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MULTIPLIERS FROM SPACES OF TEST FUNCTIONS TO AMALGAMS

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Abstract

In this paper we study the space of multipliers $M(r, s : p, q)$ from the space of test functions $\Phi_{rs}(G)$, on a locally compact abelian group G, to amalgams $(L^p, l^q)(G)$; the former includes (when $r = s = \infty$) the space of continuous functions with compact support and the latter are extensions of the $L^p(G)$ spaces. We prove that the space $M(\infty : p)$ is equal to the derived space $(L^p)_0$ defined by Figá-Talamanca and give a characterization of the Fourier transform for amalgams in terms of these spaces of multipliers.

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1. Introduction

The space of test functions $\Phi_{\infty s}$ $(1 \leq s \leq \infty)$, on the real line, was originally defined by H. Holland [10]. The definition of $\Phi_{rs}(G)$ on a locally compact abelian group G, is due to Bertrandias and Dupuis [2]. The amalgam spaces (L^p, l^q) $(1 \leq p, q \leq \infty)$ are Banach spaces of functions which belong locally to $L^p(G)$ and globally to l^q. If $p = q$ then (L^p, l^q) is the usual $L^p(G)$ space. The purpose of this paper is to study the space $M(r, s : p, q)$ $(1 \leq r, s, p, q \leq \infty)$ of multipliers from $\Phi_{rs}(G)$ to $(L^p, l^q)(G)$. We prove the following.

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1. For $1 \le r, s, p \le \infty$ and $1 \le q < 2$, the space $M(r, s : p, q)$ is trivial.
2. For $r = s = \infty$ and $p = q$, the space $M(\infty : p)$ is equal to the derived space $(L^p)_0$ defined by Figá-Talamanca in [6].
3. for $r = s = \infty$ and $1 \le r, p, q \le \infty$, the space $M(r, s : p, q)$ contains or is equal to a set of Fourier transforms of measures. In particular a measure μ is the Fourier transform of a function in L^p, for $1 \le p \le 2$, if and only if μ is a multiplier in $M(p' : \infty)$.

2. Notation and preliminary results

Throughout this paper G is a locally compact abelian group with dual group Γ. The elements of Γ are denoted by $\hat{x}$ and we write $[x, \hat{x}]$ instead of $\hat{x}(x)$ $(x \in G)$. As usual $C_c(G)$ $(C_0(G))$ is the space of continuous functions on G with compact support (which vanish at infinity). For a function f on G, we use f' to denote the reflection $f'(y) = f(-y)$, and for x in G, the translation operator τ_x is defined by $\tau_x f(y) = f(y - x)$. If μ is a measure on G, then its reflection μ' and its translation $\tau_x\mu$ are defined by $\mu'(f) = \mu(f')$ and $\tau_x\mu(f) = \mu(\tau_x f)(f \in C_c(G))$ respectively. The pairing between a linear space B and its dual B^* is given by $\langle f, \sigma\rangle = \sigma(f)$ for σ in B^*, and f in B. We use J. Stewart's definition of the amalgam spaces $(L^p, l^q)(G) = (L^p, l^q)$, $(C_0, l^q)(G) = (C_0, l^q)$, $(L^p, c_0)(G) = (L^p, c_0)$ $(1 \le p, q \le \infty)$ and the space of measures $M_q(G) = M_q (1 \le q \le \infty)$ [12]. We assume all the properties of inclusion, duality, and convolution product of these spaces, Hölder and Young's inequalities, and the Hausdorff-Young theorem for amalgams as given in [14], and all the properties of the Segal algebra $S_0(G)$ given in [4] and [14]. We denote by A any of the amalgams (L^p, l^q), (L^p, c_0) $(1 \le p < \infty)$, $(C_0, l^s)(1 \le s \le \infty)$. We use H. Feichtinger's definition of the Fourier transform as an element of $S_0(G)^*$ [4, 14 Definition 2.3]. We write $\hat{\mu}$ $(\check{\mu})$ for the Fourier transform (inverse Fourier transform) of an element μ of $S_0(G)^*(S_0(\Gamma)^*)$. If M is a subset of $S_0(G)^*$, then $M^\frown$ denotes the set of Fourier transforms of element M. We let $\mathscr{M}_T$ be the space of transformable measures [1], and as usual p' is the conjugate of the number p. We finish this section with two preliminary results.

PROPOSITION 1. *If $\sigma \in S_0(G)^*$ and $h \in S_0(G)$, then $\sigma * h$ is the element of $L^\infty(G)$ given by $\langle f, \sigma * h\rangle = \langle f * h, \sigma\rangle$ for all f in $L^1(G)$. Hence $\langle f, \sigma * h\rangle = \langle h, \sigma * f\rangle$ for all f in $L^1(G)$.*

PROOF. By [14, Proposition 2.8], $\sigma * h$ is in $S_0(G)^*$ and for g in $S_0(G)$ we have that

$$|\langle g, \sigma * h\rangle| = |\langle g * h, \sigma\rangle| \leq \|\sigma\| \, \|h\|_{S_0} \|g\|_1.$$

The conclusion follows from the density of $S_0(G)$ in $L^1(G)$ [14, Proposition 2.5 and (2.5)].

THEOREM 2. *Let S be any of the spaces $(L^p, l^1)(G)$ $(1 \leq p < \infty)$ or (C_0, l^1). If $T : S \to S_0(G)^*$ is a linear bounded operator such that $T(f*g) = Tf * g$ for all f and g in S, then there exists a unique μ in $S_0(G)^*$ such that $Tf = \mu * f$ for all f in S.*

Hence $(Tf)\hat{} = \sigma \hat{f}$ for all f in S, where $\sigma = \hat{\mu}$.

