -3, b_{n+2} = b_{n+1} - 2b_n.$

Proof. A Seifert matrix of $\mathbf{b}(7,3)$ is $V = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ (see Table II of [1]). Thus, Theorem 1 of [6] applies with $\gamma = \det(V) = 2$ and $\omega = \text{g.c.d.}(v_{11}, v_{12} + v_{21}, v_{22}) = 1$. □

The following table exhibits the torsion coefficients of $H_1(M_n)$, for $n \leq 15$.[3]

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
τ_1	7	5	21	11	35	13	21	5	77	67	315	181	637	275
τ_2	1	5	3	11	5	13	3	5	11	67	45	181	91	275

Table 1

7. How many different "cyclic" identifications can be defined on P_n?

Finally, we examined which 3-manifolds arise from different identification-systems on the boundary of P_n. Of course, the general problem is intractable; therefore, we investigated cases that were quite similar to the one studied in the previous sections.

In fact, we defined the identification-system $\imath_n$ on P_n as being *admissible* when the following conditions are fulfilled:

(a) $\imath_n$ is orientation reversing and invariant with respect to the action of G_n;

(b) the space $|\overline{K}_1| = |P_1/\imath_1|$ is homeomorphic to $\mathbf{S}^3$;

(c) the space $|\overline{K}_n| = |P_n/\imath_n|$ is homeomorphic to a 3-manifold for each $n \geq 1$.

In this way, each element of the resulting (admissible) family is a cyclic covering of $\mathbf{S}^3$, branched over a suitable subcomplex of the 1-skeleton of $\overline{K}_1$ and over the polar diameter NS. Moreover, since a cyclic covering of a graph (with some vertices of

[3] Compare Appendix of [13].

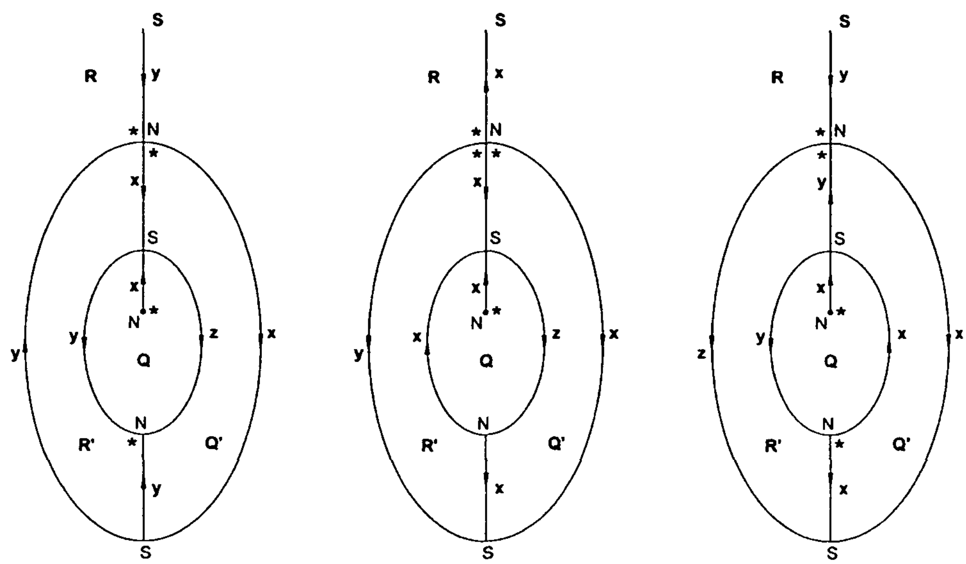

Figure 9

degree > 2) cannot be a 3-manifold, the branching set of an admissible family must be a knot or a link.

We developed this part of our research in two steps: first we studied all the possible orientation reversing identification-systems[4] on P_1, checking that only three produced the 3-sphere: K_1, K'_1 and K''_2. Figure 9 shows the corresponding identification-systems on P_1, which glue Q with Q' and R with R', matching up their starred reference points.

In each of these three cases, the cellular complex has two vertices (N and S) and three edges connecting them (x, y and z). Hence, each family arising from these cases (admissible or not) consists of branched cyclic coverings of a knot, or of a graph with two vertices of degree > 2 (the points N and S). We then checked all 12 possible cases with $n = 2$ arising from K_1, K'_1 or K''_1. In detail, starting from the complex $\partial(P_2)$ depicted in Figure 10, we tested – in the same way of Section 5 – all the spaces obtained by gluing Q_1 with T_i and Q_2 with T_{i+2}, for $i = 1, 3$ (note that the subscripts are mod 4), combined with all the identifications of R_1 and R_2, either with T_{i-1} and T_{i+1} or with T_{i+1} and T_{i-1}. Among the resulting combinations, seven give $\mathbf{S}^3$, two give the lens space $L(7, 3)$ and the other three fail to produce 3-manifolds. Because of the uniqueness of the representation of lens spaces (including the 3-sphere) as 2-fold coverings of knots or links **[10]**, these different admissible identification-systems produce no non-trivial family of 3-manifolds, except the M_n's already studied.

[4] There are 32 of them, up to symmetry.

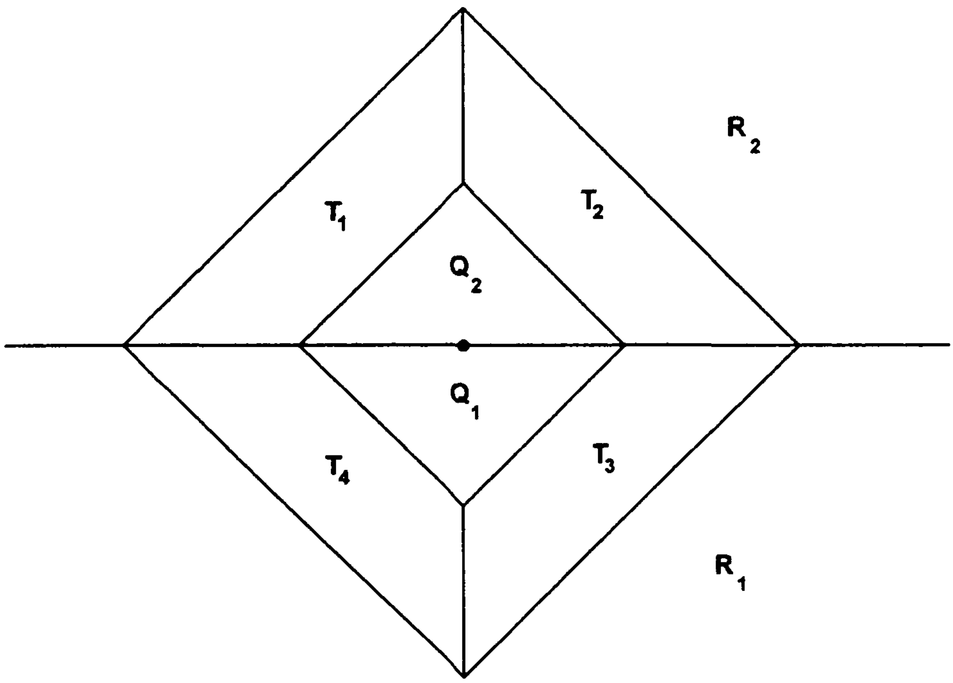

Figure 10

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