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ON PSEUDO $\mathcal{S}$-ASYMPTOTICALLY PERIODIC FUNCTIONS

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Abstract

We introduce the concept of pseudo $\mathcal{S}$-asymptotically periodic functions and study some of the qualitative properties of functions of this type. In addition, we discuss the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for abstract neutral functional differential equations. Some applications involving ordinary and partial differential equations with delay are presented.

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1. Introduction

In this paper we introduce the concept of pseudo $\mathcal{S}$-asymptotically periodic functions, establish some of the qualitative properties of functions of this type and study the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for a class of abstract neutral differential equations of the form

$$\frac{d}{dt}(u(t) - F(t, u_t)) = Au(t) + G(t, u_t), \qquad t \geq 0, \tag{1.1}$$

$$u_0 = \varphi \in \mathcal{C}, \tag{1.2}$$

where A is the generator of a uniformly exponentially stable semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on Banach space $(X, \|\cdot\|)$, the history u_t belongs to $\mathcal{C} = C([-r, 0]; X)$ and $F, G : [0, \infty) \times \mathcal{C} \to X$ are suitable functions.

A bounded continuous function $f : [0, \infty) \to X$ is called pseudo $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{l\to\infty} \frac{1}{l} \int_0^l \|f(s + \omega) - f(s)\|\, ds = 0.$$

The concept of pseudo $\mathcal{S}$-asymptotically periodic functions is a natural generalisation of the concept of $\mathcal{S}$-asymptotically periodic functions introduced recently in the literature. The literature on $\mathcal{S}$-asymptotically periodic functions is very recent and extensive. For some relevant work on $\mathcal{S}$-asymptotically periodic functions, we cite the pioneering work of Pierri *et al.* [9], the papers [8, 9, 17, 19] for qualitative properties of this class of functions, the works [1, 4, 5, 8, 9, 17] related to abstract differential equations and [15, 20, 22] for problems involving ordinary differential equations on finite-dimensional spaces.

Neutral differential equations arise in many areas of applied mathematics, and for this reason this type of equations has received much attention in recent years. Partial neutral differential equations arise, for example, in the theory elaborated by Gurtin and Pipkin [7] and Nunziato [18] for the description of heat conduction in materials with fading memory. In the classical theory of heat conduction, it is assumed that both the internal energy and the heat flux are linearly dependent on the temperature u and its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [7, 18], the internal energy and the heat flux are described as functionals of u and u_x. In connection with the above, we note that the system

$$\frac{d}{dt}\left(c_0 u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)\,ds\right) = c_1 \triangle u(t,x) + \int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)\,ds,$$
$$u(t,x) = 0, \qquad x \in \partial\Omega,$$

has frequently been used to describe this phenomenon; see, for instance, [16, 21]. In this system, $\Omega \subseteq \mathbb{R}^n$ is open and bounded with smooth boundary; $(t,x) \in [0,\infty) \times \Omega$; $u(t,x)$ represents the temperature in the position x and at the time t; c_1, c_2 are physical constants; and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation respectively. By assuming that the initial distribution of temperature u is known on $(-\infty, 0] \times \Omega$ and that $k_1 = k_2 = 0$ on a compact subset of $(-\infty, 0]$, we can transform this system into the abstract problem (1.1)–(1.2). For additional details on neutral differential equations we refer the reader to [10–13] and the references therein.

Next, we introduce some notation and concepts used in this paper. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_r(z, Z)$ denotes the closed ball with centre at $z \in Z$ and radius r in Z and the symbol $Z \hookrightarrow W$ is used in the case where Z is continuously included in W.

As usual, $C_b([a,\infty); Z)$ is the space formed by all the bounded continuous functions defined from $[a,\infty)$ into Z endowed with the uniform convergence norm denoted by $\|\cdot\|_{C_b([a,\infty);Z)}$, and $C_0([a,\infty); Z)$ denotes the subspace of $C_b([a,\infty); Z)$ formed by all the functions f such that $\lim_{t\to\infty} f(t) = 0$.

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We include now some well-known concepts on almost periodic functions.

DEFINITION 1.1. A function $f \in C_b(\mathbb{R}; Z)$ is called almost periodic if, for all $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, f)$, such that $\|f(t+\xi) - f(t)\|_Z < \varepsilon$ for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, f)$.

DEFINITION 1.2. A function $f \in C_b([0,\infty); Z)$ is said to be asymptotically almost periodic if there exist an almost periodic function g and a function $h \in C_0([0,\infty); Z)$ such that $f = g + h$. If g is ω-periodic, the function f is said to be asymptotically ω-periodic.

In the rest of this paper, $AAP_\omega(Z)$ denotes the subspace of $C_b([0,\infty); Z)$ formed by all the asymptotically ω-periodic functions.

From [9, Definition 3.1] we note the following concept.

DEFINITION 1.3. A function $f \in C_b([0,\infty); Z)$ is said to be $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that f is $\mathcal{S}$-asymptotically ω-periodic.

We denote by $\mathcal{S}AP_w(Z)$ the subspace of $C_b([0,\infty); Z)$ formed by all the $\mathcal{S}$-asymptotically ω-periodic functions.

We now introduce the concept of pseudo $\mathcal{S}$-asymptotically periodic function.

DEFINITION 1.4. A function $f \in C_b([0,\infty); Z)$ is called pseudo $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{h\to\infty} \frac{1}{h} \int_0^h \|f(s+\omega) - f(s)\|_Z \, ds = 0.$$

In this case, we say that f is pseudo $\mathcal{S}$-asymptotically ω-periodic.

We denote by $\mathcal{PS}AP_w(Z)$ the subspace of $C_b([0,\infty); Z)$ formed by all the pseudo $\mathcal{S}$-asymptotically ω-periodic functions.

To prove our main result on the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for the problem (1.1)–(1.2), we need to study conditions under which the functions $s \mapsto u_s$, $s \mapsto G(s, u_s)$ and $s \mapsto F(s, u_s)$ are pseudo $\mathcal{S}$-asymptotically ω-periodic. To this end, it is convenient to introduce a special class of pseudo $\mathcal{S}$-asymptotically ω-periodic functions.

DEFINITION 1.5. Let $p > 0$ and $u \in \mathcal{PS}AP_w(Z)$. We say that u is pseudo $\mathcal{S}$-asymptotically ω-periodic of class p if

$$\lim_{l\to\infty} \frac{1}{l} \int_p^l \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\|_Z \, ds = 0.$$

Denote by $\mathcal{PS}AP_{w,p}(Z)$ the subspace of $C_b([0,\infty); Z)$ formed by functions of this type.