PROOF. The proof is essentially the same as [14, Theorem 3.2]. Observe that the functions λ_α defined in the proof of [14, Theorem 3.2] belongs to $S_0(G)$ [13, Lemma 6.4] and $S_0(G)$ is included in $(C_0, l^0)(G)$. The second statement follows from [14, Proposition 2.8].

3. The space of multipliers

The space of test functions $\Phi_{rs}(G) = \phi_{rs}$ $(1 \leq r, s \leq \infty)$, as defined in [15, Definition 3.1] consists of continuous functions with compact support φ such that its Fourier transform $\hat{\varphi}$ belongs to $(C_0, l^s)(\Gamma)$. It is normed by $\|\hat{\varphi}\|_{rs}$ (see [14, (1.9)]). The duality between $\Phi_{rs}(G)$ and its Banach dual, $M_{s'}(\Gamma)$ if $r = \infty$, $(L^{r'}, l^{s'})(\Gamma)$ if r is finite [2, §2 c], [15, Remark 3.2ii)] will be denoted by $\langle\langle \varphi, \mu\rangle\rangle$, hence

$$\langle\langle \varphi, \mu\rangle\rangle = \int_\Gamma \hat{\varphi}(-\hat{x})\, d\mu(\hat{x}) \tag{1}$$

for $\varphi \in \Phi_{rs}$, $\mu \in M_{s'}(\Gamma)$ if $r = \infty$, $\mu \in (L^{r'}, l^{s'})(\Gamma)$ if $r < \infty$. Clearly, as sets, Φ_{rs} is equal to Φ_∞, and as normed spaces Φ_∞ is continuously embedded into Φ_{rs}. The space $\Phi_{\infty 1}$ is dense in $S_0(G)$ [123, Lemma 6.4; 5, p. 275] and it is the smallest of all the spaces Φ_{rs}.

DEFINITION 3. A multiplier from $\Phi_{rs}(G)$ $(1 \leq r, s \leq \infty)$ to the amalgam A is a bounded linear operator which is translation invariant, that is, for any $x \in G$, $\tau_x T = T\tau_x$.

The space of multipliers will be denoted by $M(r, s : p, q)$ if $A = (L^p, l^q)$, by $M(r, s : \infty, q)$ if $A = (C_0, l^q)$, and by $M(r, s : p, \infty)$ if $A = (L^p, c_0)$. If $r = s$ or $p = q$, then we write $M(r : q)$.

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If T is a multiplier from Φ_{rs} to A, then its adjoint T' is a bounded linear operator from A^* to Φ^*_{rs}, and by (1) we have for $g \in A^*$ and $\varphi \in \Phi_{rs}$ that

$$(2) \qquad \int_\Gamma \hat{\varphi}(-\hat{x})\, dT'g(\hat{x}) = \langle\langle \varphi , T'g\rangle\rangle = \langle T\varphi , g\rangle = \int_G T\varphi(x)\, dg'(x).$$

We use this to prove that T commutes with convolution.

Proposition 4. *Let T be in $M(r, s : p, q)$ $(1 \le r, s, p, q \le \infty)$. Then for all φ and ψ in Φ_{rs} we have $T(\varphi * \psi) = T\varphi * \psi$.*

Proof. Let g be in A^*. By (2) and Fubini's theorem we have that

$$\begin{aligned} \langle T\varphi * \psi , g\rangle &= \int_G \int_G T\varphi(x-s)\psi(s)\, ds\, dg'(x) \\ &= \int_G \psi(s)\langle\langle \tau_s\varphi , T'g\rangle\rangle\, ds = \int_\Gamma \hat{\varphi}(-\hat{x})\hat{\psi}(-\hat{x})\, dT'g(\hat{x}) \\ &= \langle\langle \varphi * \psi , T'g\rangle\rangle = \langle T(\varphi * \psi), g\rangle. \end{aligned}$$

If T is in $M(r, s : p, q)$ $(1 \le r, s, p, q \le \infty)$, $x \in G$, $g \in A^*$ and $\varphi \in \Phi_{rs}$, then as in the previous proof

$$\langle\langle \varphi , T'\tau_x g\rangle\rangle = \langle T\tau_x\varphi , g\rangle = \langle\langle \varphi , [x , .]T'g\rangle\rangle.$$

Hence

$$(3) \qquad T'\tau_x g = [x , .]T'g.$$

If F is the Fourier transform on Φ^*_{rs} and T is multiplier in $M(r, s, : p, q)$ $(1 \le r, s, p, q \le \infty)$, then by (2), Proposition 4, and [14, Proposition 2.5, 2.8] the composition of F and T' is a bounded linear operator which commutes with convolution. That is, for g and f in $(L^{p'}, l^1)(G)$ if $1 < p \le \infty$ and in $(C_0, l^1)(G)$ if $p = 1$ we have that

$$FT'(f * g) = FT'f * g.$$

This together with Theorem 2, Proposition 4, and [14, Remark 2.4 ii)] implies that there exists $\mu \in S_0(G)^*$, hence a unique $\sigma \in S_0(\Gamma)^*$, such that

$$(4) \qquad FT'f = \mu * f$$

$$(5) \qquad f'f = \sigma\hat{f} = (\mu * f)\hat{}.$$

Moreover, since $S_0(G)$ is included in $(L^{p'}, l^1)$ and (C_0, l^1) we have by Proposition 1, (5), and [14, (1.9)] that $\mu * f$ is a transformable measure for

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all f in $S_0(G)$. Hence by [1, Corollary 3.1], if $\varphi \in \Phi_{rs}(G)$, then $\hat{\varphi}$ belongs to $L^1(T'f)$ and therefore

$$\int_G \varphi(x)\mu * f(x)\,dx = \int_\Gamma \hat{\varphi}(-\hat{x})d(T'f)(\hat{x}).$$

By (1) and Proposition 1 we conclude that

$$\langle \mu * \varphi, f\rangle = \langle T\varphi, f\rangle$$

for all f in $S_0(G)$.

