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A NOTE ON DERIVATIONS II.

BY

I. N. HERSTEIN[1]

In a previous note on derivations [1] we determined the structure of a prime ring R which has a derivation $d \neq 0$ such that the values of d commute, that is, for which $d(x)\,d(y) = d(y)\,d(x)$ for all $x, y \in R$. Perhaps even more natural might be the question: what elements in a prime ring commute with all the values of a non-zero derivation? We address ourselves to this question here, and settle it.

We prove the

THEOREM. *Let R be a prime ring and let $d \neq 0$ be a derivation of R. Suppose that $a \in R$ is such that $ad(x) = d(x)a$ for all $x \in R$. Then:*

(1) *If R is not of characteristic 2, a must be in Z, the center of R.*

(2) *If R is of characteristic 2, then $a^2 \in Z$. Moreover, if $a \notin Z$ then d is the inner derivation given by $d(x) = (\lambda a)x - x(\lambda a)$, where λ is in the extended centroid of R, for all $x \in R$.*

Proof. Suppose that $a \notin Z$. Using the hypothesis we have, for all $x, y \in R$, that $[a, d(xy)] = 0$ where $[u, v]$ denotes the commutator $uv - vu$. Since $d(xy) = d(x)y + xd(y)$ we have $[a, d(x)y + xd(y)] = 0$. Again making use of the fact that a commutes with all $d(t)$ we obtain

(1) $$[a, x]\,d(y) + d(x)[a, y] = 0.$$

If $y \in R$ commutes with a then $[a, y] = 0$, hence (1) reduces to $[a, x]\,d(y) = 0$ for all $x \in R$. Because $a \notin Z$, by the Corollary to Lemma 1.1.7 of [2] we are forced to conclude that $d(y) = 0$. In other words, d vanishes on the centralizer, $C_R(a) = \{y \in R \mid ya = ay\}$, of a in R. But, for any $x \in R$, $d(x) \in C_R(a)$ by hypothesis; thus we get that $d^2(x) = 0$ for all $x \in R$.

However, as the proof of Lemma 1.1.9 of [2] shows, if R is prime (even semi-prime) of characteristic not 2 and d is a derivation of R such that $d^2 = 0$ then $d = 0$. Since we have supposed that $d \neq 0$, if the characteristic of R is not 2, by the results above we are led to the conclusion $a \in Z$. This settles the situation when the characteristic of R is not 2.

So, from this point on, we assume that R is of characteristic 2 and that $a \notin Z$. In this case equation (1) becomes

(2) $$[a, x]\,d(y) = d(x)[a, y] \quad \text{for all} \quad x, y \in R.$$

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Thus, if $d(y)=0$ we obtain from equation (2) that $d(x)[a, y]=0$ for all $x\in R$. The proof of Lemma 1.1.7 of [2] reveals that when R is prime we must have $[a, y]=0$, that is, $y\in C_R(a)$. Combined with what we said earlier, namely that d vanishes on $C_R(a)$, we now know that $C_R(a)$ coincides with $\{y\in R \mid d(y)=0\}$.

We return to equation (2), substituting in it xw for x, where x and w are arbitrary in R. Hence $[a, xw]\,d(y)=d(xw)[a, y]$. In this last relation we insert the explicit values $[a, xw]=[a, x]w+x[a, w]$ and $d(xw)=d(x)w+xd(w)$; we end up with $[a, x]wd(y)+x[a, w]\,d(y)=d(x)w[a, y]+xd(w)[a, y]$. However, equation (2) gives us equality for the last terms on both sides of this equation; thus we obtain

$$(3) \qquad [a, x]wd(y)=d(x)w[a, y] \quad \text{for all} \quad w\in R, \quad \text{and all} \quad x, y\in R.$$

If $[a, x]\neq 0$, using a result of Martindale (Lemma 1.3.2 in [2]) we have that $d(x)=\lambda(x)[a, x]$ where $\lambda(x)$ is in the extended centroid of R. (See p. 22 of [2] for the notion of extended centroid.) Moreover, since $C_R(a)=\{y\in R \mid d(y)=0\}$, we must have that $\lambda(x)\neq 0$ if $[a, x]\neq 0$. Also, if $[a, x]=0$ then $d(x)=0$ hence $0=d(x)=0[a, x]$. Thus for all $x\in R$, $d(x)=\lambda(x)[a, x]$ where $\lambda(x)$ is in the extended centroid of R.