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Definition 1.6. We say that a function $H \in C([0,\infty] \times Z; W)$ is uniformly (Z, W)-pseudo $\mathcal{S}$-asymptotically ω-periodic of class p if

$$\lim_{l\to\infty} \frac{1}{l} \int_p^l \sup_{\tau\in[s-p,s]} \sup_{\|z\|_Z\le R} \|H(\tau+\omega, z) - H(\tau, z)\|_W \, ds = 0,$$

for all $R > 0$. Denote by $\mathcal{PSAP}_{\omega,p}(Z, W)$ the set formed by functions of this type.

For additional details on almost periodic, asymptotically almost periodic and $\mathcal{S}$-asymptotically periodic functions, we refer the reader to [1–5, 8, 9, 14, 17, 23] and the references therein.

In the next section we study some qualitative properties of the class of pseudo $\mathcal{S}$-asymptotically periodic functions. Specifically, we discuss conditions under which a pseudo $\mathcal{S}$-asymptotically periodic function is $\mathcal{S}$-asymptotically periodic or asymptotically periodic. In Section 3 we study the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for the problem (1.1)–(1.2). In the final section we consider some applications on the existence of pseudo $\mathcal{S}$-asymptotically periodic solutions for ordinary and partial neutral differential equations.

2. On pseudo $\mathcal{S}$-asymptotically periodic functions

In this section we study some properties concerning pseudo $\mathcal{S}$-asymptotically ω-periodic functions. To begin, we establish without proof the following proposition.

Proposition 2.1. *The following conditions are satisfied.*

(a) *$\mathcal{PSAP}_w(X)$ is a Banach space and $AAP_\omega(X) \hookrightarrow \mathcal{S}AP_w(X) \hookrightarrow \mathcal{PSAP}_w(X)$.*

(b) *If $f \in \mathcal{PSAP}_w(X)$ is differentiable and f' is uniformly continuous, then $f' \in \mathcal{PSAP}_w(X)$.*

From [17] we know that $AAP_\omega(X) \neq \mathcal{S}AP_w(X)$. In the next examples we note that $\mathcal{PSAP}_w(X) \neq \mathcal{S}AP_w(X)$.

Example 2.2. Let $f : [0,\infty) \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} n^2 t - n^3 + 1, & t \in \left[n - \dfrac{1}{n^2}, n\right),\ n \in \mathbb{N}, \\ -n^2 t + n^3 + 1, & t \in \left[n, n + \dfrac{1}{n^2}\right),\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f \in C_b([0,\infty); \mathbb{R}) \bigcap L^1([0,\infty); \mathbb{R})$, we have that $f \in \mathcal{PSAP}_w(\mathbb{R})$ for all $\omega > 0$. On the other hand, for $\omega > 0$,

$$\lim_{n\to\infty} (f(n+\omega) - f(n)) = -1 \quad \text{and} \quad \lim_{n\to\infty} \left(f\left(n + \frac{1}{n^2} + \omega\right) - f\left(n + \frac{1}{n^2}\right)\right) = 0,$$

which shows that $f \notin \mathcal{S}AP_w(\mathbb{R})$.

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EXAMPLE 2.3. In Example 2.2, the function f belongs to $L^1([0,\infty);\mathbb{R})$ which implies that $f \in \mathcal{PSAP}_w(X)$. Next, we consider a function $g \in \mathcal{PSAP}_w(X)$ such that $g \notin L^1([0,\infty);\mathbb{R})$ and $\liminf_{l\to\infty}(1/l)\int_0^l g(s)\,ds \neq 0$.

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim_{n\to\infty} b_n = 0$, $b_n \neq 0$ for all $n \in \mathbb{N}$ and the sequence $(a_n)_{n\in\mathbb{N}} = (\sum_{i=1}^n b_i)_{n\in\mathbb{N}}$ is bounded, nonconvergent and $a_n \geq 1$ for all $n \in \mathbb{N}$.

Let $g : [0,\infty) \to \mathbb{R}$ be defined as $g(t) = a_{n+1} + (a_{n+1} - a_n)(t - n - 1)$ for $t \in [n, n+1]$ and $n \in \mathbb{N}$. From [17, Example 2.1], $g \in \mathcal{SAP}_w(\mathbb{R})$ for all $\omega > 0$, which implies that $g \in \mathcal{PSAP}_w(\mathbb{R})$ for all $\omega > 0$. On the other hand, by noting that $\int_n^{n+1} g(s)\,ds \geq \min\{a_n, a_{n+1}\} \geq 1$ for all $n \in \mathbb{N}$, we obtain that $\liminf_{l\to\infty}(1/l)\int_0^l g(s)\,ds \geq 1$.

In the following propositions, we establish conditions under which a pseudo $\mathcal{S}$-asymptotically ω-periodic function is $\mathcal{S}$-asymptotically ω-periodic or asymptotically ω-periodic.

DEFINITION 2.4. Let $f \in C_b([0,\infty); X)$ and $p > 0$. We say that f is asymptotically (p,ω)-norm oscillating if, for all $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that

$$\sup_{s\in[t,t+\omega]} \|f(s+\omega) - f(s)\| \leq \sup_{\tau\in[\theta,\theta+p]} \|f(\tau+\omega) - f(\tau)\| + \varepsilon,$$

for all $t \geq L_\varepsilon$ and each $\theta \in [t, t+\omega]$.

REMARK 2.5. There are many examples of asymptotically (p,ω)-norm oscillating functions. We note, for instance, that an asymptotically ω-periodic function is $(n\omega,\omega)$-norm oscillating for all $n \in \mathbb{N}$.

DEFINITION 2.6. Let $v \in C([0,\infty);\mathbb{R}^+)$ and $f \in C_b([0,\infty); X)$. We say that f is v-pseudo $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{l\to\infty} \frac{1}{l}\int_0^l \frac{\|f(s+\omega) - f(s)\|}{v(s)}\,ds = 0.$$

In this case, we say that f is v-pseudo $\mathcal{S}$-asymptotically ω-periodic.

In the next result, for $f \in C_b([0,\infty); X)$ and $p \geq 0$, we use the notation $H_{f,p}$ for the function $H_{f,p} : [0,\infty) \to \mathbb{R}^+$ given by

$$H_{f,p}(s) = \sup_{\tau\in[s,s+p]} \|f(\tau+\omega) - f(\tau)\|.$$

PROPOSITION 2.7. *Let $v \in C([0,\infty),\mathbb{R}^+)$ and $f \in \mathcal{PSAP}_w(\mathbb{R})$. Suppose that f is asymptotically (p,ω)-norm oscillating, v is nonincreasing and either of the following conditions is satisfied:*

(a) *$H_{f,p}$ is v-pseudo $\mathcal{S}$-asymptotically ω-periodic and $\{v(s)s \mid s \geq 0\}$ is bounded;*
(b) *the set $\{(1/r)\int_0^r (H_{f,p}(s)/v(s))\,ds \mid r \geq 0\}$ is bounded and $\lim_{s\to\infty} v(s)s = 0$.*

Then $f \in \mathcal{SAP}_w(\mathbb{R})$.