By the density of $S_0(G)$ in A, and [14, Theorem 1.4], we conclude that for all $\varphi \in \Phi_{rs}$

$$T\varphi = \mu * \varphi. \tag{6}$$

From (4) and (5) and the fact that Φ_{rs} is included in the amalgams $(L^{p'}, l^1)$ and (C_0, l^1) we have that

$$T\varphi = FT'\varphi \text{ and } (T\varphi)\hat{} = T'\varphi \quad \text{for all } \varphi \in \Phi_{rs}.$$

Proposition 5. *Let T be in $M(r, s : p, q)$ $(1 \le r, s, p, q \le \infty)$. The functional σ in $S_0(\Gamma)^*$ associated to T' in (5) belongs to $M_\infty(\Gamma)$. Moreover, σ belongs to*

1. $(L^1, l^\infty)(\Gamma)$ *if either r is finite or $1 \le q \le 2$.*
2. $M_2(\Gamma)$ *if $r = s = \infty$.*
3. $(L^1, l^2)(\Gamma)$ *if $r = s = \infty$ and $1 \le q \le 2$.*
4. $(L^{r'}, l^\infty)(\Gamma)$ *if $r = 2$ and r is finite.*
5. $(L^{q'}, l^\infty)(\Gamma)$ *if $1 \le q \le 2$ and $2 \le p$, $s \le \infty$.*

Proof. We take E a compact subset of Γ, h a continuous function with compact support contained in E, and g a function in $\Phi_{\infty 1}(G)$ such that $\hat{g}$ is in $C_c(G)$ and $\hat{g} \equiv 1$ on E [12, Theorem 3.1]. By [14, (2.6] we have that

$$\begin{aligned}|\langle h, \sigma\rangle| &= |\langle h\hat{g}, \sigma\rangle| = |\langle h, \sigma\hat{g}\rangle| = |\langle h, T'g\rangle| \\ &\le \|T'g\|_{r's}\|h\|_{rs} \le C_E\|h\|_\infty\end{aligned}$$

where C_E is a constant depending on E.

Therefore σ is a measure of Γ by [5, Theorem B1; 11, Theorem 5.1.4].

Now for β in I the function $\tau_\beta g$ is equal to one on L_β, (I and L_β as in [14, Remark 1.3]) and $T'([\beta, .]g) = \sigma\tau_\beta\hat{g}$ belongs to $M_{s'}(\Gamma)$ [14, (1.9)], hence

$$|\sigma|(L_\beta)^{s'} = \left[\int_{L_\beta} |\tau_\beta \hat{g}|(\hat{x}) d|\sigma|(\hat{x})\right]^{s'}$$
$$= \|\sigma\tau_\beta\hat{g}\|_{s'}^{s'} \le \|T'\|^{s'} \|[\beta,.]g\|_{p'q'}^{s'} = \|T'\|^{s'} \|g\|_{p'q'}^{s'}$$

and therefore σ is a measure in $M_\infty(\Gamma)$. To prove 1 we take a compact subset K of Γ with Haar measure zero, and a function φ in $\Phi_{rs}(G)$ such that $\hat{\varphi} \equiv 0$ on K. If r is finite, then $\sigma\hat{\varphi} = T'\varphi$ is a function in $(L^{r'}, l^{s'})(\Gamma)$ and we have that

$$\sigma(K) = \int_K \hat{\varphi}(\hat{x})\, d\sigma(\hat{x}) = \int_K T'\varphi(\hat{x}) d(\hat{x}) = 0.$$

Hence σ is absolutely continuous with respect to the Haar measure on Γ and we conclude from [3, Chapter V] that σ belongs to $(L^1, l^\infty)(\Gamma)$. If $1 \le q \le 2$, then by (6) and [14, Proposition 2.8, Remark 2.7] we have that $\sigma\hat{\varphi} = (T\varphi)\hat{}$ is a function on Γ. As before this implies that σ is in $(L^1, l^\infty)(\Gamma)$.

To prove 3 we note that Φ_∞ is equal to $\Phi_{\infty 2}$ as sets, and by (5), for any $\varphi \in \Phi_\infty$, the measure $\sigma\hat{\varphi}$ belongs to $M_1(\Gamma)$ that is, σ is a Fourier multiplier on $\Phi_{\infty 2}$ and by [15, Theorem 6.15], σ is in $M_2(\Gamma)$. Part 4 follows from 1 and 2.

Now, if r is finite and $s = 2$, then $\sigma\hat{\varphi}$ belongs to $(L^{r'}, l^2)(\Gamma)$ for all $\varphi \in \Phi_{r1}$. Hence $\nu\sigma\hat{\varphi}$ is in $L^1(\Gamma)$ for any ν in $(l^r, l^2)(\Gamma)$.

Again by [15, Theorem 6.1], $\nu\sigma$ belongs to $(L^1, l^2)(\Gamma)$ and by the converse of Hölder's inequality σ is in $(L^{r'}, l^\infty)(\Gamma)$.

