Canad. Math. Bull. Vol. **15** (2), 1972

ON THE MULTIPLICATIVE INVERSE EIGENVALUE PROBLEM

BY

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1. By "multiplicative inverse eigenvalue problem" (m.i.e.p., for short) we mean the following. Let A be an $n \times n$ matrix and let $s_1, \ldots, s_n$ be n given numbers. Under what conditions does there exist an $n \times n$ diagonal matrix V such that VA has eigenvalues $s_1, \ldots, s_n$?

In the "additive inverse eigenvalue problem" (a.i.e.p., for short) we seek the diagonal matrix V so that $A+V$ has eigenvalues $s_1, \ldots, s_n$.

In the present paper we extend to the m.i.e.p. the ideas used in [7] for the a.i.e.p.

By per X we denote the permanent of the square matrix X.

Let A be an $n \times n$ matrix and let I denote the identity matrix of the same order. Obviously,

$$f(z) = \text{per}\,(A - zI)$$

is a polynomial in z of degree n. We shall call the roots of this polynomial, permanental roots of A. If in the m.i.e.p. we replace eigenvalues by permanental roots, we obtain the "multiplicative inverse permanental root problem" (m.i.p.p.); if in the a.i.e.p. we replace eigenvalues by permanental roots we obtain the "additive inverse permanental root problem" (a.i.p.p.).

In §3 we give some results on the m.i.p.p. and a.i.p.p.

2. Let $A = [a_{ij}]$ be an $n \times n$ matrix,

$$P_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|,$$

$$V = \text{diag}\,(v_1, \ldots, v_n),$$

$$v = (v_1, \ldots, v_n)$$

and

$$s = (s_1, \ldots, s_n).$$

If a and b are two real numbers, whether $a \leq b$ or $a > b$ by $[a, b]$ we shall mean the set of real numbers between a and b, a and b included.

Note that for either of the multiplicative problems if $a_{ii} \neq 0$ $(i=1, \ldots, n)$, there is no loss of generality if we assume that $a_{ii} = 1 (i=1, \ldots, n)$.

THEOREM 2.1. *Let $A = [a_{ij}]$ be an $n \times n$ real matrix. Assume that $a_{ii} = 1$ $(i=1, \ldots, n)$, $\max_i P_i \leq \frac{1}{2}$ and no two of the intervals $[s_i(1-2P_i), s_i\{(1+P_i)/(1-P_i)\}]$ $(i=1, \ldots, n)$*

Received by the editors October, 1, 1970.

https://doi.org/10.4153/CMB-1972-034-0 Published online by Cambridge University Press

intersect where the s_i are real numbers. Then there exists a diagonal matrix $\tilde{V}$ such that $\tilde{V}A$ has eigenvalues $s_1, \ldots, s_n$.

Proof. Let v_i be real numbers satisfying

$$v_i \in \left[s_i \frac{1-2P_i}{1-P_i}, \frac{s_i}{1-P_i}\right] \quad (i = 1, \ldots, n). \tag{2.1}$$

Let $\lambda_1(v), \ldots, \lambda_n(v)$ be the eigenvalues of VA. It can be easily seen that no two of the Geršgorin circles corresponding to the matrix VA intersect. Therefore each Geršgorin circle contains exactly one eigenvalue of VA and each eigenvalue is real. Reordering, if necessary, these eigenvalues, we can write

$$\lambda_i(v) \in [v_i(1-P_i), v_i(1+P_i)] \quad (i = 1, \ldots, n). \tag{2.2}$$

Let $\lambda(v) = (\lambda_1(v), \ldots, \lambda_n(v))$.

Consider the operator T defined by

$$T(v) = v+s-\lambda(v).$$

Going over the coordinates of $T(v)$ we get

$$T_i(v) \in \left[s_i \frac{1-2P_i}{1-P_i}, \frac{s_i}{1-P_i}\right] \quad (i = 1, \ldots, n).$$

By the Brouwer fixed point theorem, there exists a vector $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)$ such that $T(\tilde{v}) = \tilde{v}$; i.e. $\lambda(\tilde{v}) = s$.

Therefore, taking $\tilde{V} = \operatorname{diag}(\tilde{v}_1, \ldots, \tilde{v}_n)$, $\tilde{V}A$ has eigenvalues $s_1, \ldots, s_n$ as required.