Proof. In both (a) and (b), for $\varepsilon > 0$, there is $L_\varepsilon > \omega$ such that

$$\sup_{\vartheta\in[t,t+\omega]} \|f(\vartheta+\omega) - f(\vartheta)\| \le \sup_{\tau\in[\theta,\theta+p]} \|f(\tau+\omega) - f(\tau)\| + \varepsilon,$$
$$\frac{\nu(s)s}{\omega}\left(\frac{1}{s}\int_0^s \frac{H_{f,p}(\tau)}{\nu(\tau)}\,d\tau\right) \le \varepsilon,$$

for all $t \ge L_\varepsilon$, every $\theta \in [t, t+\omega]$ and each $s \ge L_\varepsilon$. Then, for $s > L_\varepsilon$,

$$\begin{aligned}
\|f(s+\omega) - f(s)\| &= \frac{1}{\omega}\int_s^{s+\omega} \|f(s+\omega) - f(s)\|\,d\theta \\
&\le \frac{1}{\omega}\int_s^{s+\omega} \sup_{\tau\in[s,s+\omega]} \|f(\tau+\omega) - f(\tau)\|\,d\theta \\
&\le \frac{1}{\omega}\int_s^{s+\omega} \nu(\theta) \sup_{\tau\in[\theta,\theta+p]} \frac{\|f(\tau+\omega) - f(\tau)\|}{\nu(\theta)}\,d\theta + \frac{1}{\omega}\int_s^{s+\omega} \varepsilon\,d\theta \\
&\le \frac{\nu(s)s}{\omega}\left(\frac{1}{s}\int_0^s \frac{H_{f,p}(\theta)}{\nu(\theta)}\,d\theta\right) + \varepsilon, \\
&\le 2\varepsilon,
\end{aligned}$$

which implies that $\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0$ and $f \in \mathcal{S}AP_w(\mathbb{R})$. □

The next corollary follows directly from [19, Proposition 2.4] and Proposition 2.7. From [19] we note that a function $f \in C_b([0,\infty), X)$ is ω-normal on compact sets if for every sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ with $m_n \to \infty$ as $n \to \infty$, there exist a subsequence $(m_{n_j})_{j\in\mathbb{N}}$ and $F \in C_b([0,\infty), X)$ such that $f_{m_{n_j}\omega} \to F$ as $j \to \infty$ uniformly on compact subsets of $[0,\infty)$, where $f_{m_{n_j}\omega}(t) = f(t + m_{n_j}\omega)$ for $t \ge 0$.

Corollary 2.8. *Suppose that the assumptions in Proposition 2.7 are fulfilled, f is ω-normal on compact sets and $\sum_{j\ge0} \nu(j\omega) < \infty$. Then f is asymptotically ω-periodic.*

We conclude this section with the following result.

Proposition 2.9. *Assume that X is a Banach space over the complex field $\mathbb{C}$. If f is asymptotically almost periodic and $f \in \mathcal{PSAP}_w(X)$, then f is asymptotically ω-periodic.*

Proof. Suppose that $f = g + \varphi$, where $\varphi \in C_0([0,\infty); X)$ and g is almost periodic. Since f and φ are pseudo $\mathcal{S}$-asymptotically periodic, it follows that $g \in \mathcal{PSAP}_w(X)$. By noting that the function $G_w : [0,\infty) \to X$ given by $G_\omega(t) = g(t+\omega) - g(t)$ is almost periodic, from the results in [23] we known that G_ω has an associated Fourier series $\sum_{n=1}^{\infty} \hat{G_\omega}(\lambda_n) e^{i\lambda_n t}$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of numbers and the coefficients $\hat{G_\omega}(\lambda_n)$ are given by

$$\hat{G_\omega}(\lambda_n) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda_n s} G_\omega(s)\,ds.$$

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Moreover, from the theory of almost periodic functions (see the last part of the proof of [23, Theorem 10.1]) we see that

$$\begin{aligned}\|\hat{G_\omega}(\lambda_n)\| &= \left\| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\lambda_n s} G_\omega(s)\, ds \right\| \\ &\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|G_\omega(s)\|\, ds \\ &\leq \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \|G_\omega(s)\|\, ds,\end{aligned}$$

which implies that $\hat{G_\omega}(\lambda_n) = 0$ for all $n \in \mathbb{N}$. Finally, from the uniqueness of the Fourier series [23, p. 111] we obtain that $G_\omega(s) = 0$ for all $s \geq 0$ and that g is ω-periodic. Thus, f is asymptotically ω-periodic. □

3. Pseudo $\mathcal{S}$-asymptotically ω-periodic mild solutions for abstract neutral equations

In this section we study the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for the problem (1.1)–(1.2). In the rest of this paper, $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on $(X, \|\cdot\|)$ and M, γ are positive constants such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\gamma t}$ for all $t > 0$.

Definition 3.1. A function $u \in C([-r, \infty); X)$ is said to be a mild solution of (1.1)–(1.2) if $u_0 = \varphi$ and

$$\begin{aligned}u(t) = T(t)(\varphi(0) - F(0, \varphi)) + F(t, u_t) + \int_0^t AT(t-s)F(s, u_s)\, ds \\ + \int_0^t T(t-s)G(s, u_s)\, ds, \quad \forall t \geq 0.\end{aligned} \tag{3.1}$$

To establish our results we consider the following conditions.

(H$_1$) There exist a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X and constants $\alpha \in [0, 1)$, $C > 0$ such that $\|AT(t)\|_{\mathcal{L}(Y,X)} \leq Ce^{-\gamma t} t^{-\alpha}$ for every $t > 0$. The function F belongs to $C([0, \infty) \times \mathcal{C}; Y)$ and there exists $L_F \in C_b([0, \infty); \mathbb{R}^+)$ such that $\|F(t, \psi_1) - F(t, \psi_2)\|_Y \leq L_F(t)\|\psi_1 - \psi_2\|_{\mathcal{C}}$ for all $(t, \psi_i) \in [0, \infty) \times \mathcal{C}$.