Part 5 is similar to 4; note that $\sigma\hat{\varphi} = (T\varphi)\hat{}$ belongs to $(L^{q'}, l^2)(\Gamma)$ for all $\varphi \in \Phi_{rs}$.

From (6) and Proposition 2.4 we see that $M(r, s : p, q)$ $(1 \le r, s, p, q \le \infty)$ is isometrically isomorphic to the set of $\mu \in S_0(G)^*$ such that $\hat{\mu}$ is in $M_\infty(\Gamma)$ if $r = \infty$ and in $(L^1, l^\infty)(\Gamma)$ if r is finite, and norm equal to

$$|||\mu||| = \sup\{\|\mu * \varphi\|_{pq} | \varphi \in \Phi_{rs}, \|\hat{\varphi}\|_{rs} \le 1\}.$$

We now use the concept of set of uniqueness, to show that for $1 \le q < 2$, $(1 \le r, s, p \le \infty)$ the space $M(r, s : p, q)$ is trivial (cf. [6, Theorem 3]).

Definition 6. A subset E of Γ is a set of uniqueness for $(L^p, l^q)(G)$ $(1 \le p, q \le \infty)$, if for any f in $(L^p, l^q)(G)$ such that $\hat{f}$ vanishes outside E, then $f \equiv 0$.

Sets of uniqueness for $(L^p, l^q)(G)$ $(1 \le p, q \le 2)$ always exists [8, page 133], and also for $(L^r, l^q)(G)$ $(2 \le r \le \infty,\ 1 \le q \le 2)$ because $(L^r, l^q) \subset (L^p, l^q)$ for $1 \le p \le 2 \le r \le \infty$.

Theorem 7. *If* $1 \le p \le \infty$, $1 \le q < 2$ *and* f *is a function on* Γ *such that* $f\varphi$ *belongs to* $(L^p, l^q)(G)\hat{}$ *for all* $\varphi \in C_c(\Gamma)$. *Then* $f \equiv 0$ *locally almost everywhere.*

Proof. Suppose that f does not vanish locally almost everywhere. Then there exists a compact set K of nonnegative measure such that f does not vanish almost everywhere on K. Let ψ be a continuous function with compact support such that $\psi \equiv 1$ on K. Then ψf does not vanish locally almost everywhere. Since $\psi\varphi$ is in $C_c(\Gamma)$ for all $\varphi \in C_c(\Gamma)$, it follows that $\psi\varphi f$ belongs to $(L^p, l^q)(G)\hat{}$ for all $\varphi \in C_c(\Gamma)$. Thus without loss of generality we can assume that f vanishes off some compact set K of nonnegative measure.

If $p = q = 1$, then φf is in $L^1(G)\hat{}$ for all $\varphi \in C_c$, so $\varphi f = \hat{g}$ for some $g \in L^1(G)$. Since φf is in L^∞_c and $L^\infty_c \subset (L^2, l^1)$, the function g belongs to $L^1 \cap (L^\infty, l^2)$, then by the Riesz-Thorin theorem [13, Theorem 5.6; 10, Theorem 5], we have that g is in (L^p, l^q) for some fixed $1 < p < \infty$, $1 < q < 2$, so we can further assume that φf belongs to $(L^p, l^q)(G)\hat{}$ for some fixed $1 < p < \infty$, $1 < q < 2$.

If $p = \infty$ and $q = 1$, then as above $g \in (L^\infty, l^1)$ and φf is in $L^2(\Gamma)$, so g is in $(L^\infty, l^1) \cap L^2$. By the same argument we can assume that φf is in $(L^p, l^q)(G)\hat{}$ for some fixed $2 < p < \infty$, $1 < q < 2$.

If φ is a function in $C_c(\Gamma)$ such that $\varphi \equiv 1$ on K, then $\varphi f = f$, hence f is in $(L^p, l^q)(G)\hat{}$ and therefore f is a function in $L^{q'}$ with compact support, because f vanishes off K. Thus f belongs to $L^2(\Gamma)$.

Let S be the map defined on $C_c(\Gamma)$ by $(S\varphi)\hat{} = \varphi f$. An application of the Closed Graph Theorem shows that S restricted to $C_c(E)$, for E a compact subset of Γ, is continuous. Now we take E a compact subset of Γ and $\{\varphi_n\}$ a sequence in $C_c(\Gamma)$ such that $\varphi_n \equiv 1$ on E for all n, $0 \le \varphi_n(\hat{x}) \le 1$ for all $\hat{x}$ in Γ, and the support of each φ_n is equal to E, with $E_{n+1} \subset E_n$ and $E = \cap E_n$. Hence $\{\varphi_n\} \subset C_c(E_1)$ and converges pointwise to χ_E, the characteristic function of E. Since $E_{n+1} \subset E_n$ for all n, there is a constant C_E, depending on E_1 such that $\|\varphi_n\| \le C_E$ for all n. Hence $\|S\varphi_n\|_{pq} \le \|\varphi_n\|_\infty \le C_E$; that is, $\{S\varphi_n\}$ is a normed subset of (L^p, l^q), and therefore it has a weakly convergent subset $\{S\varphi_k\}$. Let g in (L^p, l^q) be such that $\lim\langle S\varphi_k, h\rangle = \langle g, h\rangle$ $(h \in (L^{p'}, l^{q'})(\Gamma))$. Since $|\varphi_k f| \le |f|$ on Γ, we have that for $h \in C_c(\Gamma)$

$$\langle S\varphi_k, h\rangle = \lim\langle (S\varphi_k)\hat{}, k\rangle = \lim\langle \varphi_k f, \hat{h}\rangle = \langle \chi_E, \hat{h}\rangle = \langle (\chi_E f)\hat{}, h\rangle.$$

We conclude that $(\chi_E f)\hat{} = g$. But if E is a subset for K and a set of uniqueness for (L^p, l^q), then this is a contradiction because $\chi_E f$ does not vanish almost everywhere on E.