REMARK. It is obvious that the solution $\tilde{v}$ satisfies

$$\tilde{v}_i \in \left[s_i \frac{1-2P_i}{1-P_i}, \frac{s_i}{1-P_i}\right] \quad (i = 1, \ldots, n). \tag{2.3}$$

If, however, $s_i > 0$ and $\tilde{v}_i < s_i/(1+P_i)$, (2.2) gives $\lambda_i < s_i$.

If $s_i < 0$ and $\tilde{v}_i > s_i/(1+P_i)$, (2.2) gives $\lambda_i > s_i$. Therefore (2.3) can be improved to

$$\tilde{v}_i \in \left[\frac{s_i}{1+P_i}, \frac{s_i}{1-P_i}\right] \quad (i = 1, \ldots, n).$$

The above theorem is not contained in Hadeler's results [5] since it can be applied to nonsymmetric matrices. We show with a numerical example that even for symmetric matrices our theorem is not contained in those of Hadeler.

Let

$$A = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{4} & 1 & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{8} & 1 \end{bmatrix}$$

and $s_1 = -1$, $s_2 = 1$, $s_3 = \frac{11}{2}$. The hypothesis of theorem (2.1) is satisfied but not the hypothesis of Theorem 4 in [5].

REMARK. Let T be a nonsingular diagonal matrix of order n. If $\tilde{V}$ is diagonal we have $\tilde{V}T=T\tilde{V}$ and so $\tilde{V}TAT^{-1}=T\tilde{V}AT^{-1}$; i.e. $\tilde{V}TAT^{-1}$ and $\tilde{V}A$ have the same eigenvalues. This means that the existence of a solution of the m.i.e.p. for TAT^{-1} implies the existence of a solution of the same problem for A.

THEOREM 2.2. *Let $A=[a_{ij}]$ be a nonnegative irreducible $n\times n$ matrix with dominant eigenvalue r. If $a_{ii}=1$ $(i=1,\ldots,n)$, $r\le\frac{3}{2}$ and*

$$\frac{|s_i-s_j|}{|s_i|+|s_j|} > \left|\frac{r^2-4r+3}{r^2-3r+3}\right| \quad (i,j=1,\ldots,n;\, i\neq j) \tag{2.4}$$

where the s_i are distinct real numbers, there exists a diagonal matrix $\tilde{V}$ such that $\tilde{V}A$ has eigenvalues $s_1,\ldots,s_n$.

Proof. There exists a diagonal matrix T such that TAT^{-1} has all row sums equal to r, as is well known. Of course, the diagonal elements of TAT^{-1} are equal to 1. Applying Theorem 2.1 to the matrix TAT^{-1} and noting that the condition that the intervals $[s_i(1-2P_i), s_i\{(1+P_i)/(1-P_i)\}]$ should not intersect is equivalent to (2.4), the present theorem follows.

THEOREM 2.3. *Let $A=[a_{ij}]$ be an $n\times n$ real symmetric matrix with $a_{ii}=1 (i=1,\ldots,n)$. Assume that there exists $\tilde{V}=\operatorname{diag}(\tilde{v}_1,\ldots,\tilde{v}_n)$ such that $\tilde{v}_i>0$ $(i=1,\ldots,n)$ and $\tilde{V}A$ has eigenvalues $s_1,\ldots,s_n$. Let $H(s)$ denote the convex hull of all points $(s_{\sigma(1)},\ldots,s_{\sigma(n)})$ where σ runs over the symmetric group S_n. Then $(\tilde{v}_1,\ldots,\tilde{v}_n)\in H(s)$.*

Proof. First we note that a sufficient condition for the existence of $\tilde{V}=\operatorname{diag}(\tilde{v}_1,\ldots,\tilde{v}_n)$ such that $\tilde{v}_i>0\,(i=1,\ldots,n)$ and $\tilde{V}A$ have eigenvalues $s_1,\ldots,s_n$ is that the hypothesis of Theorem 2.1 be satisfied and that the numbers s_i be positive (see the remark after the proof of Theorem 2.1). In any case, if the hypothesis of the present theorem is satisfied, the numbers s_i cannot be imaginary. In fact let $M=\operatorname{diag}(+\sqrt{\tilde{v}_1},\ldots,+\sqrt{\tilde{v}_n})$. The numbers s_i are the eigenvalues of the real symmetric matrix MAM. The principal elements of MAM are $\tilde{v}_1,\ldots,\tilde{v}_n$. If $x=(x_1,\ldots,x_n)$ (x_i real) by $\sum^{(k)}(x_1,\ldots,x_n)$ we denote the sum of the k greatest coordinates of x. If $s_1,\ldots,s_n$ are the eigenvalues of MAM, we can write

$$\sum^{(k)}(\tilde{v}_1,\ldots,\tilde{v}_n) \le \sum^{(k)}(s_1,\ldots,s_n) \quad (k=1,\ldots,n)$$

with equality for $k=n$ [4]. These conditions mean that $(\tilde{v}_1,\ldots,\tilde{v}_n)\in H(s)$ [6].