(H$_2$) The function G belongs to $C([0, \infty) \times \mathcal{C}; X)$ and there exists $L_G \in C_b([0, \infty); \mathbb{R}^+)$ such that $\|G(t, \psi_1) - G(t, \psi_2)\| \leq L_G(t)\|\psi_1 - \psi_2\|_{\mathcal{C}}$ for all $(t, \psi_i) \in [0, \infty) \times \mathcal{C}$.

In the next two lemmas we study conditions under which the functions $s \mapsto u_s$, $s \mapsto G(s, u_s)$ and $s \mapsto F(s, u_s)$ are pseudo $\mathcal{S}$-asymptotically ω-periodic.

Lemma 3.2. *Let $u \in C_b([-r, \infty); X)$ and assume that $u|_{[0,\infty)} \in \mathcal{PSAP}_{\omega,p}(X)$. Then the function $s \mapsto u_s$ belongs to $\mathcal{PSAP}_{\omega,p}(\mathcal{C})$. Similarly, if $u \in C_b([-r, \infty); Y)$ and $u|_{[0,\infty)} \in \mathcal{PSAP}_{\omega,p}(Y)$, then the function $s \mapsto u_s$ belongs to $\mathcal{PSAP}_{\omega,p}(\mathcal{C})$.*

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Proof. We consider separately the cases $p \geq r$ and $p < r$. First, we assume $r > p$. Let $k \in \mathbb{N}$ such that $kp > r$. For $l > p + r$

$$\begin{aligned}
&\frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \|u_{\tau+\omega} - u_\tau\|_C \, ds \\
&\quad \leq \frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \sup_{\theta\in[-r,0]} \|u(\tau+\omega+\theta) - u(\tau+\theta)\| \, ds \\
&\quad \leq \frac{1}{l}\int_p^l \sup_{\tau\in[s-p-r,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\quad \leq \frac{1}{l}\int_p^l \Big(\sum_{i=-1}^{k-1} \sup_{\tau\in[s-r+ip,s-r+(i+1)p]} \|u(\tau+\omega) - u(\tau)\|\Big) \, ds \\
&\quad \leq \sum_{i=-1}^{k-1} \frac{1}{l}\int_{p-r+(i+1)p}^{l-r+(i+1)p} \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\quad \leq \sum_{i=-1}^{k-1} \frac{1}{l}\int_{p-r}^{l+kp-r} \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\quad \leq \sum_{i=-1}^{k-1} \frac{1}{l}\int_{p-r}^{p} \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\qquad + \sum_{i=-1}^{k-1} \frac{1}{l}\int_{p}^{l+kp-r} \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\quad \leq 2(k+1)\frac{r}{l}\|u\|_{C_b([0,\infty);X)} + (k+1)\frac{(l+kp-r)}{l}\,\frac{1}{l+kp-r} \\
&\qquad \times \int_p^{l+kp-r} \sup_{\tau\in[s-p,s]} \|u(\tau+\omega) - u(\tau)\| \, ds,
\end{aligned}$$

from which we infer that

$$\lim_{l\to\infty} \frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \|u_{\tau+\omega} - u_\tau\|_C \, ds = 0$$

and that the function $s \mapsto u_s$ belongs to $\mathcal{PSAP}_{\omega,p}(C)$.

Now we suppose that $p > r$. Let $k \in \mathbb{N}$ such that $kr > 2p$. For $l > kr + p$, we see that

$$\begin{aligned}
&\frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \|u_{\tau+\omega} - u_\tau\|_C \, ds \\
&\quad \leq \frac{1}{l}\int_p^l \sup_{\tau\in[s-p-r,s]} \|u(\tau+\omega) - u(\tau)\| \, ds \\
&\quad \leq \frac{1}{l}\int_p^{rk} \sup_{\tau\in[s-p-r,s]} \|u(\tau+\omega) - u(\tau)\|
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{l}\int_{rk}^{l}\sup_{\tau\in[s-p-r,s]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\le\frac{2(rk-p)}{l}\|u\|_{C_b([0,\infty);X)}+\frac{1}{l}\int_{rk}^{l}\sup_{\tau\in[s-2p,s]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\le\frac{2(rk-p)}{l}\|u\|_{C_b([0,\infty);X)}+\frac{1}{l}\int_{rk}^{l}\sup_{\tau\in[s-2p,s-p]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\quad+\frac{1}{l}\int_{rk}^{l}\sup_{\tau\in[s-p,s]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\le\frac{2(rk-p)}{l}\|u\|_{C_b([0,\infty);X)}+\frac{1}{l}\int_{rk-p}^{l-p}\sup_{\tau\in[s-p,s]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\quad+\frac{1}{l}\int_{p}^{l}\sup_{\tau\in[s-p,s]}\|u(\tau+\omega)-u(\tau)\|\,ds\\
&\le\frac{2(rk-p)}{l}\|u\|_{C_b([0,\infty);X)}+\frac{2}{l}\int_{p}^{l}\sup_{\tau\in[s-p,s]}\|u(\tau+\omega)-u(\tau)\|\,ds,
\end{aligned}
$$

which implies that $\lim_{l\to\infty}(1/l)\int_p^l \sup_{\tau\in[s-p,s]}\|u_{\tau+\omega}-u_\tau\|_C\,ds=0$ and $s\mapsto u_s$ belongs to $\mathcal{PSAP}_{\omega,p}(C)$. This completes the proof. □

Lemma 3.3. *Assume that condition* $\mathbf{H_1}$ *is satisfied and* $F\in\mathcal{PSAP}_{\omega,p}(C,Y)$. *If* $u\in C_b([-r,\infty);X)$ *is such that* $u|_{[0,\infty)}\in\mathcal{PSAP}_{\omega,p}(X)$, *then the function* $s\mapsto F(s,u_s)$ *belongs to* $\mathcal{PSAP}_{\omega,p}(Y)$. *Similarly, if condition* $\mathbf{H_2}$ *is satisfied,* $G\in\mathcal{PSAP}_{\omega,p}(C,X)$ *and* $u\in C_b([-r,\infty);X)$ *is such that* $u|_{[0,\infty)}\in\mathcal{PSAP}_{\omega,p}(X)$, *then the function* $s\mapsto G(s,u_s)$ *belongs to* $\mathcal{PSAP}_{\omega,p}(X)$.