PROPOSITION 8. *If* μ *is a multiplier in* $M(r, s : p, q)$ *for* $1 \le r, s, p, \le \infty$, $1 \le q < 2$, *then*

$$\hat{\mu}h \in (L^p, l^q)(G)^\wedge \text{ for all } h \in C_c(\Gamma).$$

PROOF. By Proposition 5, $\hat{\mu}$ is a function in $(L^1, l^\infty)(\Gamma)$. By [2, §2, c)] for $h \in C_c(\Gamma)$, there is a sequence $\{h_n\}$ in $\Phi_{\infty 1}(G)$ such that $\lim \|\hat{h}_n - h\|_{\infty 1} = 0$. Since $\|\mu * h_n\|_{pq} \le |||\mu||| \|\hat{h}_n\|_{\infty 1}$, the sequence $\{\mu * h_n\}$ is Cauchy in $(L^p, l^q)(\Gamma)$, so $\lim \|\mu * h_n - g\|_{pq} = 0$ for some g in $(L^p, l^q)(G)$. Since $S_0(G)$ is a subspace of $(C_0, l^1)(G)$, the pointwise product of ψ and $h_n - h$ belongs to $L^1(\hat{\mu}) = (C_0, l^1)(G)$, [13, Proposition 4.1]. Hence for ψ in $S_0(G)$

$$\begin{aligned} |\langle \psi, \hat{\mu}(\hat{h}_n)\rangle| &= |\langle \psi(\hat{h}_n - h), \hat{\mu}\rangle| \\ &\le \|\hat{\mu}\|_\infty \|\psi(\hat{h}_n - h)\|_{\infty 1} \le \|\hat{\mu}\|_\infty \|\psi\|_\infty \|\hat{h}_n - h\|_{\infty 1} \\ &\le \|\hat{\mu}\|_\infty \|\psi\|_{\infty k'} \|\hat{h}_n - h\|_{\infty 1} \end{aligned}$$

where k is equal to p' if $1 \le p \le 2$ and to 2 if $2 \le p \le \infty$. By the density of $S_0(\Gamma)$ in $(C_0, l^{k'})(\Gamma)$ we conclude that $\hat{\mu}(\hat{h}_n - h)$ is a function in $M_k(\Gamma)$ and therefore $\lim \|\hat{\mu}(\hat{h}_n - h)\|_{1k} = 0$ [14, page 125]. Since $\hat{h} - \hat{g}$ belongs to $(L^{q'}, l^k)(\Gamma)$ and $\hat{\mu}\hat{h}_n = (\mu * h_n)^\wedge$ (cf. (5) and (6)) we have by the continuity of the Fourier transform that

$$\begin{aligned} \|\hat{\mu}h - \hat{g}\|_{1k} &\le \|\hat{\mu}\hat{h}_n - \hat{\mu}h\|_{1k} + \|\hat{\mu}\hat{h}_n - \hat{g}\|_{1k} \\ &\le \|\hat{\mu}(\hat{h}_n - h)\|_{1k} + \|\hat{\mu}\hat{h}_n - \hat{g}\|_{q'k} \\ &\le \|\hat{\mu}(\hat{h}_n - h)\|_{1k} + C\|\mu * h_n - g\|_{pq} \end{aligned}$$

where C is a constant depending on G, p and q. This implies that $\hat{\mu}h = \hat{g}$.

COROLLARY 9. *The space* $M(r, s : p, q)$ *for* $1 \le r, s, p \le \infty$, $1 \le q < 2$ *is trivial.*

PROOF. Theorem 7, Proposition 8, and the inclusions $M(r, s : p, q) \subset M(\infty, r : p, q) \subset M(\infty, 1 : p, q)$.

This last result is for any locally compact abelian group, and this improves [11, Theorems 4.6.5 and 4.6.6] because as we will see in the next section, the derived space $(L^p)_0$ defined in [6] is equal to $M(\infty : p)$.

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4. Special infinite cases

In this section we give necessary and sufficient conditions for an element of $S_0(G)^*$ to be a multiplier.

PROPOSITION 10. *Let* μ *be an element of* $S_0(G)^*$ *with the Fourier transform* $\hat{\mu}$ *in* $(L^{r'}, l^{\infty})(\Gamma)$ *for* $1 \le r < \infty$ *(in* $M_{\infty}(\Gamma)$*). If* $\hat{\mu}h$ *is an element of* A *for each* h *in* $(L^r, l^s)(\Gamma)$ *(in* $(C_0, l^s)(\Gamma)$*)* $(1 \le s \le \infty)$*, then* μ *belongs to* $M(r, s : p, q)$ *(to* $M(\infty, s : p, q)$*)* $(1 \le p, q \le \infty)$.