Finally we note that the hypothesis of Theorem 2.3 is not so restrictive as it seems to be. In fact if $A=[a_{ij}]$ is a real symmetric matrix satisfying only $a_{ii}>0$ $(i=1,\ldots,n)$ we can always reduce the problem to the case of a real symmetric matrix with all diagonal elements equal to 1. Let

$$D = \operatorname{diag}(1/a_{11},\ldots,1/a_{nn})$$

and

$$D^{1/2} = \operatorname{diag}(1/+\sqrt{a_{11}},\ldots,1/+\sqrt{a_{nn}}).$$

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The matrices DA and $D^{1/2}AD^{1/2}$ are diagonally similar.

It is sufficient to solve the problem for $D^{1/2}AD^{1/2}$ (see the remark before Theorem 2.2) and this matrix is real symmetric with diagonal elements equal to 1.

3. Now we show that some results on the m.i.e.p. and a.i.e.p. carry over to the m.i.p.p. and a.i.p.p. respectively. First we prove some results on the permanental roots.

THEOREM 3.1. *Let $A=[a_{ij}]$ be an $n\times n$ arbitrary complex matrix. The permanental roots of A lie in the union of the n circles*

$$|z-a_{ii}| \leq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad (i = 1,\ldots,n). \tag{3.1}$$

Proof. This theorem is an immediate consequence of the fact that a diagonal dominant matrix has nonzero permanent [2].

THEOREM 3.2. *Assume that m of the circles (3.1) do not intersect the remaining $n-m$ circles. Then those m circles contain exactly m permanental roots of A.*

The proof of this theorem depends on a continuity argument and follows that given in [1] for the eigenvalues.

COROLLARY TO THEOREM 3.2. *Suppose the ith of the circles (3.1) does not intersect the remaining ones, A has real principal elements and the polynomial $f(z)=\text{per}$ $(A-zI)$ has real coefficients. Then the unique permanental root of A contained in the ith of the circles (3.1) is real.*

Proof. Bearing in mind that the complex permanental roots have to lie symmetrically about the real axis, this corollary is an immediate consequence of Theorem 3.2.

Clearly if A is real and no two of the circles (3.1) intersect, the permanental roots of A are distinct and real.

THEOREM 3.3. *Let A be a nonnegative irreducible $n\times n$ matrix with dominant eigenvalue r. Then its permanental roots lie in the circle*

$$|z-a_{mm}| \leq r-a_{mm}$$

where

$$a_{mm} = \min_i a_{ii}.$$

Proof. There exists a nonnegative diagonal matrix T such that $TAT^{-1}=S=[s_{ij}]$ with $\sum_{j=1}^{n} s_{ij}=r$. Obviously $s_{ii}=a_{ii}$ $(i=1,\ldots,n)$. Since T is diagonal, the permanental roots of A and S coincide. Applying Theorem 3.1 to S we have that the permanental roots of A lie in the union of the circles

$$|z-a_{ii}| \leq r-a_{ii} \quad (i = 1,\ldots,n).$$

Since all these circles are contained in $|z-a_{mm}| \leq r-a_{mm}$, the theorem is true.

Theorems 3.1, 3.2, and 3.3 improve results presented in [3].

In view of Theorems 3.1 and 3.2, the Corollary to Theorem 3.2 and the remark before Theorem 2.2, which is also valid for the m.i.p.p., it can be easily seen that Theorems 2.1 and 2.2 carry over to the m.i.p.p. Similarly Theorems 1 and 2 of [7] carry over to the a.i.p.p.

REMARK. To prove Theorem 2.1 we applied to A the Geršgorin circle theorem by rows. We could, of course, have applied this theorem by columns. In this case we would have obtained other results which we do not state here because they are more involved.

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https://doi.org/10.4153/CMB-1972-034-0 Published online by Cambridge University Press

https://doi.org/10.4153/CMB-1972-034-0 Published online by Cambridge University Press