Proof. We only prove the first assertion. Let $R=\|u\|_{C_b([-r,\infty);X)}$. Since the function $s\mapsto u_s$ belongs to $\mathcal{PSAP}_{\omega,p}(C)$ (see Lemma 3.2), for $\varepsilon>0$ there exists $L_\varepsilon>0$ such that

$$
\begin{aligned}
\frac{1}{l}\int_p^l\sup_{\tau\in[s-p,s]}\sup_{\|x\|_C\le R}\|F(\tau+\omega,x)-F(\tau,x)\|_Y\,ds\le\varepsilon,\\
\|L_F\|_{C_b([0,\infty);\mathbb{R})}\frac{1}{l}\int_p^l\sup_{\tau\in[s-p,s]}\|u_{\tau+\omega}-u_\tau\|_C\,ds\le\varepsilon,
\end{aligned}
$$

for all $l\ge L_\varepsilon$. Under these conditions, for $l\ge L_\varepsilon$,

$$
\begin{aligned}
&\frac{1}{l}\int_p^l\sup_{\tau\in[s-p,s]}\|F(\tau+\omega,u_{\tau+\omega})-F(\tau,u_\tau)\|_Y\,ds\\
&\quad\le\frac{1}{l}\int_p^l\sup_{\tau\in[s-p,s]}\|F(\tau+\omega,u_{\tau+\omega})-F(\tau,u_{\tau+\omega})\|_Y\,ds\\
&\qquad+\frac{1}{l}\int_p^l\sup_{\tau\in[s-p,s]}\|F(\tau,u_{\tau+\omega})-F(\tau,u_\tau)\|_Y\,ds
\end{aligned}
$$

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$$
\begin{aligned}
&\le \frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]}\ \sup_{\|x\|_C\le R} \|F(\tau+\omega,x)-F(\tau,x)\|_Y\,ds\\
&\qquad + \|L_F\|_{C_b([0,\infty);\mathbb{R})}\frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \|u_{\tau+\omega}-u_\tau\|_Y\,ds\\
&\le 2\varepsilon,
\end{aligned}
$$

which proves the assertion. □

The next result is the key to proving our main theorem.

Lemma 3.4. *Let $u\in C_b([-r,\infty);Y)$ and $v:[-r,\infty)\to X$ be the function defined by $v(t)=0$ for $t\in[-r,0]$ and $v(t)=\int_0^t AT(t-\tau)u(\tau)\,d\tau$ for $t\ge 0$. If $u|_{[0,\infty)}\in\mathcal{PSAP}_{\omega,p}(Y)$ then $v\in\mathcal{PSAP}_{\omega,p}(X)$.*

Proof. We analyse separately the cases $p<1$ and $p\ge 1$, and for the sake of simplicity we consider the function $h:[0,\infty)\to\mathbb{R}^+$ defined by $h(\xi)=\|u(\xi+\omega)-u(\xi)\|_Y$.

To begin, we note that from the Bochner's criteria for integrable functions and the estimate

$$
\begin{aligned}
\left\|\int_0^t AT(t-s)u(s)\,ds\right\| &\le \int_0^t \|AT(t-s)\|_{\mathcal{L}(Y,X)}\|u(s)\|_Y\,ds\\
&\le C\int_0^t \frac{e^{-\gamma(t-s)}}{(t-s)^\alpha}\|u(s)\|_Y\,ds\\
&\le C\|u\|_{C([0,\infty),Y)}\Big(\frac{1}{\gamma}+\frac{1}{1-\alpha}\Big),
\end{aligned}
$$

it follows that v is well defined and $v\in C_b([-r,\infty);Y)$.

Assume that $p<1$. Let $q>1+p$ and $k\in\mathbb{N}$ such that $kp>1$. For $l\ge q$,

$$
\begin{aligned}
&\frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]}\|v(\tau+\omega)-v(\tau)\|\,ds\\
&\quad\le \frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]}\int_0^\omega \|AT(\tau+\omega-\xi)u(\xi)\|\,d\xi\,ds\\
&\qquad + \frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]}\int_0^\tau \|AT(\tau-\xi)(u(\xi+\omega)-u(\xi))\|\,d\xi\,ds\\
&\quad\le \frac{C}{l}\int_p^l \sup_{\tau\in[s-p,s]}\int_0^\omega \frac{e^{-\gamma(\tau+\omega-\xi)}}{(\tau+\omega-\xi)^\alpha}\|u(\xi)\|_Y\,d\xi\,ds\\
&\qquad + \frac{C}{l}\int_p^q \sup_{\tau\in[s-p,s]}\int_0^\tau \frac{e^{-\gamma(\tau-\xi)}}{(\tau-\xi)^\alpha}h(\xi)\,d\xi\,ds\\
&\qquad + \frac{C}{l}\int_q^l \sup_{\tau\in[s-p,s]}\int_0^\tau \frac{e^{-\gamma(\tau-\xi)}}{(\tau-\xi)^\alpha}h(\xi)\,d\xi\,ds\\
&\quad= I_1(l)+I_2(l)+I_3(l).
\end{aligned}
$$

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We estimate the terms $I_i(l)$ separately. First, we see that

$$\begin{aligned} I_1(l) &\le \frac{C}{l}\int_p^l e^{-\gamma(s-p)} \sup_{\tau\in[s-p,s]}\int_0^\omega \frac{e^{-\gamma(\omega-\xi)}}{(\omega-\xi)^\alpha}\|u(\xi)\|_Y\,d\xi\,ds \\ &\le \frac{C}{l}\int_p^l e^{-\gamma(s-p)}\|u\|_{C_b([0,\infty);Y)}\frac{w^{1-\alpha}}{1-\alpha}\,ds \\ &\le C\frac{e^{\gamma p}}{\gamma}\|u\|_{C_b([0,\infty);Y)}\frac{w^{1-\alpha}}{1-\alpha}\frac{1}{l}, \end{aligned}$$

from which we infer that $I_1(l)\to 0$ as $l\to\infty$. Next,

$$\begin{aligned} I_2(l) &\le \frac{C}{l}\int_p^q \sup_{\tau\in[s-p,s]}\int_0^\tau \frac{h(\xi)}{(\tau-\xi)^\alpha}\,d\xi\,ds \\ &\le \frac{C}{l}\int_p^q \|h\|_{C_b([0,\infty);Y)}\frac{s^{1-\alpha}}{1-\alpha}\,ds \\ &\le 2C\|u\|_{C_b([0,\infty);Y)}\frac{q^{1-\alpha}}{1-\alpha}\frac{q-p}{l}, \end{aligned}$$