PROOF. We define the map S on $(L^r, l^s)(\Gamma)$ by $(Sh)\hat{} = \hat{\mu}h$. Let $\{h_n\}$ be a sequence in (L^r, l^s) such that $\lim \|h_n - h\|_{rs} = 0$ and suppose that $\lim \|h_n - g\|_A = 0$. For $\psi \in S_0(G)$ we have that

$$\begin{aligned}
|\langle \psi, (Sh)\hat{} - \hat{g}\rangle| &\le |\langle \psi, (Sh_n)\hat{} - (Sh)\hat{}\rangle| + |\langle \psi, (Sh_n)\hat{} - \hat{g}\rangle| \\
&\le |\langle \psi, \hat{\mu}h_n - \hat{\mu}h\rangle| + |\langle \hat{\psi}, Sh_n - g\rangle| \\
&\le |\langle \psi(h_n - h), \hat{\mu}\rangle| + \|\hat{\psi}\|_{A^*}\|Sh_n - g\|_A \\
&\le \|\hat{\mu}\|_{r'\infty}\|\psi\|_{\infty s'}\|h_n - h\|_{rs} + \|\hat{\psi}\|_{A^*}\|Sh_n - g\|_A.
\end{aligned}$$

From [14, Remark 2.4 iii)], the density of $S_0(G)$ in A, and the Closed Graph Theorem, the map S is continuous. Now, if $\psi \in \Psi_{rs}$, then by [14, Proposition 2.8] we have that

$$\|\mu * \psi\|_A = \|S\hat{\psi}\|_A \le \|S\| \, \|\hat{\psi}\|_{rs}.$$

The proof for $r = \infty$ is similar.

REMARKS. The space $\mathscr{R}(\Phi_{rs})$ $(1 \le r, s \le \infty)$ of resonant classes of measures relative to Φ_{rs} [15, Definition 3.3] consists of transformable measures whose Fourier transform belongs to $(L^{r'}, l^{s'})(\Gamma)$ if $1 \le r < \infty$ to $M_{\infty s'}(\Gamma)$ if $r = \infty$.

From Proposition 10, Corollary 7, [15, Corollary 3.5; 1, Theorem 2.5] we have that

1. if $\mu \in \mathscr{M}_T$ with $\hat{\mu}h \in \mathscr{R}(\Phi_{p'q'})$ $(1 < p \le \infty, 1 \le q < 2)$ for each $h \in (C_0, l^1)(\Gamma)$, then $\mu \equiv 0$.
2. if $f \in (L^p, l^q)(G)$ $(1 \le q < 2, 1 \le p \le \infty)$ and $\hat{f}h \in (L^p, l^q)(\Gamma)\hat{}$ for each $h \in (C_0, l^1)(\Gamma)$, then $f \equiv 0$. That is, the subspace of $(L^p, l^q)(G)$ invariant under the product of Fourier transforms by elements of $(C_0, l^2)(\Gamma)$ is trivial.

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When $p = q$, this improves Figá-Talamanca's result in [6] because (C_0, l^1) is a subspace of $C_0 \cap L^1$.

THEOREM 11. *An element* $\mu \in S_0(G)^*$ *is a multiplier in* $M(r, s : p, \infty)$ $(1 \le r, s, p \le \infty)$ *if and only if for each* g *in* $(L^{p'}, l^1)(G)$ *there exists a measure* ν_g *in* $M_{s'}(\Gamma)$ *if* $r = \infty$ *and in* $(L^{r'}, l^{s'})(\Gamma)$ *if* r *is finite, such that* $\mu * g = \hat{\nu}_g$.

PROOF. The necessity part follows from (4) and (6). We now assume that r is finite and let R be the Segal algebra (L^p, l^1) if $1 < p \le \infty$ and (C_0, l^1) if $p = 1$. We define the map S on R by $Sg = \nu_g$. As in the previous proposition an application of the Closed Graph Theorem shows that S is continuous. By Proposition 1 and the fact that $(\mu * g)\hat{} = Sg (g \in S_0(G))$ [14, Remark 2.4ii)] the convolution $\mu * g$ is a transformable measure. Hence by [1, Corollary 3.1] for $\psi \in \Phi_{rs}(G)$ and $g \in S_0(G)$ we have that

$$\begin{aligned}|\langle g, \mu * \psi\rangle| &= |\langle \psi, \mu * g\rangle| = |\langle \psi, (\mu * g)\hat{}\rangle| \\ &= |\langle \hat{\psi}, Sg\rangle| \le \|Sg\|_{r's'} \|\hat{\psi}\|_{rs} \le \|S\| \, \|g\|_R \|\hat{\psi}\|_{rs}.\end{aligned}$$

Since $S_0(G)$ is dense in R and $\hat{\mu}\hat{\psi} = S\psi$ for all $\psi \in S_0(G)$, we conclude as in the proof of Proposition 5 that μ is a multiplier. The case $r = \infty$ is similar.

By [14, Theorem 6.2] we see that if $T \in M(\infty : p, q)$, then the element μ associated to FT' in (4) belongs to $(L^p, l^q)(\Gamma)$. Hence by (6), $M(\infty : p, q) \subset (L^p, l^q)$, but this is not always the case, as we will see in §4. The next theorem gives necessary and sufficient conditions for a function in $(L^p, l^q)(G)$ to be a multiplier.

THEOREM 12. *A function* f *in* $(L^p, l^q)(G)$ *belongs to* $M(r, s : p, q)$ $(1 \le r, s, p, q \le \infty)$ *if and only if for each* g *in* $(L^{p'}, l^{q'})(G)$, *there exists a unique measure* ν_g *in* $M_{s'}(\Gamma)$ *if* $r = \infty$ *in* $(L^{r'}, l^{s'})(\Gamma)$ *if* r *is finite, such that* $f * g = \check{\nu}_g$.