We claim that $[a,[a, x]]=0$ for all $x\in R$. Clearly, if $[a, x]=0$ then $[a,[a, x]]=0$. On the other hand, if $[a, x]\neq 0$ then, by the above, $d(x)=\lambda(x)[a, x]$ where $\lambda(x)\neq 0$, so since $[a, d(x)]=0$ we have that $\lambda(x)[a,[a, x]]=0$. Because $\lambda(x)\neq 0$ is invertible we end up with $[a,[a, x]]=0$ for all $x\in R$. Writing this out $a(ax+xa)=(ax+xa)a$ we see that $a^2\in Z$.

Now to the final part of the theorem. We just saw that if $a\notin Z$ then $d(x)=\lambda(x)[a, x]$, with $\lambda(x)$ in the extended centroid, for all $x\in R$. We want to prove that $\lambda(x)$ is a constant.

Let $x, y\in R$; then $d(xy)=\lambda(xy)[a, xy]$, that is, $d(x)y+xd(y)=\lambda(xy)[a, x]y+\lambda(xy)x[a, y]$. Because $d(x)=\lambda(x)[a, x]$, $d(y)=\lambda(y)[a, y]$ we get $\lambda(x)[a, x]y+\lambda(y)x[a, y]=\lambda(xy)[a, x]y+\lambda(xy)x[a, y]$. Hence, if $\mu=\lambda(x)+\lambda(xy)$ and $\nu=\lambda(y)+\lambda(xy)$, the above boils down to

$$\mu[a, x]y=\nu x[a, y] \quad \text{for all} \quad x, y\in R.$$

Since $a^2\in Z$, $[a[a, x]]=0$, we obtain from this, by commuting it with a, that

$$(\mu+\nu)[a, x][a, y]=0 \quad \text{for all} \quad x, y\in R.$$

Thus, if $[a, x][a, y]\neq 0$ we have $\mu+\nu=0$, that is, $\lambda(x)+\lambda(xy)+\lambda(y)+\lambda(xy)=0$, and so, $\lambda(x)=\lambda(y)$. Suppose now that $[a, x]\neq 0$, $[a, y]\neq 0$. We claim that there is a $w\in R$ such that both $[a, x][a, w]\neq 0$ and $[a, w][a, y]\neq 0$. If this were so we would have by the above that $\lambda(x)=\lambda(w)$ and $\lambda(w)=\lambda(y)$, hence $\lambda(x)=\lambda(y)$. This would tell us that λ would be constant on all elements failing to commute with a. Knowing further that

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$C_R(a)=\{y \in R \mid d(y)=0\}$ would then tell us that $d(x)=[\lambda a, x]$ for all $x \in R$, for some λ in the extended centroid. This is, of course, our desired result.

So, to finish, we must show the existence of such a $w \in R$. In fact, we shall show a little more, namely, that there is an element $w \in R$ such that $[a, x][a, w][a, y] \neq 0$. If this were not true then $[a, x][a, z][a, y]=0$ for all $z \in R$, that is, $[a, x]az[a, y]=[a, x]za[a, y]$. By the result of Martindale quoted earlier, $[a, x]a=\mu[a, x]$ where μ is in the extended centroid of R. Since $a^2=\sigma \in Z$ we have that $\mu^2=\sigma$ and since the extended centroid is a field and is of characteristic 2, μ is uniquely determined by σ, hence does not depend on x. But then $[a, x](a+\mu)=0$ for all x such that $[a, x]\neq 0$; if $[a, x]=0$ this relation is certainly true. So $[a, x](a+\mu)=0$ for all $x \in R$. But then this carries over to all x in the central closure T of R, which itself is a prime ring. Since $a \notin Z$ and $[a, x](a+\mu)=0$ for all $x \in T$, by the Corollary to Lemma 1.1.7 of [2] we deduce that $a+\mu=0$, and so $a \in Z$. With this contradiction the theorem is proved.

Bibliography

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Department of Mathematics
The University of Chicago
5734 University Avenue
Chicago, Illinois 60637