which implies that $I_2(l)\to 0$ as $l\to\infty$. Finally,

$$\begin{aligned} I_3(l) &\le \frac{C}{l}\int_q^l \sup_{\tau\in[s-p,s]}\bigg(\int_0^{\tau-1}\frac{e^{-\gamma(\tau-\xi)}}{(\tau-\xi)^\alpha}h(\xi)\,d\xi + \int_{\tau-1}^\tau \frac{e^{-\gamma(\tau-\xi)}}{(\tau-\xi)^\alpha}h(\xi)\,d\xi\bigg)\,ds \\ &\le \frac{C}{l}\int_q^l \sup_{\tau\in[s-p,s]}\int_0^{\tau-1} e^{-\gamma(\tau-\xi)}h(\xi)\,d\xi\,ds + \frac{C}{l}\int_q^l \sup_{\tau\in[s-p,s]}\int_{\tau-1}^\tau \frac{h(\xi)}{(\tau-\xi)^\alpha}\,d\xi\,ds \\ &\le \frac{C}{l}\int_q^l e^{-\gamma(s-p)}\sup_{\tau\in[s-p,s]}\int_0^{\tau-1} e^{\gamma\xi}h(\xi)\,d\xi\,ds + \frac{C}{(1-\alpha)l}\int_q^l \sup_{\xi\in[s-p-1,s]} h(\xi)\,ds \\ &\le \frac{Ce^{\gamma p}}{l}\int_0^l\int_0^s e^{-\gamma(s-\xi)}h(\xi)\,d\xi\,ds + \frac{C}{(1-\alpha)l}\int_q^l \sum_{i=-1}^{k-1}\sup_{\tau\in[s-1+ip,(s-1)+(i+1)p]} h(\xi)\,ds \\ &\le \frac{Ce^{\gamma p}}{l}\int_0^l\int_\xi^l e^{-\gamma(s-\xi)}h(\xi)\,ds\,d\xi + \frac{C}{(1-\alpha)l}\sum_{i=-1}^{k-1}\int_{q-1+(i+1)p}^{l-1+(i+1)p}\sup_{\tau\in[s-p,s]} h(\xi)\,ds \\ &\le \frac{Ce^{\gamma p}}{l}\int_0^l e^{\gamma\xi}h(\xi)\Big(\frac{e^{-\gamma\xi}-e^{-\gamma l}}{\gamma}\Big)\,d\xi + \frac{(k+1)C}{(1-\alpha)l}\int_p^{l+kp}\sup_{\tau\in[s-p,s]} h(\xi)\,ds \\ &\le \frac{Ce^{\gamma p}}{\gamma l}\Big(\int_0^p h(\xi)\,d\xi + \int_p^l h(\xi)\,d\xi\Big) + \frac{(k+1)C}{(1-\alpha)l}\int_p^{l+kp}\sup_{\tau\in[s-p,s]} h(\xi)\,ds \\ &\le \frac{2Ce^{\gamma p}p}{\gamma l}\|u\|_{C_b([0,\infty);Y)} + \frac{Ce^{\gamma p}}{\gamma}\frac{1}{l}\int_p^l h(\xi)\,d\xi \\ &\quad + \frac{(k+1)C(l+kp)}{(1-\alpha)l}\frac{1}{l+kp}\int_p^{l+kp}\sup_{\tau\in[s-p,s]} h(\xi)\,ds, \end{aligned}$$

which shows that $\lim_{l\to\infty} I_3(l) = 0$ and completes the proof that $v\in\mathcal{PSAP}_{\omega,p}(X)$.

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To finish the proof, we next assume that $p \geq 1$. Let $q > 2p + 1$. If $I_i(l)$, $i = 1, 2, 3$, are defined as above, for $l \geq q$ it is easy to see that

$$\frac{1}{l}\int_p^l \sup_{\tau\in[s-p,s]} \|v(\tau+\omega) - v(\tau)\|\,ds \leq I_1(l) + I_2(l) + I_3(l)$$

and $\lim_{l\to\infty}(I_1(l) + I_2(l)) = 0$. Thus, to complete the proof it remains to show that $\lim_{l\to\infty} I_3(l) = 0$.

From a review of our estimates above for $I_3(l)$, it is easy to see that

$$I_3(l) \leq \frac{C}{l}\int_q^l e^{-\gamma(s-p)} \sup_{\tau\in[s-p,s]} \int_0^{\tau-1} e^{\gamma\xi} h(\xi)\,d\xi\,ds + \frac{C}{(1-\alpha)l}\int_q^l \sup_{\xi\in[s-p-1,s]} h(\xi)\,ds,$$
$$\frac{C}{l}\int_q^l e^{-\gamma(s-p)} \sup_{\tau\in[s-p,s]} \int_0^{\tau-1} e^{\gamma\xi} h(\xi)\,d\xi\,ds \leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi.$$

Therefore,

$$\begin{aligned}
I_3(l) &\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi + \frac{C}{(1-\alpha)l}\int_q^l \sup_{\xi\in[s-p-1,s]} h(\xi)\,ds \\
&\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi + \frac{C}{(1-\alpha)l}\int_q^l \sup_{\xi\in[s-2p,s]} h(\xi)\,ds \\
&\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi + \frac{C}{(1-\alpha)l}\int_q^l \Big(\sup_{\xi\in[s-2p,s-p]} h(\xi) + \sup_{\xi\in[s-p,s]} h(\xi)\Big)\,ds \\
&\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi + \frac{C}{(1-\alpha)l}\Big(\int_{q-p}^{l-p} \sup_{\xi\in[s-p,s]} h(\xi)\,ds + \int_q^l \sup_{\xi\in[s-p,s]} h(\xi)\,ds\Big) \\
&\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^l h(\xi)\,d\xi + \frac{2C}{(1-\alpha)l}\int_p^l \sup_{\xi\in[s-p,s]} h(\xi)\,ds \\
&\leq \frac{Ce^{\gamma p}}{\gamma l}\int_0^p h(\xi)\,d\xi + \Big(\frac{Ce^{\gamma p}}{\gamma} + \frac{2C}{(1-\alpha)}\Big)\frac{1}{l}\int_p^l \sup_{\xi\in[s-p,s]} h(\xi)\,ds \\
&\leq \frac{2pCe^{\gamma p}}{\gamma l}\|u\|_{C_b([0,\infty);Y)} + C\Big(\frac{e^{\gamma p}}{\gamma} + \frac{2}{(1-\alpha)}\Big)\frac{1}{l}\int_p^l \sup_{\xi\in[s-p,s]} h(\xi)\,ds,
\end{aligned}$$

which implies that $\lim_{l\to\infty} I_3(l) = 0$. This completes the proof. □

The following lemma is proved by arguing as in the proof of Lemma 3.4.

LEMMA 3.5. *Let $u \in C_b([-r,\infty); X)$ and $v : [-r,\infty) \to X$ be the function defined by $v(t) = 0$ for $t \in [-r, 0]$ and $v(t) = \int_0^t T(t-\tau)u(\tau)\,d\tau$ for $t \geq 0$. If $u|_{[0,\infty)} \in \mathcal{PSAP}_{\omega,p}(X)$, then $v \in \mathcal{PSAP}_{\omega,p}(X)$.*

We can now establish the main result of this paper. In the next theorem, i_c denotes the inclusion map from Y into X and Y is the space in condition $\mathbf{H_1}$.