PROOF. Suppose that f is in $M(r, s : p, q)$ and define the function F on $\Phi_{rs}(G)$ by $F(\psi) = f * g * \psi(0)$. Clearly F is linear and $|F(\psi)| \le |||f||| \, \|g\|_{p'q} \|\hat{\psi}\|_{rs}$. By [15, Remark 3.2] there exists ν_g in $\Phi_{rs}(G)^*$ such that $\langle \psi, f * g\rangle = \langle\langle \psi, \nu_g\rangle\rangle$. This implies that $f * g$ is transformable and $(f * g)\hat{} = \nu_g$ [1, §2], hence $f * g = \hat{\nu}_g$ [14, Remark 2.4 ii)]. To prove the converse we define the function S on $(L^{p'}, l^{q'})(G)$ by $(Sg)\hat{} = \nu_g$ and, as in Theorem 11, the function S is continuous. Now for $\psi \in \Phi_{rs}$ we have that

$$\|f * \psi\|_{pq} = \sup\{|\langle g, f * \psi\rangle| \mid g \in B \text{ and } \|g\|_B \leq 1\}$$

where B is the amalgam $(L^{p'}, l^{q'})(G)$ if $1 < p, q \leq \infty$ $(C_0, l^{q'})$ if $p = 1$, $1 < q \leq \infty$ or (L^p, c_0) if $1 < p \leq \infty$, $q = 1$. Since

$$|\langle g, f * \psi\rangle| = |\langle \psi, f * g\rangle| = |\langle \psi, (Sg)\rangle||\langle \hat{\psi}, Sg\langle| \leq \|S\| \, \|g\|_B \|\hat{\psi}\|_{r's'}$$

we conclude as in the proof of Theorem 11 that f is a multiplier.

Remark. From Theorem 12 and [6, Lemma 1] the space $(L^p)_0$ is equal to $M(\infty : p)$. Moreover $M(\infty : p, q) \subset (L^p, l^q) \cap \mathscr{M}_{\mathscr{T}}$ for $1 \leq p$, $q \leq \infty$ [1, Theorem 2.3].

5. Spaces of Fourier transform of measures

In [6, §4] Figá-Talamanca showed that $(L^{p'})^\vee \subset M(\infty : p)$ $(2 \leq p \leq \infty)$ and $M_1^\vee = M(\infty : \infty)$. Similarly, in this section, we consider the problem of finding a space of measures M such that $M^\vee \subset M(r, s : p, q)$.

Theorem 13. 1. *Let* $2 \leq p$, $q \leq \infty$, $1 \leq s \leq \infty$, $1 \leq r < \infty$. *If* $1/x = 1/q + 1/r \leq 1$ *and* $1/y = 1/p + 1/s \leq 1$, *then* $(L^{x'}, l^{y'})(\Gamma)^\vee \subset M(r, s : p, q)$ *and* $(L^{r'}, l^{s'})(\Gamma)^\vee = M(r, s : \infty)$.

2. *Let* $2 \leq q < \infty$, $2 \leq p \leq \infty$, $1 \leq s \leq \infty$. *If* y *is as in part* 1, *then* $(L^{q'}, l^{y'})(\Gamma)^\vee \subset M(\infty, s : p, q)$, $M_{y'}(\Gamma)^\vee \subset M(\infty, s : p, \infty)$ *and* $M_{s'}(\Gamma)^\vee = M(\infty, s : \infty)$.

3. *Let* $2 \leq q \leq \infty$, $1 \leq p \leq 2$, $1 \leq r < \infty$, $1 \leq s \leq \infty$. *If* x *is as in part* 1 *and* $1/y = 1/2 + 1/s \leq 1$, *then* $(L^{x'}, l^{y'})(\Gamma)^\vee \subset M(r, s : p, q)$.

4. *Let* $2 \leq q < \infty$, $1 \leq p \leq 2$, $1 \leq s \leq \infty$. *If* y *is as in part* 3, *then* $(L^{q'}, l^{y'})(\Gamma)^\vee \subset M(\infty, s : p, q)$, $M_1(\Gamma)^\vee \subset M(\infty, s : p, \infty)$ *and* $(L^1, l^2)(\Gamma)^\vee \subset M(\infty : p, \infty)$.

Proof. 1. Let $f \in (L^{x'}, l^{y'})(\Gamma)$, $h \in S_0(G)$ and $\psi \in \Phi_{rs}(G)$. By [14, Definition 2.3 and (2.5)] we have that

$$\begin{aligned} |\langle h, \check{f} * \psi\rangle| &= |\langle \check{h}\check{\psi}, f\rangle| \leq \|f\|_{x'y'} \|\check{h}\check{\psi}\|_{xy} \\ &\leq \|f\|_{x'y'} \|\hat{h}\|_{pq} \|\hat{\psi}\|_{rs} \leq \|f\|_{x'y'} C \|h\|_{p'q'} \|\check{\psi}\|_{rs} \end{aligned}$$

where C is a constant depending on G, p and q, given by the Hausdorff-Young theorem for amalgams [14, Remark 2.7].

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Since $S_0(G)$ is dense in $(L^{p'}, l^{q'})$ we conclude by [14, Remark 2.4 ii)] that f is in $M(r, s : p, q)$ and

$$|||f||| \leq C\|f\|_{x'y'}. \tag{7}$$

The inclusion $(L^{r'}, l^{s'})(\Gamma)^{\vee} \subset M(r, s : \infty)$ is proven in a similar manner.

If f is in $M(r, s : \infty)$, then clearly the map $F(\psi) = \langle \psi, f \rangle$ is a functional on $\Phi_{rs}(G)$. Hence by [15, Remark 3.2 ii)] there exists $\mu \in (L^{r'}, l^{s'})(\Gamma)$ such that $\langle \psi, f \rangle = \langle\langle \psi, \mu \rangle\rangle = \langle \psi, \check{\mu} \rangle$ for all $\psi \in \Phi_{rs}(G)$, and in particular for all $\psi \in \Phi_{\infty 1}$. Since $\Phi_{\infty 1}$ is dense in $S_0(G)$, we conclude that $f = \check{\mu}$.

The proofs for 2, 3, and 4 are similar.

The amalgam (L^1, l^2) is the biggest space of functions whose Fourier transform is also a function [9]. Thus we see from Theorem 13, that if $y' > 2$, then $M(r, s : p, q)$ contains elements of $S_0(G)^*$ which are not functions. We will show that for certain values of p, q, r, s, the space $M(r, s : p, q)$ is included in an amalgam space, and contains a space of Fourier transforms. The constant which appears in the next result is given by the Hausdorff-Young theorem.

COROLLARY 14. 1. *If* $2 \leq q < \infty$ *and* $2 \leq p \leq \infty$, *then*

(a) $(L^{q'}, l^{p;})(\Gamma)^{\vee} \subset M(\infty : p, q) \subset (L^p, l^q)(G)$ *and* $\|f\|_{pq} \leq |||\check{f}||| \leq C\|f\|_{q'p'}$,

(b) $M_{p'}(\Gamma)^{\vee} \subset M(\infty : p, \infty) \subset (L^p, l^{\infty})(G)$ *and* $\|\check{\mu}\|_{p\infty} \leq |||\check{\mu}|||C\|\mu\|_{p'}$,

where C *is a constant depending on* G, p *and* q.

2. *If* $2 \leq q < \infty$ *and* $1 \leq p \leq 2$, *then*

(a) $(L^{q'}, l^2)(G)^{\vee} \subset M(\infty : p, q) \subset (L^2, l^q)(G)$ *and* $\|\check{f}\|_{2q} \leq |||\check{f}||| \leq C\|f\|_{q'2}$,

(b) $M_2(\Gamma)^{\vee} \subset M(\infty : p, \infty) \subset (L^2, l^{\infty})(G)$ *and* $\|\check{\mu}\|_{2\infty} \leq |||\check{\mu}||| \leq C\|\mu\|_2$,

where C *is a constant depending on* G *and* q.

3. *If* $2 \leq r$, $s \leq \infty$, $2 \leq q < \infty$ *and* $1/x = 1/q + 1/s \leq 1$, *then* $(L^{x'}, l^{s'})(\Gamma)^{\vee} \subset M(r, s : \infty, q) \subset (L^s, l^r)(G)$ *and* $\|\check{f}\|_{sr} \leq C|||\check{f}||| \leq C^2\|f\|_{x's'}$ *where* C *is a constant depending on* G, r, *and* s.

4. *If* $1 \leq r \leq 2 \leq s \leq \infty$, $2 \leq q < \infty$ *and* x *is as part* 3), *then* $(L^{x'}, l^{s'})(\Gamma)^{\vee} \subset M(r, s : \infty, q) \subset (L^p, l^q)(G)s$ *and* $\|\check{f}\|_{s2} \leq C|||\check{f}||| \leq C^2\|f\|_{sx}$ *where* C *is a constant depending on* G *and* s.

5. *If* $2 \leq s \leq \infty$, *then* $M_{s'}(\Gamma)^{\vee} \subset M(\infty, s : \infty) \subset (L^s, l^{\infty})(G)$ *and* $\|\check{\mu}\|_{s\infty} \leq |||\mu||| \leq C^2\|\mu\|_{s'}$ *where* C *is a constant depending on* G *and* s.

PROOF. 1. Let $\{\psi_U\}$ be the approximate identity of $L^1(G))$ defined in [15, page 462]. Since $S_0(G)$ is a Segal algebra we have for $\mu \in M(\infty : p, q)$ and $h \in S_0(G)$ that

$$\begin{aligned}|\langle h, \mu\rangle| &= \lim |\langle h * \psi_U, \mu\rangle| = \lim |\langle h, \mu * \phi_U\rangle| \\ &\leq \lim |||\mu||| \, \|\hat{\psi}_U\|_\infty \|h\|_{p'q'} \leq |||\mu||| \, \|h\|_{p'q'}.\end{aligned}$$

By [14, Proposition 2.6] we conclude that μ is in (L^p, l^q) and $\|\mu\|_{pq} \leq |||\mu|||$. The rest of the proof follows from (7) above. Part b) and 2 are proven in a similar manner.

3. Let $\mu \in M(r, s : \infty, q)$ and $h \in \Phi_{\infty 1}(G)$. As in the proof of part 1 using [14, Theorem 1.6] we have that

$$\begin{aligned}|\langle h, \mu\rangle| &\leq \lim |\langle \psi_U, \mu * h\rangle| \leq \lim \|\psi_U\|_{1q'} \|\mu * h\|_{\infty q} \\ &\leq \|\mu * h\|_{\infty q} \leq |||\mu||| \, \|\hat{h}\|_{rs} \leq |||\mu||| C \|h\|_{s'r'}.\end{aligned}$$

By the density of $\Phi_{\infty 1}$ in $(L^{s'}, l^{r'})(G)$ [14, Proposition 2.5] we conclude that $\|\mu\|_{rs} \leq |||\mu|||$. The rest of the proof follows from (7) above. The proofs of 4 and 5 are similar.

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