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THEOREM 3.6. *Assume that conditions* $\mathbf{H_1}$, $\mathbf{H_2}$ *are satisfied,* $G \in \mathcal{PSAP}_{\omega,p}(C, X)$*,* $F \in \mathcal{PSAP}_{\omega,p}(C, Y)$ *and the functions* $\|G(\cdot, 0)\|$*,* $\|F(\cdot, 0)\|_Y$ *belong to* $C_b([0, \infty); \mathbb{R})$*. If*

$$\Theta = \|L_F\|_{C_b([0,\infty);\mathbb{R})}\Big(\|i_c\|_{\mathcal{L}(Y,X)} + C\Big(\frac{1}{\gamma} + \frac{1}{1-\alpha}\Big)\Big) + \frac{M}{\gamma}\|L_G\|_{C_b([0,\infty);\mathbb{R})} < 1,$$

then there exists a unique mild solution $u \in \mathcal{PSAP}_{\omega,p}(X)$ *of the problem (1.1)–(1.2).*

PROOF. Let $\mathfrak{B} = \{u : [-r, \infty) \to X \mid u_0 = \varphi,\ u|_{[0,\infty)} \in \mathcal{PSAP}_{w,p}(X)\}$ endowed with the metric $d(u, z) = \|u - z\|_{C([0,\infty);X)}$ and let $\Gamma : \mathfrak{B} \to \mathfrak{B}$ be the map defined by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)(\varphi(0) - F(0, \varphi)) + F(t, u_t) + \int_0^t AT(t-s)F(s, u_s)\, ds + \int_0^t T(t-s)G(s, u_s)\, ds, \quad t \geq 0.$$

Next, we prove that Γ is a contraction on $\mathfrak{B}$. Let $u \in \mathfrak{B}$. From the estimate

$$\begin{aligned}
\Big\|\int_0^t T(t-s)G(s, u_s)\, ds\Big\| &+ \Big\|\int_0^t AT(t-s)F(s, u_s)\, ds\Big\| \\
&\leq \int_0^t Me^{-\gamma(t-s)}L_G(s)\|u_s\|_C\, ds + \int_0^t Me^{-\gamma(t-s)}\|G(s, 0)\|\, ds \\
&\quad + \int_0^t Ce^{-\gamma(t-s)}\frac{L_F(s)}{(t-s)^\alpha}\|u_s\|_C\, ds + \int_0^t C\frac{\|F(s, 0)\|_Y}{(t-s)^\alpha}\, ds \\
&\leq \frac{M}{\gamma}\|L_G\|_{C_b([0,\infty);\mathbb{R})}\|u\|_{C_b([-r,\infty);X)} + \frac{M}{\gamma}\|G(\cdot, 0)\|_{C_b([0,\infty);X)} \\
&\quad + \|L_F\|_{C_b([0,\infty);\mathbb{R})}C\Big(\frac{1}{\gamma} + \frac{1}{1-\alpha}\Big)\|u\|_{C_b([-r,\infty);X)} \\
&\quad + C\Big(\frac{1}{\gamma} + \frac{1}{1-\alpha}\Big)\|F(\cdot, 0)\|_{C_b([0,\infty);Y)},
\end{aligned}$$

we infer that $\Gamma u \in C_b([0, \infty); X)$, and from Lemmas 3.2–3.5 it follows that $\Gamma u \in \mathfrak{B}$. Moreover, for $u, z \in \mathfrak{B}$ and $t \geq 0$, we see that

$$\begin{aligned}
\|\Gamma u(t) - \Gamma z(t)\| \leq \|i_c\|_{\mathcal{L}(Y,X)}L_F(t)\|u - z\|_{C_b([0,\infty);X)} &+ \int_0^t \frac{Ce^{-\gamma(t-s)}}{(t-s)^\alpha}L_F(s)\, ds\|u - z\|_{C_b([0,\infty);X)} \\
&+ \int_0^t Me^{-\gamma(t-s)}L_G(s)\, ds\|u - z\|_{C_b([0,\infty);X)},
\end{aligned}$$

which implies that $d(\Gamma u, \Gamma z) \leq \Theta d(u, z)$ and there exists a unique mild solution $u \in \mathcal{PSAP}_{\omega,p}(X)$ of (1.1)–(1.2). The proof is complete. □

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4. Applications

In this section we discuss the existence of pseudo $\mathcal{S}$-Asymptotically ω-periodic mild solutions for neutral differential equations. To begin, we consider the ordinary neutral differential equation

$$\frac{d}{dt}\Big(u(t) - \lambda \int_{t-r}^{t} C(t-s)u(s)\,ds\Big) = Au(t) + \lambda \int_{t-r}^{t} B(t-s)u(s)\,ds - \eta(t) + q(t), \qquad t \geq 0, \tag{4.1}$$

$$u(\theta) = \varphi(\theta), \qquad \theta \in [-r, 0], \tag{4.2}$$

which arises in the study of the dynamics of income, employment, value of capital stock, and cumulative balance of payments; see [6] for details. In this system, $\lambda \in \mathbb{R}$, the state $u(t) \in \mathbb{R}^n$, C, B are $n \times n$ matrix continuous functions, A is a constant $n \times n$ matrix, η represents government intervention and q private initiative.

In this case, our results are easily applicable since the assumption $\mathbf{H_1}$ is automatically satisfied with $Y = X = \mathbb{R}^n$ and $\alpha = 0$. To represent this system in the abstract form (1.1)–(1.2) we introduce the functions $F, G : [0, a] \times C \to X$ defined by

$$F(t, \psi) = -\lambda \int_{-r}^{0} C(-s)\psi(s)\,ds \quad \text{and} \quad G(t, \psi)(t) = \lambda \int_{-r}^{0} B(-s)\psi(s)\,ds - \eta(t) + g(t).$$

In the next result, which is a consequence of Theorem 3.6, we say that a function $u \in C_b([-r, \infty); X)$ is a mild solution of (4.1)–(4.2) if u is a mild solution of the associated abstract problem (1.1)–(1.2). In the rest of this section p, w are positive real numbers.

Proposition 4.1. *Assume that $\eta, q \in \mathcal{PSAP}_{w,p}(\mathbb{R})$, there are positive constants M, γ such that $\|e^{tA}\| \leq Me^{-\gamma t}$ and*

$$\|C\|_{L^1([0,r];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))}\Big(1 + M\|A\|\Big(\frac{1}{\gamma} + 1\Big)\Big) + \frac{M}{\gamma}\|B\|_{L^1([0,r];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))} < 1.$$

Then there exists a unique mild solution $u \in \mathcal{PSAP}_{\omega,p}(\mathbb{R}^n)$ of (4.1)–(4.2).

We now consider some examples involving delayed partial differential equations. Suppose that $X = L^2([0, \pi])$ and $A : D(A) \subset X \to X$ is the operator given by $Ax = x''$ on $D(A) := \{x \in X \mid x'' \in X, \ \ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and associated normalised eigenvectors $z_n(\xi) = (2/\pi)^{1/2} \sin(n\xi)$. We note that $\{z_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of X, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ and $\| T(t) \| \leq e^{-t}$ for all $x \in X$ and each $t \geq 0$. Moreover, in this case we note that $(-A)^{-1/2}x = \sum_{n=1}^{\infty}(1/n)\langle x, z_n\rangle z_n$ for $x \in X$, $(-A)^{1/2}x = \sum_{n=1}^{\infty} n\langle x, z_n\rangle z_n$ for $x \in D((-A)^{1/2}) = \{x \in X \mid \sum_{n=1}^{\infty} n\langle x, z_n\rangle z_n \in X\}$. In addition, $\|(-A)^{-1/2}\|_{\mathcal{L}(X)} = 1$ and

$$\| (-A)^{1/2}T(t) \|_{\mathcal{L}(X)} \leq \frac{1}{\sqrt{2}} e^{-t/2} t^{-1/2} \quad \text{for all } t > 0.$$

Consider the delayed partial differential equation

$$\frac{\partial}{\partial t}u(t,\xi) = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \int_{t-r}^{t} b(s-t)u(s,\xi)\,ds + c(t)g(u(t,\xi)), \tag{4.3}$$

$$u(t,0) = u(t,\pi) = 0, \tag{4.4}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad -r \le \theta \le 0, \tag{4.5}$$

for $t > 0$ and $\xi \in [0,\pi]$, where r is a positive real number, $\varphi \in C([-r,0];X)$, $c \in \mathcal{PSAP}_{w,p}(\mathbb{R})$, $b \in C([-r,0];\mathbb{R})$ and $g:\mathbb{R}\to\mathbb{R}$ is Lipschitz with Lipschitz constant L_g.

Let $C = C([-r,0];X)$. By defining the function $G:[0,\infty)\times C \to X$ by

$$G(t,\psi)(\xi) = \int_{-r}^{0} b(s)\psi(s,\xi)\,ds + c(t)g(\psi(0,\xi)),$$

we represent (4.3)–(4.5) in the abstract form (1.1)–(1.2). Moreover, it is easy to see that

$$\|G(t,\psi_1) - G(t,\psi_2)\| \le (\|b\|_{L^2([-r,0];\mathbb{R})}r^{1/2} + |c(t)|L_g)\|\psi_1-\psi_2\|_C, \quad \psi_i \in C, \quad t \ge 0.$$

As above, we say that $u \in C_b([-r,\infty);X)$ is a mild solution of (4.3)–(4.5) if u is a mild solution of the associated problem (1.1)–(1.2). From Theorem 3.6, we have the following result.

Proposition 4.2. *If* $(\|b\|_{L^2([-r,0];\mathbb{R})}r^{1/2} + \|c\|_{C_b([0,\infty);\mathbb{R})}L_g) < 1$, *then there exits a unique mild solution* $u \in \mathcal{PSAP}_{\omega,p}(X)$ *of the problem (4.3)–(4.5).*

We conclude this paper with an application to partial neutral differential equations which arise in control systems described by abstract retarded functional-differential equations with feedback control governed by a proportional integro-differential law; see [12, Examples 4.2] for details. Specifically, we study the problem

$$\frac{\partial}{\partial t}\left(u(t,\xi) + \int_{-r}^{0}\int_{0}^{\pi} b(s,\eta,\xi)u(t+s,\eta)\,d\eta\,ds\right)$$
$$= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(t)u(t,\xi) + \int_{-r}^{0} a(t,s)u(t+s,\xi)\,ds, \quad t \ge 0,\ \xi \in [0,\pi], \tag{4.6}$$

$$u(t,0) = u(t,\pi) = 0, \tag{4.7}$$

where $a_0 \in \mathcal{PSAP}_{\omega,p}(\mathbb{R})$, $a \in \mathcal{PSAP}_{\omega,p}(L^2([-r,0];\mathbb{R}))$, the functions $b, (\partial^i/\partial\zeta^i)$ $b(\tau,\eta,\zeta)$, $i = 1, 2$, are measurable, $b(\tau,\eta,\pi) = b(\tau,\eta,\ 0) = 0$ for every (τ,η) and

$$N_1 := \max\left\{\int_0^{\pi}\int_{-r}^{0}\int_0^{\pi}\left(\frac{\partial^i}{\partial \zeta^i}b(\tau,\eta,\zeta)\right)^2\,d\eta\,d\tau\,d\zeta \mid i = 0,1\right\} < \infty.$$

Let $C = C([-r,0];X)$ and let $F, G:[0,a]\times C \to X$ be functions defined by

$$F(t,\psi)(\xi) := \int_{-r}^{0}\int_{0}^{\pi} b(\tau,\eta,\xi)\psi(\tau,\eta)\,d\eta\,d\tau,$$

$$G(t,\psi)(\xi) := a_0(t)\psi(0,\xi) + \int_{-r}^{0} a(t,s)\psi(s,\xi)\,ds.$$

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It is easy to see that $F \in C([0, a]; \mathcal{L}(C; X_{1/2}))$, $G \in C([0, a]; \mathcal{L}(C; X))$, $\| F(t, \cdot) \|_{\mathcal{L}(C;X_{1/2})} \leq (N_1 r)^{1/2}$ and $\| G(t, \cdot) \|_{\mathcal{L}(C,X)} \leq |a_0(t)| + \|a(t)\|_{L^2([-r,0];\mathbb{R})}$ for every $t \geq 0$. The next result follows from Theorem 3.6 with $\alpha = \frac{1}{2}$.

THEOREM 4.3. *If*

$$(N_1 r)^{1/2}\left(1 + \frac{4}{\sqrt{2}}\right) + \|a_0\|_{C_b([0,\infty);\mathbb{R})} + \|a\|_{C_b([0,\infty);L^2([-r,0];\mathbb{R}))} < 1,$$

then there exists a unique mild solution $u \in \mathcal{PSAP}_{\omega,p}(X)$ *of the system (4.6)–(4.7).*

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